Modern Power Systems Analysis
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Preface

The power industry, a capital and technology intensive industry, is a basic national infrastructure. Its security, reliability, and economy have enormous and far-reaching effects on a national economy. An electrical power system is a typical large-scale system. Questions such as how to reflect accurately the characteristics of modern electrical power systems, how to analyze effectively their operating features, and how to improve further the operating performance are always at the forefront of electrical power systems research.

Electrical power system analysis is used as the basic and fundamental measure to study planning and operating problems. In the last century, electrical power researchers have undertaken a great deal of investigation and development in this area, have made great progress in theoretical analysis and numerical calculation, and have written excellent monographs and textbooks.

Over the last 20 years, the changes in electrical power systems and other relevant technologies have had a profound influence on the techniques and methodologies of electrical power system analysis.

First, the development of digital computer technology has significantly improved the performance of hardware and software. Now, we can easily deal with load flow issues with over ten thousand nodes. Optimal load flow and static security analysis, which were once considered hard problems, have attained online practical applications.

Second, the applications of HVDC and AC flexible transmission technologies (FACTS) have added new control measures to electrical power systems, and have increased power transmission capacity, enhanced control capability, and improved operating characteristics. However, these technologies bring new challenges into the area of electrical power system analysis. We must build corresponding mathematical models for these new devices and develop algorithms for static and dynamic analysis of electrical power systems including these devices.

In addition, the rapid development of communication technology has enabled online monitoring of electrical power systems. Therefore, the demand for online software for electrical power system analysis becomes more and more pressing.

Furthermore, worldwide power industry restructuring and deregulation has separated the former vertically integrated system into various parts, and the once
unified problem of power system dispatching is now conducted via complicated bilateral contracts and spot markets. New issues such as transmission ancillary service and transmission congestion have emerged.

In recent years, several power blackouts have taken place worldwide, especially the “8.13” blackout on the eastern grid of USA and Canada and the blackouts that occurred successively in other countries have attracted a great deal of attention.

All of these aspects require new theories, models, and algorithms for electrical power system analysis. It is within such an environment that this book has been developed. The book is written as a textbook for senior students and postgraduates as well as a reference book for power system researchers.

We acknowledge the support from various research funding organizations, their colleagues, and students, especially, the special funds for Major State Basic Research Projects of China “Research on Power System Reliability under Deregulated Environment of Power Market” (2004CB217905). We express our special gratitude to Professor Wan-Liang Fang and Professor Zheng-Chun Du for providing the original materials of Chaps. 5 and 6, and 7 and 8, respectively. We also express our sincere gratitude to the following colleagues for their contributions to various chapters of the book: Professor Zhao-Hong Bie for Chaps. 1 and 3; Professor Xiu-Li Wang for Chaps. 2 and 4; Dr. Ze-Chun Hu for Chap. 3; Dr. Xiao-Ying Ding for Chap. 4; Dr. Lin Duan for Chaps. 5 and 6; Professor De-Chiang Gang for Chap. 7; and Professor Hai-Feng Wang for Chaps. 6 and 8.

Xi’an, China
Liverpool, UK
London, UK

Xi-Fan Wang
Yonghuna Song
Malcolm Irving
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Chapter 1
Mathematical Model and Solution of Electric Network

1.1 Introduction

The mathematical model of an electric network is the basis of modern power system analysis, which is to be used in studies of power flow, optimal power flow, fault analysis, and contingency analysis. The electric network is constituted by transmission lines, transformers, parallel/series capacitors, and other static elements. From the viewpoint of electrical theory, no matter how complicated the network is, we can always establish its equivalent circuit and then analyze it according to the AC circuit laws. In this chapter, the electric network is represented by the linear lumped parameter model that is suitable for studies at synchronous frequency. For electromagnetic transient analysis, the high frequency phenomena and wave processes should be considered. In that situation, it is necessary to apply equivalent circuits described by distributed parameters.

Generally speaking, an electric network can be always represented by a nodal admittance matrix or a nodal impedance matrix. A modern power system usually involves thousands of nodes; therefore methods of describing and analyzing the electric network have a great influence on modern power system analysis. The nodal admittance matrix of a typical power system is large and sparse. To enhance the computational efficiency, sparsity techniques are extensively employed. The nodal admittance matrix and associated sparsity techniques will be thoroughly discussed in this chapter.

The nodal impedance matrix is widely applied in the fault analysis of power systems and will be introduced in Sect. 1.5.

The equivalent circuits of the transformer and phase-shifting transformer are also presented in Sect. 1.1 because they require special representation methods.
1.2 Basic Concepts

1.2.1 Node Equation and Loop Equation

There are two methods usually employed in analyzing AC circuits, i.e., the node voltage method and loop current method. Both methods require the solution of simultaneous equations. The difference between them is that the former applies node equations while the latter applies loop equations. At present, node equations are more widespread in analyzing power systems, and loop equations are used sometimes as an auxiliary tool.

In the following, we use a simple electric network as an example to illustrate the principle and characteristics of the node equation method.

As shown in Fig. 1.1, the sample system has two generators and an equivalent load, with five nodes and six branches whose admittances are $y_1 \sim y_6$.

Assigning the ground as the reference node, we can write the nodal equations according to the Kirchoff’s current law,

$$
\begin{align*}
&y_4(\dot{V}_2 - \dot{V}_1) + y_5(\dot{V}_3 - \dot{V}_1) - y_6\dot{V}_1 = 0 \\
y_1(\dot{V}_4 - \dot{V}_2) + y_3(\dot{V}_3 - \dot{V}_2) + y_4(\dot{V}_1 - \dot{V}_2) = 0 \\
y_2(\dot{V}_5 - \dot{V}_3) + y_3(\dot{V}_2 - \dot{V}_3) + y_5(\dot{V}_1 - \dot{V}_3) = 0 \\
y_1(\dot{V}_4 - \dot{V}_2) = \dot{I}_1 \\
y_2(\dot{V}_5 - \dot{V}_3) = \dot{I}_2
\end{align*}
$$

where $\dot{V}_1 \sim \dot{V}_5$ denote the node voltages.

Combining the coefficients of node voltages, we obtain the following equations:
\[
\begin{align*}
(y_4 + y_5 + y_6)\dot{V}_1 - y_4\dot{V}_2 - y_5\dot{V}_3 &= 0 \\
-y_4\dot{V}_1 + (y_1 + y_3 + y_4)\dot{V}_2 - y_3\dot{V}_3 - y_1V_4 &= 0 \\
-y_5\dot{V}_1 - y_3\dot{V}_2 + (y_2 + y_3 + y_5)\dot{V}_3 - y_2\dot{V}_5 &= 0 \\
-y_1\dot{V}_2 + y_1\dot{V}_4 &= I_1 \\
-y_2\dot{V}_3 + y_2\dot{V}_5 &= I_2
\end{align*}
\] (1.2)

In (1.2), the left-hand term is the current flowing from the node and the right-hand term is the current flowing into the node. The above equations can be rewritten in more general form as follows:

\[
\begin{align*}
Y_{11}\dot{V}_1 + Y_{12}\dot{V}_2 + Y_{13}\dot{V}_3 + Y_{14}\dot{V}_4 + Y_{15}\dot{V}_5 &= I_1 \\
Y_{21}\dot{V}_1 + Y_{22}\dot{V}_2 + Y_{23}\dot{V}_3 + Y_{24}\dot{V}_4 + Y_{25}\dot{V}_5 &= I_2 \\
Y_{31}\dot{V}_1 + Y_{32}\dot{V}_2 + Y_{33}\dot{V}_3 + Y_{34}\dot{V}_4 + Y_{35}\dot{V}_5 &= I_3 \\
Y_{41}\dot{V}_1 + Y_{42}\dot{V}_2 + Y_{43}\dot{V}_3 + Y_{44}\dot{V}_4 + Y_{45}\dot{V}_5 &= I_4 \\
Y_{51}\dot{V}_1 + Y_{52}\dot{V}_2 + Y_{53}\dot{V}_3 + Y_{54}\dot{V}_4 + Y_{55}\dot{V}_5 &= I_5
\end{align*}
\] (1.3)

Comparing (1.3) with (1.2), we can see

\[
\begin{align*}
Y_{11} &= y_4 + y_5 + y_6, \\
Y_{22} &= y_1 + y_3 + y_4, \\
Y_{33} &= y_2 + y_3 + y_5, \\
Y_{44} &= y_1, \\
Y_{55} &= y_2.
\end{align*}
\]

These elements are known as nodal self-admittances.

\[
\begin{align*}
Y_{12} &= Y_{21} = -y_4, \\
Y_{13} &= Y_{31} = -y_5, \\
Y_{23} &= Y_{32} = -y_3, \\
Y_{24} &= Y_{42} = -y_1, \\
Y_{35} &= Y_{53} = -y_2.
\end{align*}
\]

Similarly, the above elements are known as mutual admittances between the connected nodes. The mutual admittances of the pair of disconnected nodes are zero.

Equation (1.3) is the node equation of the electric network. It reflects the relationship between node voltages and injection currents. Here \(I_1 \sim I_5\) are the nodal injection currents. In this example, except \(I_4\) and \(I_5\), all other nodal injection currents are zero.
Equation (1.3) can be solved to get node voltages $\hat{V}_1 \sim \hat{V}_5$, then the branch currents can be obtained. Thus, we have obtained all the variables of the network.

Generally, for a $n$ node network, we can establish $n$ linear node equations in (1.3) format. In matrix notation, we have

$$I = YV,$$  \hspace{1cm} (1.4)

where

$$I = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix}, \quad V = \begin{bmatrix} \hat{V}_1 \\ \hat{V}_2 \\ \vdots \\ \hat{V}_n \end{bmatrix}.$$ 

Here $I$ is the vector of nodal injection currents and $V$ is the vector of nodal voltages; $Y$ is called the nodal admittance matrix

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1n} \\ Y_{21} & Y_{22} & \cdots & Y_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ Y_{n1} & Y_{n2} & \cdots & Y_{nn} \end{bmatrix}.$$ 

As we have seen, its diagonal element $Y_{ii}$ is the nodal self-admittance and the off diagonal element $Y_{ij}$ is the mutual admittance between node $i$ and node $j$.

Now we introduce the incidence matrix that is very important in network representations.

The incidence matrix represents the topology of an electric network. Different incidence matrices correspond to different networks configurations. The elements of the incidence matrix are only 0, +1, or -1. They do not include the parameters of network branches.

For example, there are five nodes and six branches in Fig. 1.1. Its incidence matrix is a matrix with five rows and six columns.

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 & -1 & 1 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

In the incidence matrix, the serial numbers of rows correspond to the node numbers and the serial numbers of columns correspond to the branch numbers. For example, the first row has three nonzero elements, which denotes node 1 is connected with three branches. These three nonzero elements are in the fourth, fifth, and sixth columns, which means the branches connected with node 1 are branches 4, 5, and 6.
If the branch current flows into the node, the nonzero element equals $-1$; if the branch current flows out of the node, the nonzero element equals $1$. The positions of the nonzero elements in each column denote the two node numbers of the relevant branch. For example, in the fifth column the nonzero elements are in the first and third row, which means the fifth branch connects node 1 and 3. In the sixth column, there is only one nonzero element in the first row, which means the sixth branch is a grounded branch.

From the above discussion we see that an incidence matrix can uniquely determine the topology of a network configuration.

The incidence matrix has a close relationship with the network node equation. If there are $n$ nodes and $b$ branches in an electric network, the state equation for every branch is

$$I_{Bk} = y_{Bk} \hat{V}_{Bk}, \quad (1.5)$$

where $y_{Bk}$ is the admittance of branch $k; I_{Bk}$ the current flowing in branch $k$; and $\hat{V}_{Bk}$ is the voltage difference of branch $k$, whose direction is determined by $I_{Bk}$.

If branch $k$ includes a voltage source, as shown in Fig. 1.2a, it should be transformed to the equivalent current source as shown in Fig. 1.2b.

$$y_{Bk} = 1/z_{Bk}$$
$$\hat{a}_{Bk} = \hat{e}_{Bk}/z_{Bk} = y_{Bk}\hat{e}_{Bk}$$

The current source can be treated as current injecting into the electric network, thus the branch can also be represented by (1.5). In matrix notation, the equation of a $b$ branch network is

$$I_B = Y_B V_B, \quad (1.6)$$

![Fig. 1.2 Transformation from voltage source to current source](image-url)
where \( \mathbf{I}_B \) is the vector of the currents in branches, \( \mathbf{V}_B \) the vector of the branch voltage differences, and \( \mathbf{Y}_B \) is a diagonal matrix constituted by the branch admittances.

According to Kirchoff’s current law, the injection current \( \mathbf{i}_i \) of node \( i \) in an electric network can be expressed as follows

\[
\mathbf{i}_i = \sum_{k=1}^{b} a_{ik} \mathbf{i}_{Bk} \quad (i = 1, 2, \ldots, n), \tag{1.7}
\]

where \( a_{ik} \) is a coefficient. If branch current \( \mathbf{i}_{Bk} \) directs toward node \( i \), \( a_{ik} = -1 \); if branch current \( \mathbf{i}_{Bk} \) directs away from the node \( i \), \( a_{ik} = 1 \); and if branch \( k \) does not connect to node \( i \), \( a_{ik} = 0 \). It is easy to get the relationship between nodal current vector \( \mathbf{i} \) and branch current vector \( \mathbf{i}_B \) as follows,

\[
\mathbf{i} = \mathbf{AI}_B, \tag{1.8}
\]

where \( \mathbf{A} \) is the incidence matrix of the network.

Assuming the power consumed in the whole network is \( S \), we can obtain the following equation,

\[
S = \sum_{k=1}^{b} \hat{\mathbf{i}}_{Bk} \hat{\mathbf{V}}_{Bk} = \hat{\mathbf{I}}_B^* \hat{\mathbf{V}}_B, \tag{1.9}
\]

where \( \hat{\mathbf{i}}_{Bk} \) and \( \hat{\mathbf{I}}_B \) are the conjugate of the corresponding vector and * is the scalar product of the two vectors.

From the viewpoint of the nodal input power, we have

\[
S = \sum_{i=1}^{n} \hat{\mathbf{i}}_i \hat{\mathbf{V}}_i = \hat{\mathbf{I}}^* \hat{\mathbf{V}}. \]

Obviously,

\[
\hat{\mathbf{I}}^* \hat{\mathbf{V}} = \hat{\mathbf{I}}_B^* \hat{\mathbf{V}}_B. \tag{1.9}
\]

From (1.8), we see

\[
\hat{\mathbf{I}} = \hat{\mathbf{I}}_B \mathbf{A}^T. \]

Substituting it into (1.9), we obtain,

\[
\hat{\mathbf{I}}_B \mathbf{A}^T \hat{\mathbf{V}} = \hat{\mathbf{I}}_B \hat{\mathbf{V}}_B. \]
Therefore,

$$A^T \hat{V} = \hat{V}_B.$$  \hfill (1.10)

Substituting (1.6) and (1.10) into (1.8) sequentially, we can get

$$\dot{I} = AY_B A^T \hat{V} = Y \hat{V},$$  \hfill (1.11)

where $Y$ is the nodal admittance matrix of the electric network

$$Y = AY_B A^T.$$  \hfill (1.12)

Thus the nodal equations of an electric network can be obtained from its incidence matrix.

In the following, the network shown in Fig. 1.1 is used again to illustrate the basic principle of analyzing the electric network by the loop current equations. In the loop equation method, the network elements are often represented in impedance form. The equivalent circuit is shown in Fig. 1.3. There are three independent loops in the network and the loop currents are $I_1, I_2,$ and $I_3,$ respectively. According to Kirchoff’s voltage law, the voltage equations of the loops are

$$\begin{align*}
\dot{V}_4 &= (z_1 + z_4 + z_6) \dot{I}_1 + z_6 \dot{I}_2 - z_4 \dot{I}_3 \\
\dot{V}_5 &= z_6 \dot{I}_1 + (z_2 + z_5 + z_6) \dot{I}_2 + z_5 \dot{I}_3 \\
0 &= -z_4 \dot{I}_1 + z_5 \dot{I}_2 + (z_3 + z_4 + z_5) \dot{I}_3
\end{align*}$$

(1.13)

Rewrite the above equation into the normative form,

$$\begin{align*}
\dot{E}_1 &= Z_{11} \dot{I}_1 + Z_{12} \dot{I}_2 + Z_{13} \dot{I}_3 \\
\dot{E}_2 &= Z_{21} \dot{I}_1 + Z_{22} \dot{I}_2 + Z_{23} \dot{I}_3 \\
\dot{E}_3 &= Z_{31} \dot{I}_1 + Z_{32} \dot{I}_2 + Z_{33} \dot{I}_3
\end{align*}$$

(1.14)

Fig. 1.3 Sample system with loop currents
where
\[
\dot{E}_1 = \dot{V}_4, \dot{E}_2 = \dot{V}_5, \dot{E}_1 = 0
\]
are voltage potentials of three loops, respectively,\n\[
Z_{11} = z_1 + z_4 + z_6, Z_{22} = z_2 + z_5 + z_6, Z_{33} = z_3 + z_4 + z_5
\]
are loop self-impedances,\n\[
Z_{12} = Z_{21} = z_6, Z_{13} = Z_{31} = -z_4, Z_{23} = Z_{32} = z_5
\]
are the loop mutual impedances.

If we know loop voltage \(\dot{E}_1, \dot{E}_2, \) and \(\dot{E}_3\), we can solve the loop current \(\dot{i}_1, \dot{i}_2, \) and \(\dot{i}_3\) from (1.14), and then obtain the branch current,\n\[
\dot{i}_1 = \dot{i}_1, \quad \dot{i}_2 = \dot{i}_2, \quad \dot{i}_3 = \dot{i}_3,
\]
\[
\dot{i}_4 = \dot{i}_1 - \dot{i}_3, \quad \dot{i}_5 = \dot{i}_2 + \dot{i}_3, \quad \dot{i}_6 = \dot{i}_1 + \dot{i}_2.
\]

And the node voltages are\n\[
\dot{V}_1 = z_6 \dot{i}_6, \quad \dot{V}_2 = \dot{V}_4 - z_1 \dot{i}_1, \quad \dot{V}_3 = \dot{V}_5 - z_2 \dot{i}_2.
\]

Thus all the variables of the electric network are solved.

Generally, an electric network with \(m\) independent loops can be formulated by \(m\) loop equations. In matrix notation, we have\n\[
E_1 = Z_1 I_1,
\]
where\n\[
I_1 = \begin{bmatrix}
\dot{i}_1 \\
\dot{i}_2 \\
\vdots \\
\dot{i}_m
\end{bmatrix}, \quad E_1 = \begin{bmatrix}
\dot{E}_1 \\
\dot{E}_2 \\
\vdots \\
\dot{E}_m
\end{bmatrix}
\]
are vectors of the loop currents and voltage phasors, respectively;
\[
Z_1 = \begin{bmatrix}
Z_{11} & Z_{12} & \cdots & Z_{1m} \\
Z_{21} & Z_{22} & \cdots & Z_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
Z_{m1} & Z_{m2} & \cdots & Z_{mm}
\end{bmatrix}
\]
is the loop impedance matrix, where \(Z_{ii}\) is the self-impedance of the loop \(i\) and equals the sum of the branch impedances in the loop; \(Z_{ij}\) is the mutual impedance between loop \(i\) and loop \(j\), and equals the sum of the impedances of their common branches. The sign of \(Z_{ij}\) depends on the directions of loop currents of loop \(i\) and loop \(j\). If their directions are identical, \(Z_{ij}\) is positive, and if their directions are different, \(Z_{ij}\) is negative.
For the example shown in Fig. 1.3 we can write the basic loop incidence matrix according to the three independent loops,

$$B = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & -1 & 1 & 0
\end{bmatrix}.$$  

The serial numbers of rows correspond to the loop numbers and the serial numbers of columns correspond to the branch numbers. For example, in the third row, there are three nonzero elements in the third, fourth, and fifth columns which means loop 3 includes branches 3, 4, and 5. If the branch current has the same direction as the basic loop current, the corresponding nonzero element equals $+1$; if the directions of branch current and loop current are different the corresponding nonzero element equals $-1$.

It should be noted that a basic loop incidence matrix cannot uniquely determine a network configuration. In other words, there may be different configurations corresponding to the same basic loop incidence matrix.

Similarly to the discussion on the node incidence matrix above, we can get the basic loop equations of an electric network from its basic loop incidence matrix $B$, 

$$Z_L = BZ_B B^T,$$  
(1.17)

where $Z_B$ is a diagonal matrix composed of the branch impedances.

The application of incidence matrices is quite extensive. If we have the above basic concepts, network analysis problems can be dealt with more flexibly. The details will be discussed in the relevant later sections.

### 1.2.2 Equivalent Circuit of Transformer and Phase-Shift Transformer

The equivalent circuit of an electric network is established by the equivalent circuits of its elements such as transmission lines and transformers. The AC transmission line is often described by the nominal $\Pi$ equivalent circuit which can be found in other textbooks. In this section, only the equivalent circuits of the transformer and the phase-shift transformer are discussed, especially the transformer with off-nominal turns ratios. Flexible AC Transmission Systems (FACTS) are increasingly involved in power systems, and we will discuss the equivalent circuit of FACTS elements in Chap. 5.

When the exciting circuit is neglected or treated as a load (or an impedance), a transformer can be represented by its leakage impedance connected in series with an ideal transformer as shown in Fig. 1.4a. The relation between currents and voltages can be formulated as follows:
Fig. 1.4 Transformer equivalent circuit

\[
\begin{align*}
\dot{I}_i + K\dot{I}_j &= 0 \\
\dot{V}_i - z_T\dot{I}_i &= \frac{\dot{V}_i}{K}
\end{align*}
\]

Solving the above equation, we can obtain

\[
\begin{align*}
\dot{I}_i &= \frac{1}{z_T}\dot{V}_i - \frac{1}{Kz_T}\dot{V}_j, \\
\dot{I}_j &= -\frac{1}{Kz_T}\dot{V}_i + \frac{1}{K^2z_T}\dot{V}_j.
\end{align*}
\]

Rewrite (1.18) as follows

\[
\begin{align*}
I_i &= \frac{K-1}{Kz_T}\dot{V}_i + \frac{1}{Kz_T}(\dot{V}_i - \dot{V}_j) \\
I_j &= \frac{1-K}{K^2z_T}\dot{V}_j + \frac{1}{Kz_T}(\dot{V}_j - \dot{V}_i)
\end{align*}
\]

According to (1.19), we can get the equivalent circuit as shown in Fig. 1.4b. If the parameters are expressed in terms of admittance, the equivalent circuit is shown in Fig. 1.4c, where

\[
y_T = \frac{1}{z_T}.
\]

It should be especially noted in Fig. 1.4a the leakage impedance \(z_T\) is at the terminal where the ratio is 1. When the leakage impedance \(z_T\) is at the terminal where ratio is \(K\), we should transform it to \(z_T'\) by using the following equation, so that the equivalent circuit shown in Fig. 1.4 also can be applied in this situation

\[
z_T' = z_T/K^2.
\]

The equivalent circuit of a two-winding transformer has been discussed above. A similar circuit can be used to represent a three-winding transformer. For example, Fig. 1.5 shows the equivalent circuit of a three-winding transformer that can be transformed into two two-winding transformers’ equivalent circuits.
After obtaining the transformer equivalent circuit, we can establish the equivalent circuit for a multivoltage network. For example, an electric network shown in Fig. 1.6 can be represented by the equivalent circuit shown in Fig. 1.6b or c when the leakage impedances of transformer $T_1$ and $T_2$ have been normalized to side ı and side ı. It can be proved that the two representations have an identical ultimate equivalent circuit as shown in the Fig. 1.6d.

When we analysis the operation of a power system, the per-unit system is extensively used. In this situation, all the parameters of an electric network are denoted in the per-unit system. For example, in the Fig. 1.6, if the voltage base at side ı is $V_{j1}$, at sides ı and ı is $V_{j2}$ and at side ı is $V_{j4}$, then the base ratio (nominal turns ratio) of transformer $T_1$ and $T_2$ are

$$K_{j1} = \frac{V_{j2}}{V_{j1}}, \quad K_{j2} = \frac{V_{j2}}{V_{j4}}.$$ (1.21)

The ratios of transformer $T_1$ and $T_2$ on a per-unit base (off-nominal turns ratio) are

$$K_{s1} = \frac{K_1}{K_{j1}}, \quad K_{s2} = \frac{K_2}{K_{j2}}.$$ (1.22)

Therefore, the ratio of the transformer should be $K_{s1}$ or $K_{s2}$ when its equivalent circuit is expressed in a per-unit system.

In modern power systems, especially in the circumstances of deregulation, the power flow often needs to be controlled. Therefore the application of the phase-shifting transformer is increasing. As we know, a transformer just transforms the voltages of its two terminals and its turn ratio is a real number. The phase-shifting transformer can also change the phase angle between voltages of its two terminals. Thus its turn ratio is a complex number. When the exciting current is neglected or treated as a load (or an impedance), a phase-shifting transformer can be represented
by its leakage impedance, which is connected in series with an ideal transformer having a complex turns ratio as shown in Fig. 1.7. From this figure, we can obtain the equations as follows,

\[ \frac{V_i}{I_i} = \frac{V_j}{I_j} + \frac{V_0}{J_0} = 0 \]  
\[ (1.23) \]

Apparently, the two terminal voltages are related by

\[ V'_j = \frac{V_j}{K}. \]  
\[ (1.24) \]

Since there is no power loss in an ideal autotransformer,

\[ V'_j I'_j = V_j I_j, \]
where \( \hat{I}_j \) and \( \hat{I}_j \) are the conjugates of \( \hat{I}_j \) and \( \hat{I}_j \), respectively. It follows from the above equations that

\[ \hat{I}_j = \hat{K} \hat{I}_j. \]  

Substituting (1.24) and (1.25) into (1.23)

\[ \hat{I}_i = \frac{\hat{V}_i}{z_T} - \frac{\hat{V}_j}{K z_T} = Y_{ii} \hat{V}_i + Y_{ij} \hat{V}_j \]
\[ \hat{I}_j = -\frac{\hat{V}_i}{K z_T} + \frac{\hat{V}_j}{K^2 z_T} = Y_{ji} \hat{V}_i + Y_{jj} \hat{V}_j, \]  

where

\[ Y_{ii} = \frac{1}{z_T}, \quad Y_{ij} = -\frac{1}{K z_T}, \quad Y_{ji} = -\frac{1}{K z_T}, \quad Y_{jj} = \frac{1}{K^2 z_T}. \]

Equation (1.26) is the mathematical model of the phase-shifting transformer. It is easy to be proved that (1.26) is the same as (1.18) when the turn ratio is a real number. This illustrates that the transformer is a particular case of the phase-shifting transformer. Because the ratio of a phase-shifting transformer is complex number, and \( Y_{ij} \neq Y_{ji} \), it has no equivalent circuit and the admittance matrix of the electric network with the phase-shifting transformer is not symmetric.

1.3 Nodal Admittance Matrix

1.3.1 Basic Concept of Nodal Admittance Matrix

As mentioned above, the node equation (1.3) is usually adopted in modern power system analysis. If the number of nodes in a network is \( n \), we have the following general simultaneous equations:
\[
\begin{align*}
\dot{I}_1 &= Y_{11}V_1 + Y_{12}V_2 + \cdots + Y_{1i}V_i + \cdots + Y_{1n}V_n \\
\dot{I}_2 &= Y_{21}V_1 + Y_{22}V_2 + \cdots + Y_{2i}V_i + \cdots + Y_{2n}V_n \\
&\vdots \\
\dot{I}_i &= Y_{ii}V_1 + Y_{i2}V_2 + \cdots + Y_{ii}V_i + \cdots + Y_{in}V_n \\
&\vdots \\
\dot{I}_n &= Y_{n1}V_1 + Y_{n2}V_2 + \cdots + Y_{ni}V_i + \cdots + Y_{nn}V_n \\
\end{align*}
\]

(1.27)

The matrix constituted by the coefficients of (1.27) is the nodal admittance matrix

\[
Y = \begin{bmatrix}
Y_{11} & Y_{12} & \cdots & Y_{1i} & \cdots & Y_{1n} \\
Y_{21} & Y_{22} & \cdots & Y_{2i} & \cdots & Y_{2n} \\
& \vdots & \ddots & \vdots & \ddots & \vdots \\
Y_{ii} & Y_{i2} & \cdots & Y_{ii} & \cdots & Y_{in} \\
& \vdots & \ddots & \vdots & \ddots & \vdots \\
Y_{n1} & Y_{n2} & \cdots & Y_{ni} & \cdots & Y_{nn}
\end{bmatrix}
\]

(1.28)

A nodal admittance matrix reflects the topology and parameters of an electric network, so it can be regarded as a mathematical abstraction of the electric network. The node equation based on the admittance matrix is a widely used mathematical model of electric networks. Next we will introduce some physical meaning of the matrix elements.

If we set a unit voltage at node \(i\) and ground other nodes, i.e.,

\[
\begin{align*}
\dot{V}_i &= 1 \\
\dot{V}_j &= 0 \quad (j = 1, 2, \ldots, n, j \neq i),
\end{align*}
\]

then the following relationships hold according to (1.27),

\[
I_j = Y_{ji} \quad j = 1, 2, \ldots, n.
\]

(1.29)

From (1.29) we can see the physical meaning of the \(i\)th column elements in the admittance matrix: the diagonal element \(Y_{ii}\) in the \(i\)th column, the self-admittance of node \(i\), is equal to the injection current of the node \(i\); the off-diagonal elements \(Y_{ij}\) in the \(i\)th column, the mutual-admittance of node \(i\) and node \(j\), is equal to the injection current of node \(j\) in this situation.

We will further illustrate these concepts by a simple network shown in Fig. 1.8. The network has three nodes (plus ground), thus the dimension of its admittance matrix is \(3 \times 3\),
According to the above discussion, we can get the elements of the first column: $Y_{11}, Y_{21},$ and $Y_{31}$, by setting a unit voltage on node 1 and grounding node 2 and node 3 as shown in Fig. 1.8b. Evidently,

$$
\begin{align*}
\dot{i}_1 &= \dot{i}_{12} + \dot{i}_{13} + \dot{i}_{10} = \frac{1}{z_{12}} + \frac{1}{z_{10}} + \frac{1}{z_{13}} = Y_{11}, \\
\dot{i}_2 &= -\dot{i}_{12} = -\frac{1}{z_{12}} = Y_{21}, \\
\dot{i}_3 &= -\dot{i}_{13} = -\frac{1}{z_{13}} = Y_{31}.
\end{align*}
$$

Similarly, setting a unit voltage at node 2 and grounding node 1 and node 3 as shown in Fig. 1.8c, we can get the elements of the second column:

$$
\begin{align*}
\dot{i}_1 &= -\dot{i}_{21} = -\frac{1}{z_{12}} = Y_{12}, \\
\dot{i}_2 &= \dot{i}_{21} = \frac{1}{z_{12}} = Y_{22}, \\
\dot{i}_3 &= 0 = Y_{32}.
\end{align*}
$$
For the elements of the third column we have (see Fig. 1.8d),

\[ i_1 = -i_{31} = -\frac{1}{z_{31}} = Y_{13}, \]
\[ i_2 = 0 = Y_{23}, \]
\[ i_3 = i_{31} = \frac{1}{z_{13}} = Y_{33}. \]

Finally, the admittance matrix of the above simple network becomes

\[ Y = \begin{bmatrix}
\frac{1}{z_{12}} + \frac{1}{z_{10}} + \frac{1}{z_{13}} & -\frac{1}{z_{12}} & -\frac{1}{z_{13}} \\
-\frac{1}{z_{12}} & \frac{1}{z_{12}} & 0 \\
-\frac{1}{z_{13}} & 0 & \frac{1}{z_{13}}
\end{bmatrix}. \quad (1.30) \]

If we change the node numbers in Fig. 1.8a, e.g., exchange the number ordering of node 1 with node 2, as shown in Fig. 1.8e, then the admittance matrix becomes,

\[ Y' = \begin{bmatrix}
\frac{1}{z_{12}} & -\frac{1}{z_{12}} & 0 \\
-\frac{1}{z_{12}} & \frac{1}{z_{12}} + \frac{1}{z_{20}} + \frac{1}{z_{23}} & -\frac{1}{z_{23}} \\
0 & -\frac{1}{z_{23}} & \frac{1}{z_{23}}
\end{bmatrix}. \]

The above matrix can be obtained through exchanging the first row with the second row, and at the same time exchanging the first column with the second column of the matrix shown in (1.30). The exchange of the rows and columns of the admittance matrix corresponds to the exchange of the sequence of node equations and their variables.

The properties of the admittance matrix can be summarized as follows:

1. The admittance matrix is symmetric if there is no phase-shifting transformer in the network. From (1.30) we have

\[ Y_{12} = Y_{21} = -\frac{1}{z_{12}}, Y_{13} = Y_{31} = -\frac{1}{z_{13}}, Y_{23} = Y_{32} = 0. \]

Generally, according to the reciprocity of the network,

\[ Y_{ij} = Y_{ji}. \]

Therefore, the admittance matrix is symmetric. We will discuss the networks with phase-shifting transformers later.
2. The admittance matrix is sparse. From the discussion above, we know that \( Y_{ij} \) and \( Y_{ji} \) will be zero if node \( i \) does not directly connect with node \( j \). For example, in Fig. 1.8a, node 2 does not directly connect with node 3, so both of \( Y_{23} \) and \( Y_{32} \) are zero. In general, the number of nonzero off-diagonal elements of each row is equal to the number of branches that are incident to the corresponding node. Usually, the number of branches connected to one node is 2–4, thus there are only 2–4 nonzero off-diagonal elements in each row. The property that only a few nonzero elements exist in a matrix is called sparsity. This phenomenon will be more remarkable with increase of the power system scale. For instance, for a network with 1,000 nodes, if each node directly connects three branches on average, the total number of nonzero elements for the network is 4,000, which is only 0.4% of the total elements in the admittance matrix.

The symmetry and sparsity of an admittance matrix are very important features for large-scale power systems. If we make full use of these two properties, the computation speed will be accelerated and the computer memory will be saved dramatically.

### 1.3.2 Formulation and Modification of Nodal Admittance Matrix

Now we discuss formulation of an admittance matrix by inspection first. When an electric network is composed of only transmission lines, the principles of constructing its admittance matrix can be summarized as follows:

1. The order of the admittance matrix is equal to the number of the nodes of the electric network.
2. The number of the nonzero off-diagonal elements in each row is equal to the number of the ungrounded branches connected to the corresponding node.
3. The diagonal elements of the admittance matrix, i.e., the self-admittance of the node, is equal to the sum of all the admittances of the incident branches of the corresponding node. Thus

\[
Y_{ii} = \sum_{j \in I} y_{ij},
\]  

(1.31)

where \( y_{ij} \) is the reciprocal of \( z_{ij} \), which is the branch impedance between node \( i \) and node \( j \). ‘‘\( j \) I’’ denotes that only the incident branches of node \( i \) (including the grounding branch) are included to the summation. For example, in Fig. 1.8, the self-admittance of node 1, i.e., \( Y_{11} \), should be

\[
Y_{11} = \frac{1}{z_{12}} + \frac{1}{z_{10}} + \frac{1}{z_{13}} = y_{12} + y_{10} + y_{13}.
\]

The self-admittance of node 2, i.e., \( Y_{22} \), should be

\[
Y_{22} = \frac{1}{z_{12}} = y_{12}.
\]
4. The off-diagonal element of the admittance matrix, $Y_{ij}$, is equal to the negative of the admittance between node $i$ and node $j$

$$Y_{ij} = -\frac{1}{z_{ij}} = -y_{ij}.$$  \hspace{1cm} (1.32)

For example, in Fig. 1.8a,

$$Y_{12} = -\frac{1}{z_{12}} = -y_{12},$$

$$Y_{13} = -\frac{1}{z_{13}} = -y_{13}.$$

Therefore, no matter how complicated the configuration of an electric network is, its admittance matrix can be established directly by inspection according to the parameters and the topology of the network.

When the electric network involves transformers or phase-shifting transformers, they need special treatment.

When branch $ij$ is a transformer, the admittance matrix certainly can be formed following the above steps if the transformer is substituted beforehand by the $\Pi$ equivalent circuit as shown in Fig. 1.4a. However, in practical application the transformer is often treated directly in forming the admittance matrix. If branch $ij$ is a transformer, as shown in Fig. 1.4a, the elements of the admittance matrix related to the branch can be obtained as follows:

1. Add two nonzero off-diagonal elements into the admittance matrix

$$Y_{ij} = Y_{ji} = -\frac{y_T}{K}.$$  \hspace{1cm} (1.33)

2. Add to the self-admittance of node $i$ by,

$$\Delta Y_{ii} = \frac{K - 1}{K}y_T + \frac{1}{K}y_T = y_T.$$  \hspace{1cm} (1.34)

3. Add to the self-admittance of node $j$ by

$$\Delta Y_{jj} = \frac{1}{K}y_T + \frac{1 - K}{K^2}y_T = \frac{y_T}{K^2}.$$  \hspace{1cm} (1.35)

When branch $ij$ is a phase-shifting transformer, its equivalent circuit is Fig. 1.7. Then the corresponding matrix elements are obtained as follows:

1. Add two nonzero off-diagonal elements into the admittance matrix

$$Y_{ij} = -\frac{1}{Kz_T}.$$  \hspace{1cm} (1.36)
\[
Y_{ji} = \frac{1}{Kz_T}, \quad (1.37)
\]

2. Add to the self-admittance of node \(i\) by

\[
\Delta Y_{ii} = \frac{1}{z_T}. \quad (1.38)
\]

3. Add to the self-admittance of node \(j\) by

\[
\Delta Y_{jj} = \frac{1}{K^2z_T}. \quad (1.39)
\]

It can be seen from (1.36) and (1.37) that \(Y_{ij} \neq Y_{ji}\), thus the admittance matrix is not symmetric any more although its structure is still symmetric.

Studies of different system operation states, such as transformer or transmission line outages, play an important part in modern power system analysis. Because the outage of branch \(ij\) only affects the self and mutual admittance of node \(i\) and node \(j\), we can obtain the new admittance matrix for the contingency state by modifying the original admittance matrix. The modification methods for different situations are introduced as follows:

1. To add a new node with a new branch for the original network as shown in Fig. 1.9a.

Assume that \(i\) is a node of the original network and \(j\) is the new node; \(z_{ij}\) is the impedance of the new branch. The dimension of the admittance matrix becomes \(N + 1\) because of the new node. There is only one branch connected to node \(j\), therefore, its self-admittance is,

\[
Y_{jj} = \frac{1}{z_{ij}},
\]

The self-admittance of node \(i\) should be modified (added) by,

\[
\Delta Y_{ii} = \frac{1}{z_{ij}}.
\]