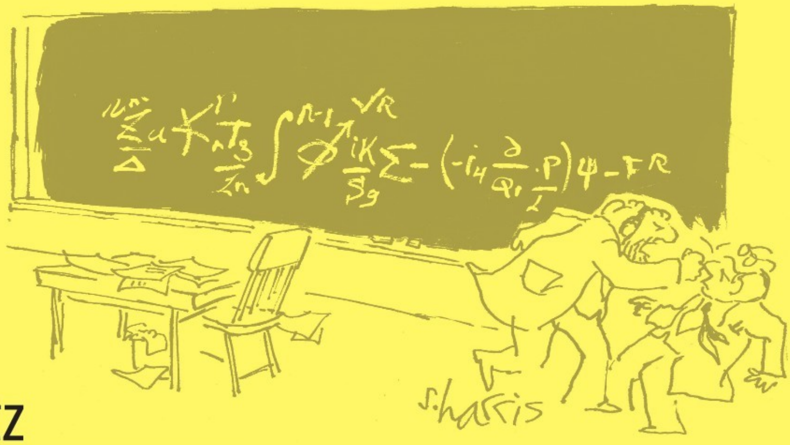

"YOU WANT PROOF? I'LL GIVE YOU PROOF!"

Steven G. Krantz

The Proof is in the Pudding

The Changing Nature of Mathematical Proof

Springer

The Proof is in the Pudding

Steven G. Krantz

The Proof is in the Pudding

The Changing Nature of Mathematical Proof

Springer

Steven G. Krantz
Department of Mathematics
Washington University in St. Louis
St. Louis, MO 63130
USA
sk@math.wustl.edu

ISBN 978-0-387-48908-7 e-ISBN 978-0-387-48744-1
DOI 10.1007/978-0-387-48744-1
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2011928557

Mathematics Subject Classification (2010): 01-XX, 03-XX, 65-XX, 97A30

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

To Jerry Lyons, mentor and friend

Preface

The title of this book is not entirely frivolous. There are many who will claim that the correct aphorism is “The proof of the pudding is in the eating.”—that it makes no sense to say, “The proof is in the pudding.” Yet people say it all the time, and the intended meaning is usually clear. So it is with mathematical proof. A *proof* in mathematics is a psychological device for convincing some person, or some audience, that a certain mathematical assertion is true. The structure, and the language used, in formulating such a proof will be a product of the person creating it; but it also must be tailored to the audience that will be receiving it and evaluating it. Thus there is no “unique” or “right” or “best” proof of any given result. A proof is part of a situational ethic: situations change, mathematical values and standards develop and evolve, and thus the very *way* that mathematics is done will alter and grow.

This is a book about the changing and growing nature of mathematical proof. In the earliest days of mathematics, “truths” were established heuristically and/or empirically. There was a heavy emphasis on calculation. There was almost no theory, no formalism, and there was little in the way of mathematical notation as we know it today. Those who wanted to consider mathematical questions were thereby hindered: they had difficulty expressing their thoughts. They had particular trouble formulating general statements about mathematical ideas. Thus it was virtually impossible for them to state theorems and prove them.

Although there are some indications of proofs even on ancient Babylonian tablets (such as Plimpton 322) from 1800 BCE, it seems that it is in ancient Greece that we find the identifiable provenance of the concept of proof. The earliest mathematical tablets contained numbers and elementary calculations. Because of the paucity of texts that have survived, we do not know how it came about that someone decided that some of these mathematical procedures required *logical justification*. And we really do not know how the formal concept of proof evolved. The *Republic* of Plato contains a clear articulation of the proof concept. The *Physics* of Aristotle not only discusses proofs, but treats minute distinctions of proof methodology. Many of the ancient Greeks, including Eudoxus, Theaetetus, Thales, Euclid, and Pythagoras, either used proofs or referred to proofs. Protagoras was a sophist, whose work was recognized by Plato. His *Antilogies* were tightly knit, rigorous arguments that could be thought of as the germs of proofs.

It is acknowledged that Euclid was the first to systematically use precise definitions, axioms, and strict rules of logic, and to carefully enunciate and *prove* every statement (i.e., every theorem). Euclid’s formalism, as well as his methodology, has become the model—even to the present day—for establishing mathematical facts.

What is interesting is that a mathematical statement of fact is a freestanding entity with intrinsic merit and value. But a proof is a device of communication. The creator or discoverer of this new mathematical result wants others to believe it and accept it. In the physical sciences—chemistry, biology, or physics for example—the method for achieving this end is the *reproducible experiment*.[1] For the mathematician, the reproducible experiment is a proof that others can read and understand and validate.

The idea of "proof" appears in many aspects of life other than mathematics. In the courtroom, a lawyer (either for the prosecution or the defense) must establish a case by means of an accepted version of proof. For a criminal case this is "beyond a reasonable doubt," while for a civil case it is "the preponderance of evidence shows." Neither of these is mathematical proof, or anything like it. The real world has no formal definitions and no axioms; there is no sense of establishing facts by strict exegesis. The lawyer uses logic—such as "the defendant is blind so he could not have driven to Topanga Canyon on the night of March 23," or "the defendant has no education and therefore could not have built the atomic bomb that was used to ... "—but the principal tools are *facts*. The lawyer proves the case beyond a reasonable doubt by amassing an overwhelming preponderance of evidence in favor of that case.

At the same time, in ordinary, family-style parlance there is a notion of proof that is different from mathematical proof. A husband might say, "I believe that my wife is pregnant," while the wife may *know* that she is pregnant. Her pregnancy is not a permanent and immutable fact (like the Pythagorean theorem), but instead is a "temporary fact" that will be false after several months. So, in this context, the concept of truth has a different meaning from the one used in mathematics, and the means of verification of a truth are also rather different. What we are really seeing here is the difference between knowledge and belief—something that never plays a formal role in mathematics.

In modern society it takes far less "proof" to convict someone of speeding than to get a murder conviction. But, ironically, it seems to take even less evidence to justify waging war.[2] A great panorama of opinions about the modern concept of proof—in many different contexts—may be found in [NCBI]. It has been said—see [MCI]—that in some areas of mathematics (such as low-dimensional topology) a proof can be conveyed by a combination of gestures.

The French mathematician Jean Leray (1906–1998) perhaps sums up the value system of the modern mathematician:

> ... all the different fields of mathematics are as inseparable as the different parts of a living organism; as a living organism mathematics has to be permanently recreated; each generation must reconstruct it wider, larger and more beautiful. The death of mathematical research would be the death of mathematical thinking which constitutes the structure of scientific language itself and by consequence the death of our scientific civilization. Therefore we must transmit to our children strength of character, moral values and drive towards an endeavouring life.

---

[1] More precisely, it is the reproducible experiment *with control*. For the careful scientist compares the results of his/her experiment with some standard or norm. That is the means of evaluating the result.

[2] See [SWD] for the provenance of these ideas. Fascinating related articles are [ASC], [MCI]. An entire issue of the *Philosophical Transactions of the Royal Society* in the fall of 2005 is devoted to articles of this type, all deriving from a meeting in Britain to discuss issues such as those treated in this book. See [PTRS].

What Leray is telling us is that mathematical ideas travel well and stand up under the test of time, because we have such a rigorous and well-tested standard for formulating and recording the ideas. It is a grand tradition, and one well worth preserving.

There is a human aspect to proof that cannot be ignored. The acceptance of a new mathematical truth is a sociological process. It is something that takes place in the mathematical community. It involves understanding, internalization, rethinking, and discussion. And even the most eminent mathematicians make mistakes, and announce new results that in fact they *do not* know how to prove. In 1879, A. Kempe published a proof of the four-color theorem that stood for eleven years before P. Heawood found a fatal error in the work. The first joint work of G. H. Hardy and J. E. Littlewood was announced at the June, 1911 meeting of the London Mathematical Society. The result was never published because they later discovered that their proof was incorrect. A. L. Cauchy, G. Lamé, and E. E. Kummer all thought at one point or another in their careers that they had proved Fermat's last theorem; they were all mistaken. H. Rademacher thought in 1945 that he had disproved the Riemann hypothesis; his achievement was even written up in *Time Magazine*. He later had to withdraw the claim because C. L. Siegel found an error. Considerable time is spent here in this book exploring the social workings of the mathematical discipline, and how the interactions of different mathematicians and different mathematical cultures serve to shape the subject. Mathematicians' errors are corrected, not by formal mathematical logic, but by other mathematicians. This is a seminal point about this discipline.[3]

The early twentieth century saw L. E. J. Brouwer's dramatic proof of his fixed-point theorem followed by his wholesale rejection of proof by contradiction (at least in the context of existence proofs—which is precisely what the proof of his fixed-point theorem is an instance of) and his creation of the intuitionist movement. This program was later taken up by Errett Bishop, and his *Foundations of Constructive Analysis* (written in 1967) has made quite a mark (see also the revised version, written jointly with Douglas Bridges, published in 1985). These ideas are of particular interest to the theoretical computer scientist, for proof by contradiction has questionable meaning in computer science (this despite the fact that Alan Turing cracked the Enigma code by applying ideas of proof by contradiction in the context of computing machines).

In the past thirty years or so it has come about that we have rethought, and reinvented, and decisively amplified our concept of proof. Computers have played a strong and dynamic role in this reorientation of the discipline. A computer can make hundreds of millions of calculations in a second. This opens up possibilities for trying things, calculating, and visualizing things that were unthinkable fifty years ago. It should be borne in mind that mathematical thinking involves concepts and reasoning, while a computer is a device for manipulating data, two quite different activities. It appears unlikely (see Roger Penrose's remarkable book *The Emperor's New Mind*) that a computer will ever be able to think and prove mathematical theorems in the way a human being performs these activities. Nonetheless, the computer can provide valuable information and insights. It can enable the user to see things that would not be envisioned otherwise. It is a valuable tool. We shall

[3]It is common for mathematicians to make errors. Probably every published mathematical paper has errors in it. The book [LEC] documents many important errors in the literature up to the year 1935.

definitely spend a good deal of time in this book pondering the role of the computer in modern human thought.

In trying to understand the role of the computer in mathematical life, it is perhaps worth drawing an analogy with history. Tycho Brahe (1546–1601) was one of the great astronomers of the Renaissance. Through painstaking scientific procedure, he recorded reams and reams of data about the motions of the planets. His gifted student Johannes Kepler (1571–1630) was anxious to get his hands on Brahe's data because he had ideas about formulating mathematical laws about the motions of the planets. But Brahe and Kepler were both strong-willed men. They did not see eye-to-eye on many things. And Brahe feared that Kepler would use his data to confirm the Copernican theory about the solar system (namely that the *sun*, not the earth, was the center of the system—a notion that ran counter to Christian religious dogma). As a result, during Brahe's lifetime Kepler did not have access to Brahe's numbers.

But providence intervened in a strange way. Tycho Brahe had been given an island by his sponsor on which to build and run his observatory. As a result, he was obliged to attend certain social functions—just to show his appreciation and to report on his progress. At one such function, Brahe drank an excessive amount of beer, his bladder burst, and he died. Kepler was then able to negotiate with Brahe's family to get the data that he so desperately needed. And thus the course of scientific history was forever altered.

Kepler did *not* use deductive thinking or reasoning, or the axiomatic method, or the strategy of mathematical proof to derive his three laws of planetary motion. Instead he simply stared at the hundreds of pages of planetary data that Brahe had provided, and he performed numerous calculations.

At around this same time John Napier (1550–1617) was developing his theory of logarithms. These are terrific calculational tools, which would have simplified Kepler's task immensely. But Kepler could not understand the derivation of logarithms and refused to use them. He did everything the hard way. Imagine what Kepler could have done with a computer!—but he probably would have refused to use one just because he would not have understood how the central processing unit (CPU) worked.

In any event, we tell here of Kepler and Napier because the situation is perhaps a harbinger of modern agonizing over the use of computers in mathematics. There are those who argue that the computer can enable us to see things—both calculationally and visually—that we could not see before. And there are those who say that all those calculations are well and good, but they do not constitute a mathematical proof. Nonetheless it seems that the first can inform the second, and a productive symbiosis can be created. We shall discuss these matters in detail as the book develops.

Now let us return to our consideration of changes that have come about in mathematics in the past thirty years, in part because of the advent of high-speed digital computers. Here is a litany of some of the components of this process:

(a) In 1974, Appel and Haken [APH1] announced a proof of the four-color conjecture. This is the question of how many colors are needed to color any map, so that adjacent countries are colored differently. Their proof used 1200 hours of computer time on a supercomputer at the University of Illinois. Mathematicians found this event puzzling because this "proof" was not something that anyone could study or check. Or understand. To this day there does not exist a proof of the four-color theorem that can be read and checked by humans.

(b) Over time, people have become more and more comfortable with the use of computers in proofs. In its early days, the theory of wavelets (for example) depended on the estimation of a certain constant—something that could be done only with a computer. De Branges's original proof of the Bieberbach conjecture [DEB2] seemed to depend on a result from special function theory that could be verified only with the aid of a computer (it was later discovered to be a result of Askey and Gasper that was proved in the traditional manner).

(c) The evolution of new teaching tools such as the software `The Geometer's Sketchpad` has suggested to many—including Fields Medalist William Thurston—that traditional proofs may be set aside in favor of experimentation, that is, testing of thousands or millions of examples, on the computer.

Thus the use of the computer has truly reoriented our view of what a proof might comprise. Again, the point is to convince someone else that something is true. There are evidently many different means of doing this.

Perhaps more interesting are some of the new social trends in mathematics and the resulting construction of nonstandard proofs (we shall discuss these in detail in the text that follows):

(a) One of the great efforts of twentieth-century mathematics has been the classification of the finite simple groups. Daniel Gorenstein, of Rutgers University, was in some sense the lightning rod who orchestrated the effort. It is now considered to be complete. What is remarkable is that this is a single theorem that is the aggregate effort of many hundreds of mathematicians. The "proof" is in fact the union of hundreds of papers and tracts spanning more than 150 years. At the moment this proof comprises over 10,000 pages. It is still being organized and distilled today. The final "proof for the record" will consist of several volumes, and it is not clear that the living experts will survive long enough to see the fruition of this work.

(b) Thomas Hales's resolution of the Kepler sphere-packing problem uses a great deal of computer calculation, much as with the four-color theorem. It is particularly interesting that his proof supplants the earlier proof of Wu-Yi Hsiang that relied on spherical trigonometry and *no computer calculation*. Hales allows that his "proof" cannot be checked in the usual fashion. He has organized a worldwide group of volunteers called `FlySpeck` to engage in a checking procedure for his computer-based arguments.

(c) Grisha Perelman's "proof" of the Poincaré conjecture and the geometrization program of Thurston are currently in everyone's focus. In 2003, Perelman wrote three papers that describe how to use Richard Hamilton's theory of Ricci flows to carry out Thurston's idea (called the "geometrization program") of breaking up a 3-manifold into fundamental geometric pieces. One very important consequence of this result would be a proof of the important Poincaré conjecture. Although Perelman's papers are vague and incomplete, they are full of imaginative and deep geometric ideas. This work set off a storm of activity and speculation about how the program might be assessed and validated. There have been huge efforts by John Lott and Bruce Kleiner (at the University of Michigan) and Gang Tian (Princeton) and John Morgan (Columbia) to complete the Hamilton/Perelman program and produce a bona fide, recorded proof that others can study and verify.

(d) In fact, Thurston's geometrization program is a tale in itself. He announced in the early 1980s that he had a result on the structure of 3-manifolds, at least for certain important subclasses of the manifolds, and he knew how to prove it. The classical Poincaré conjecture would be an easy corollary of Thurston's geometrization program. He wrote an extensive set of notes [THU3]—of book length—and these were made available to the world by the Princeton mathematics department. For a nominal fee, the department would send a copy to anyone who requested it. These notes, entitled *The Geometry and Topology of Three-Manifolds* [THU3], were extremely exciting and enticing. But the notes, for all the wealth of good mathematics that they contained, were written in a rather informal style. They were difficult to assess and evaluate.

The purpose of this book is to explore all the ideas and developments outlined above. Along the way, we are able to acquaint the reader with the culture of mathematics: who mathematicians are, what they care about, and what they do. We also give indications of why mathematics is important, and why it is having such a powerful influence in the world today. We hope that by reading this book the reader will become acquainted with, and perhaps charmed by, the glory of this ancient subject, and will realize that there is so much more to learn.

December, 2010

Steven G. Krantz
St. Louis, Missouri

Contents

Acknowledgments

One of the pleasures of the writing life is getting criticism and input from other scholars. I thank Jesse Alama, David H. Bailey, John Bland, Jonathan Borwein, Robert Burckel, David Collins, E. Brian Davies, Keith Devlin, Ed Dunne, Michael Eastwood, Jerry Folland, H. Gopalkrishna Gadiyar, Jeremy Gray, Jeff C. Lagarias, Barry Mazur, Robert Strichartz, Eric Tressler, James Walker, Russ Woodroofe, and Doron Zeilberger for careful readings of my drafts and for contributing much wisdom and useful information. Robert Burckel and David Collins each gave my draft an extraordinarily incisive and detailed reading and contributed many useful ideas and corrections. I thank Ed Dunne of the American Mathematical Society for many insights and for suggesting the topic of this book. Finally, I thank Sidney Harris for permission to use six of his cartoons.

Ann Kostant of Birkhäuser/Springer was, as always, a proactive and supportive editor. She originally invited me to write a book for the Copernicus series, and gave terrific advice and encouragement during its development. Edwin Beschler served as editor extraordinaire, and helped me sharpen the prose and enliven the exposition. David Kramer was, as always, a splendid copyeditor. I am proud of the result. Finally, I thank Randi D. Ruden for all her help and support during the writing of this book.

1

What Is a Proof and Why?

The proof of the pudding is in the eating.

—Miguel Cervantes

In mathematics there are no true controversies.

—Carl Friedrich Gauss

Logic is the art of going wrong with confidence.

—Anonymous

To test man, the proofs shift.

—Robert Browning

Newton was a most fortunate man because there is just one universe and Newton had discovered its laws.

—Pierre-Simon Laplace

The chief aim of all investigations of the external world should be to discover the rational order and harmony which has been imposed on it by God and which He revealed to us in the language of mathematics.

—Johannes Kepler

A terrier may not be able to define a rat, but a terrier knows a rat when he sees it.

—A. E. Housman

1.1 What Is a Mathematician?

A well-meaning mother was once heard telling her child that a mathematician is someone who does "scientific arithmetic." Others think that a mathematician is someone who spends all day hacking away at a computer.

Neither of these contentions is incorrect, but they do not begin to penetrate all that a mathematician really is. Paraphrasing mathematician/linguist Keith Devlin, we note that a mathematician is someone who:

S.G. Krantz, *The Proof is in the Pudding: The Changing Nature of Mathematical Proof*,
DOI 10.1007/978-0-387-48744-1_1, © Springer Science+Business Media, LLC 2011

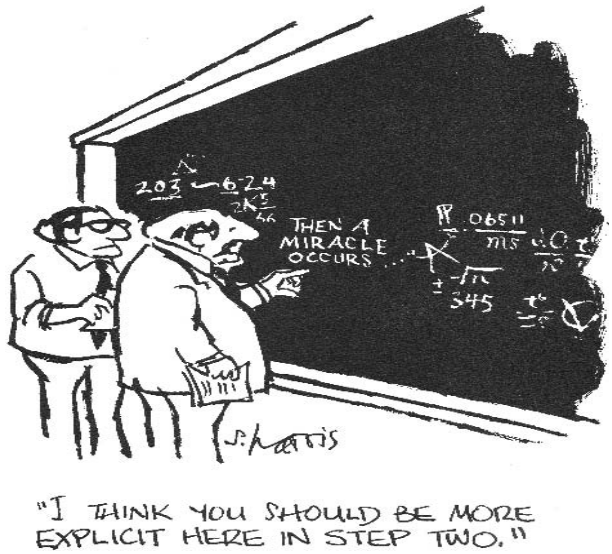

Figure 1.1.

- observes and interprets phenomena
- analyzes scientific events and information
- formulates concepts
- generalizes concepts
- performs inductive reasoning
- performs analogical reasoning
- engages in trial and error (and evaluation)
- models ideas and phenomena
- formulates problems
- abstracts from problems
- solves problems
- uses computation to draw analytical conclusions
- makes deductions
- makes guesses
- proves theorems

And even this list is incomplete. A mathematician has to be a master of critical thinking, analysis, and deductive logic. These skills are robust, and can be applied in a large variety of situations—and in many different disciplines. Today, mathematical skills are being put to good use in medicine, physics, law, commerce, Internet design, engineering, chemistry, biological science, social science, anthropology, genetics, warfare, cryptography, plastic surgery, security analysis, data manipulation, computer science, and in many other disciplines and endeavors as well.

One of the astonishing and dramatic new uses of mathematics that has come about in the past twenty years is in finance. The work of Fischer Black of Harvard and Myron

Scholes of Stanford gave rise to the first-ever method for option pricing. This methodology is based on the theory of stochastic integrals—a part of abstract probability theory. As a result, investment firms all over the world now routinely employ Ph.D. mathematicians. When a measure theory course is taught in the math department—something that was formerly the exclusive province of graduate students in mathematics studying for the qualifying exams—we find that the class is unusually large, and most of the students are from economics and finance.

Another part of the modern world that has been strongly influenced by mathematics, and which employs a goodly number of mathematicians with advanced training, is genetics and the genome project. Most people do not realize that a strand of DNA can have billions of gene sites on it. Matching up genetic markers is *not* like matching up your socks; in fact things must be done probabilistically. A good deal of statistical theory is used. So these days many Ph.D. mathematicians work on the genome project.

The focus of this book is on the concept of *mathematical proof*. Although it is safe to say that most mathematical scientists do not spend the bulk of their time proving theorems,[1] it is nevertheless the case that *proof* is the *lingua franca* of mathematics. It is the web that holds the enterprise together. It is what makes the subject live on, and guarantees that mathematical ideas will have some immortality (see [CEL] for a philosophical consideration of the proof concept).

There is no other scientific or analytical discipline that uses proof as readily and routinely as does mathematics. This is the device that makes theoretical mathematics special: the carefully crafted path of inference, following strict analytical rules that leads inexorably to a particular conclusion. *Proof* is our device for establishing the absolute and irrevocable truth of statements in mathematics. This is the reason we can depend on the mathematics of Euclid 2300 years ago as readily as we believe in the mathematics that is done today. No other discipline can make such an assertion (but see Section 1.10).

This book will acquaint the reader with some mathematicians and what they do, using the concept of "proof" as a touchstone. Along the way, we will become acquainted with foibles and traits of particular mathematicians, and of the profession as a whole. It is an exciting journey, full of rewards and surprises.

1.2 The Concept of Proof

We begin this discussion with an inspiring quotation from mathematician Michael Atiyah (1929–) [ATI2]:

> We all know what we like in music, painting or poetry, but it is much harder to explain why we like it. The same is true in mathematics, which is, in part, an art form. We can identify a long list of desirable qualities: beauty, elegance,

[1] This is because a great many mathematical scientists do not work at universities. They work instead (for instance) for the National Security Agency (NSA), or the National Aeronautics and Space Administration (NASA), or Hughes Aircraft, or Lawrence Berkeley Labs, or Microsoft. It is in fact arguable that most mathematical scientists are *not* pedigreed mathematicians.

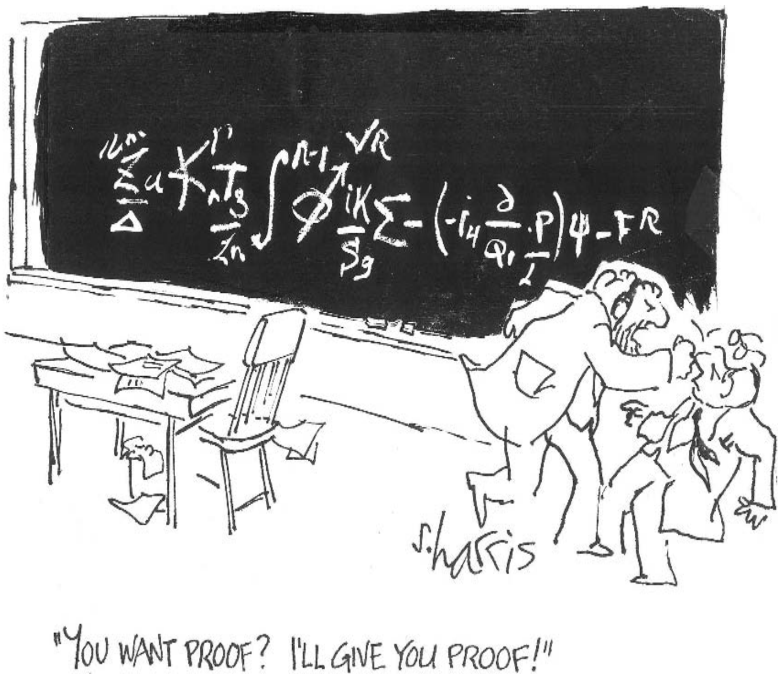

Figure 1.2.

> importance, originality, usefulness, depth, breadth, brevity, simplicity, clarity. However, a single work can hardly embody them all; in fact, some are mutually incompatible. Just as different qualities are appropriate in sonatas, quartets or symphonies, so mathematical compositions of varying types require different treatment. Architecture also provides a useful analogy. A cathedral, palace or castle calls for a very different treatment from an office block or private home. A building appeals to us because it has the right mix of attractive qualities for its purpose, but in the end, our aesthetic response is instinctive and subjective. The best critics frequently disagree.

The tradition of mathematics is a long and glorious one. Along with philosophy, it is the oldest venue of human intellectual inquiry. It is in the nature of the human condition to want to understand the world around us, and mathematics is a natural vehicle for doing so. But, for the ancients, mathematics was also a subject that was beautiful and worthwhile in its own right, a scholarly pursuit possessing intrinsic merit and aesthetic appeal. Mathematics was worth studying for its own sake.

In its earliest days, mathematics was often bound up with practical questions. The Egyptians, as well as the Greeks, were concerned with surveying land. Refer to Figure 1.3. Questions of geometry and trigonometry were natural considerations. Triangles and rectangles arose in a natural way in this context, so early geometry concentrated on these constructs. Circles too were natural to consider—for the design of arenas and water tanks and

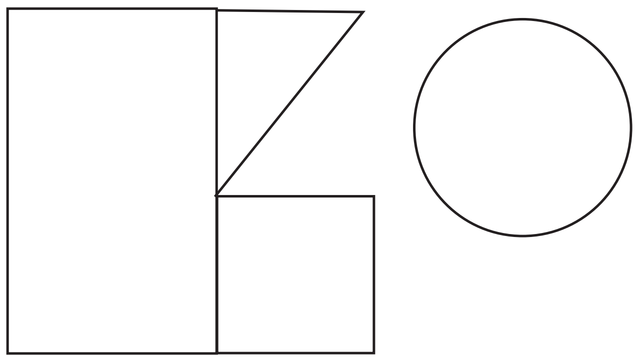

Figure 1.3. Surveying land.

other practical projects. So ancient geometry (and Euclid's axioms for geometry) discussed circles.

The earliest mathematics was phenomenological. If one could draw a plausible picture, then that was all the justification needed for a mathematical "fact." Sometimes one argued by analogy, or by invoking the gods. The notion that mathematical statements could be *proved* was not yet a developed idea. There was no standard for the concept of proof. The analytical structure, the "rules of the game," had not yet been created. If one ancient Egyptian were to say to another, "I don't understand why this mathematical statement is true. Please prove it," his request would have been incomprehensible. The concept of proof was not part of the working vocabulary of an ancient mathematician.

Well then, what is a proof? Heuristically, a proof is a rhetorical device for convincing someone else that a mathematical statement is true or valid. And how might one do this? A moment's thought suggests that a natural way to prove that something new (call it **B**) is true is to relate it to something old (call it **A**) that has already been accepted as true. In this way the concept arises of *deriving* a new result from an old result. See Figure 1.4. The next question is, "How was the old result verified?" Applying this regimen repeatedly, we find ourselves considering a sequence of logic as in Figure 1.5. But then one cannot help but ask, "Where does the chain begin?" And this is a fundamental issue.

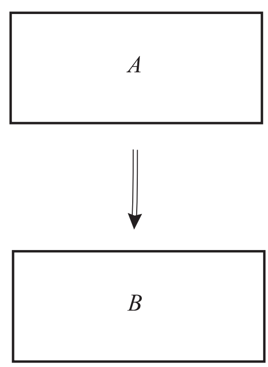

Figure 1.4. Logical derivation.

It will not do to say that the reasoning path has no beginning: that it extends infinitely far back into the mists of time because, if that were the case, it would undercut our thinking of what a proof should be. We are trying to justify new mathematical facts in terms of old mathematical facts. But if the inference regresses infinitely far back into the past, then we cannot actually ever grasp a basis or initial justification for our logic.

As a result of these questions, ancient mathematicians had to think hard about the nature of mathematical proof. Thales (640–546 BCE), Eudoxus (408–355 BCE), and Theaetetus of Athens (417–369 BCE) actually formulated theorems as formal enunciations of certain ideas that they wished to establish as facts or truths. Evidence exists that Thales proved some of these theorems in geometry (and these were later put into a broader context by Euclid). A theorem is the mathematician's formal enunciation of a fact or truth. But Eudoxus fell short in finding means to prove his theorems. His work had a distinctly practical bent, and he was particularly fond of calculations.

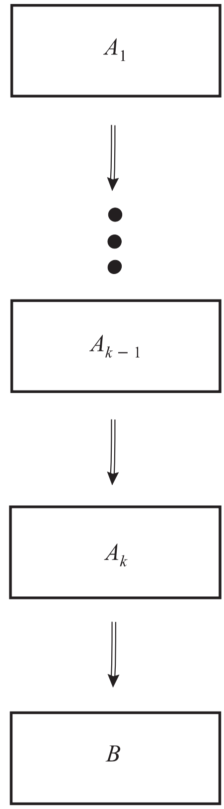

Figure 1.5. A sequence of logical steps.

Euclid of Alexandria first formalized the way we now think about mathematics. Euclid had definitions and axioms and then theorems—in that order. There is no gainsaying the assertion that Euclid set the paradigm by which we have been practicing mathematics for 2300 years. This was mathematics done right. Now, following Euclid, in order to address the issue of the infinitely regressing chart of logic, we begin our studies by putting into place a set of *definitions* and a set of *axioms*.

What is a definition? A definition explains the meaning of a piece of terminology. There are analytical problems with even this simple idea; consider the first definition that we are going to formulate. Suppose that we wish to define a *rectangle*. This will be the first piece of terminology in our mathematical system. What words can we use to define it? Suppose that we define rectangle in terms of points and lines and planes. That begs the questions: What is a point? What is a line? What is a plane?

Thus we see that our *first* definition(s) must be formulated in terms of commonly accepted words that require no further explanation. Aristotle (384–322 BCE) insisted that a definition must describe the concept being defined in terms of other concepts already known. This is often quite difficult. As an example, Euclid defined a *point* to be that which has no part. Thus he used words *outside of mathematics*, words that are a commonly accepted part of everyday jargon, to explain the precise mathematical notion of "point."[2] Once "point" is defined, then one can use that term in later definitions. And one can then use everyday language that does not require further explication. This is how we build up a system of definitions.

The definitions give us then a language for doing mathematics. We formulate our results, or *theorems*, using the words that have been established in the definitions. But wait, we are

[2]It is quite common among those who study the foundations of mathematics to refer to terms that are defined in nonmathematical language—that is, which cannot be defined in terms of other mathematical terms—as *undefined terms*. The concept of "set," which is discussed elsewhere in this book, is an undefined term. So is "point."

Figure 1.6.

not yet ready for theorems, because we have to lay cornerstones upon which our inference can develop. This is the purpose of axioms.

What is an axiom? An axiom[3] (or postulate[4]) is a mathematical statement of fact, formulated using terminology that has been defined in the definitions, that is taken to be self-evident. An axiom embodies a crisp, clean mathematical assertion. One does not *prove* an axiom. One takes an axiom to be given, and to be so obvious and plausible that no proof is required.

Axioms can also be used to explain primitives. These are ideas that are at the basis of the subject, and whose properties are considered to be self-explanatory or self-evident. Again, one cannot verify the assertions in an axiom. They are presented for the reader's enjoyment, with the understanding that they will be used in what follows to prove mathematical results.

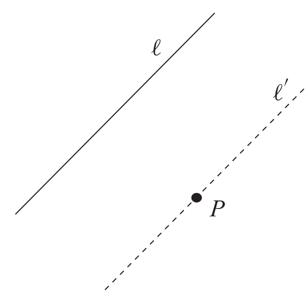

Figure 1.7. The parallel postulate.

One of the most famous axioms in all of mathematics is the *parallel postulate* of Euclid. The parallel postulate (in Playfair's formulation)[5] asserts that if P is a point, and if ℓ is a line not

[3]The word "axiom" derives from a Greek word meaning "something worthy."

[4]The word "postulate" derives from a medieval Latin word meaning "to nominate" or "to demand."

[5]John Playfair (1748–1819) was an eighteenth century geometer at the university of St. Andrews. People don't realize that Euclidean geometry as we use it today is *not* as Euclid wrote it. Playfair and Hilbert and many others worked to modernize it and make it coherent.

passing through that point, then there is a *unique* second line ℓ' passing through P that is parallel to ℓ. See Figure 1.7. The parallel postulate is part of Euclid's geometry of 2300 years ago. And people wondered for over 2000 years whether this assertion should actually be an axiom. Perhaps it could be proved from the other four axioms of geometry (see Section 2.2 for a detailed treatment of Euclid's axioms). There were mighty struggles to provide such a proof, and many famous mistakes made (see [GRE] for some of the history). But, in the 1820s, János Bolyai and Nikolai Lobachevsky showed that the parallel postulate can never be proved. The surprising reason was that there exist models for geometry in which all the other axioms of Euclid are true, yet the Parallel Postulate is false. So the parallel postulate now stands as one of the axioms of our most commonly used geometry.

Generally speaking, in any subject area of mathematics, one begins with a brief list of definitions and a brief list of axioms. Once these are in place, accepted and understood, then one can begin posing and proving theorems. A proof can take many different forms. The most traditional form of mathematical proof is a precise sequence of statements linked together by strict rules of logic. But the purpose of this book is to discuss and consider what other forms a proof might take. Today, a proof could (and often does) take the traditional form that goes back to Euclid's time. But today there are dozens of proof techniques: direct proof, induction, proof by enumeration, proof by exhaustion, proof by cases, and proof by contradiction, to name just a few. A proof could also consist of a computer calculation. Or it could consist of the construction of a physical model. Or it could consist of a computer *simulation* or *model*. Or it could consist of a computer algebra computation using `Mathematica` or `Maple` or `MATLAB`. It could also consist of an agglomeration of these various techniques.

One of the main purposes of this book is to present and examine the many forms of mathematical proof and the role that they play in modern mathematics. In spite of numerous changes and developments in the way we view the technique of proof, this fundamental methodology remains a cornerstone of the infrastructure of mathematical thought. As we have indicated, a key part of any proof—no matter what form it may take—is logic. And what is logic? This is the subject of the next section.

The philosopher Karl Popper believed [POP] that *nothing* can ever be known with absolute certainty. He instead had a concept of "truth up to falsifiability." Mathematics, in its traditional modality, rejects this point of view. Mathematical assertions that are proved according to the accepted canons of mathematical deduction are believed to be irrefutably true. And they will continue to be true. This permanent nature of mathematics makes it unique among human intellectual pursuits.

The paper [YEH] has a stimulating discussion of different modes of proof and their role in our thinking. What is a proof, why is it important, and why do we need to continue to produce proofs?

1.3 How Do Mathematicians Work?

We all have a pretty good idea how a butcher does his work, or how a physician or bricklayer does his work. In fact we have *seen* these people ply their craft. There is little mystery or doubt as to what they do.

But mathematicians are different. They may work without any witnesses, and often prefer to be in private. Many mathematicians sit quietly in their offices or homes and think. Some have favorite objects that they fondle or toss in the air. Others doodle. Some have dartboards. In fact Field Medalist Paul J. Cohen (1924–2007) used to throw darts fiercely at a dart board, and he claimed that he envisioned throwing the darts at his brother (against whom his parents pitted him in competition).

Some mathematical scientists have had rather unpredictable and delightfully surprising ways of doing their stuff. Mathematical physicists Richard Feynman (1918–1988) liked to think about physics while being at a strip club near Caltech. He went there every afternoon. When the strip club got in trouble with the law, the only one of its esteemed customers (which included doctors, lawyers, and even priests) who felt comfortable in testifying on its behalf was Richard Feynman!

Physics Nobel Laureate Steven Weinberg (1933–) used to formulate his theories of cosmology while watching soap operas on television. He was absolutely addicted to *As the World Turns* and some other favorites.

Although we often tell ourselves that a mathematician just sits and thinks, this is far from the whole truth. Mathematicians take walks, play pingpong, lift weights, meditate, talk to people, give lectures, and engage in discussion and debate. They show people half-proofs and half-truths in the hope of getting some help and working the result up into a full-blown theorem. They work through ideas with their students. They run seminars. They write up sets of notes. They publish research announcements. They go to conferences and kick ideas around. They listen to others lecture.[6] They read. They search the Web. They calculate and experiment. Some mathematicians will do sophisticated simulations on the computer. Others may build physical models. My teacher Fred Almgren was fond of dipping bent wires into a soap solution and examining the resulting soap bubbles. My view is that whatever works is good. It doesn't really matter how you get there, as long as you reach the goal.

1.4 The Foundations of Logic

Today mathematical logic is a subject unto itself. It is a full-blown branch of mathematics, like geometry, or differential equations, or algebra. But, for the purposes of practicing mathematicians, logic is a brief and accessible set of rules by which we live our lives.

The father of logic as we know it today was Aristotle (384–322 BCE). His *Organon* laid the foundations of what logic should be. We consider here some of Aristotle's precepts.

[6] I once got a great idea that helped me solve a problem of long standing by listening to a semi-incoherent lecture by a famous French mathematician. I cannot say too much of what his lecture was about, but there was one step of his proof that clinched it for me.