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Probability and Partial Differential Equations in Modern Applied Mathematics

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Probability and Partial Differential Equations in Modern Applied Mathematics

With 22 Illustrations

Springer

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FOREWORD

This IMA Volume in Mathematics and its Applications

PROBABILITY AND PARTIAL DIFFERENTIAL EQUATIONS IN MODERN APPLIED MATHEMATICS

contains a selection of articles presented at 2003 IMA Summer Program with the same title.

We would like to thank Jinqiao Duan (Department of Applied Mathematics, Illinois Institute of Technology) and Edward C. Waymire (Department of Mathematics, Oregon State University) for their excellent work as organizers of the two-week summer workshop and for editing the volume.

We also take this opportunity to thank the National Science Foundation for their support of the IMA.

Series Editors

Douglas N. Arnold, Director of the IMA

Fadil Santosa, Deputy Director of the IMA

PREFACE

The IMA Summer Program on Probability and Partial Differential Equations in Modern Applied Mathematics took place July 21-August 1, 2003, a fitting segue to the IMA annual program on Probability and Statistics in Complex Systems: Genomics, Networks, and Financial Engineering which was to begin September, 2003. In addition to the outstanding resources and staff at IMA, the summer program was developed with the assistance of the following members of the organizing committee: Rabi N. Bhattacharya, Larry Chen, Jinqiao Duan, Ronald B. Guenther, Peter E. Kloeden, Salah Mohammed, Sri Namachchivaya, Mina Ossiander, Bjorn Schmalfuss, Enrique Thomann, and Ed Waymire.

The program was devoted to the role of probabilistic methods in modern applied mathematics from perspectives of both a tool for analysis and as a tool in modeling. Researchers involved in contemporary problems concerning dispersion and flow, e.g. fluid flow, cash flow, genetic migration, flow of internet data packets, etc., were selected as speakers and to lead discussion groups. There is a growing recognition in the applied mathematics research community that stochastic methods are playing an increasingly prominent role in the formulation and analysis of diverse problems of contemporary interest in the sciences and engineering. In organizing this program an explicit effort was made to bring together researchers with a common interest in the problems, but with diverse mathematical expertise and perspective.

A probabilistic representation of solutions to partial differential equations that arise as deterministic models, e.g. variations on Black-Scholes options equations, contaminant transport, reaction-diffusion, non-linear equations of fluid flow, Schrodinger equation etc. allows one to exploit the power of stochastic calculus and probabilistic limit theory in the analysis of deterministic problems, as well as to offer new perspectives on the phenomena for modeling purposes. In addition such approaches can be effective in sorting out multiple scale structure and in the development of both non-Monte Carlo as well as Monte Carlo type numerical methods.

There is also a growing recognition of a role for the inclusion of stochastic terms in the modeling of complex flows. The addition of such terms has led to interesting new mathematical problems at the interface of probability, dynamical systems, numerical analysis, and partial differential equations. During the last decade, significant progress has been made towards building a comprehensive theory of random dynamical systems, statistical cascades, stochastic flows, and stochastic pde's. A few core problems in the modeling, analysis and simulation of complex flows under uncertainty are: Find appropriate ways to incorporate stochastic effects into models; Analyze and express the impact of randomness on the evolution of complex systems in ways useful to the advancement of science and engineering; Design efficient numerical algorithms to simulate random phenomena. There is also a need for new ways in which to incorporate the impact of probability, statistics, pde's and numerical analysis in the training of present and future PhD students in the mathematical sciences. The engagement of graduate students was an important feature of this summer program. Stimulating poster sessions were also included as a significant part of the program.

The editors thank the IMA leadership and staff, especially Doug Arnold and Fadil Santosa, for their tremendous help in the organization of this workshop and in the subsequent editing of this volume. The editors hope this volume will be useful to researchers and graduate students who are interested in probabilistic methods, dynamical systems approaches and numerical analysis for mathematical modeling in engineering and science.

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CONTENTS

NONNEGATIVE MARKOV CHAINS WITH APPLICATIONS*

K.B. ATHREYA†

Abstract. For a class of Markov chains that arise in ecology and economics conditions are provided for the existence, uniqueness (and convergence to) of stationary probability distributions. Their Feller property and Harris irreducibility are also explored.

Key words. Population models, stationary measures, random iteration, Harris irreducibility, Feller property.

AMS(MOS) subject classifications. 60J05, 92D25, 60F05.

1. Introduction.

The evolution of many populations in ecology and that of some economies exhibit the following characteristics: a) It is random but the stochastic transition mechanism displays a time stationary behavior, b) for small population size (and in small and fledgling economies) the growth rate is proportional to the current size with a random proportionality constant, c) for large populations the above growth rate is curtailed by competition for resources (diminishing return in economies). This leads to considering the following class of stochastically recursive time series model

$$X_{n+1} = C_{n+1} X_n \, g(X_n) \; , \qquad n \geq 0 \tag{1}$$

where $g: [0,\infty) \to [0,1]$ is continuous and decreasing, $g(0) = 1$, and $\{C_n\}_{n\geq 1}$ are i.i.d. and independent of the initial value X_0.
These are called density dependent models (Vellekoop and Hügnas (1997), Hassel (1974)).

It is clear that $\{C_n\}_{n\geq 0}$ defined by the above random iteration scheme is a Markov chain with stated space $S = [0,\infty)$ and transition function

$$P(x, A) = P(Cx\, g(x) \in A). \tag{2}$$

The goals of this paper are to describe some recent results on the existence of nontrivial stationary distributions, convergence to them, their uniqueness, etc.

2. Examples.

a) *Random logistic maps.* The logistic model has been quite popular in the ecology literature to capture the density dependence as will as prey-predator interaction (May (1976)). In the present context the parameter

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C is allowed to vary in an i.i.d. fashion over time. Thus the model (1) becomes

$$X_{n+1} = C_{n+1} X_n (1 - X_n) \ , \qquad n \geq 0 \tag{3}$$

with $X_n \in [0,1]$, $C_n \in [0,4]$. Thus, the state space $S = [0,1]$ and $g(x) \equiv 1 - x$ has compact support.

b) *Random Ricker maps.* Ricker (1954) proposed the following model for the evolution of fish population in Canada:

$$X_{n+1} = C_{n+1} X_n e^{-dX_n} \tag{4}$$

with $X_n \in [0,\infty)$, $C_n \in [0,\infty)$, $0 < d < \infty$. Thus, the state space $S = [0,\infty)$ and $g(x) \equiv e^{-dx}$ has exponential decay.

c) *Random Hassel maps.* Hassel (1974) proposed a model with a polynomial decay for large values. Here

$$X_{n+1} = C_{n+1} X_n (1 + X_n)^{-d} \tag{5}$$

with $X_n \in [0,\infty)$, $C_n \in [0,\infty)$, $0 < d < \infty$. Here $S = [0,\infty)$, $g(x) = (1+x)^{-d}$.

d) *Vellekoop-Högnas maps.* A model that includes all the previous cases was proposed by Vellekoop and Högnas (1997)

$$X_{n+1} = C_{n+1} X_n (h(X_n)^{-b} \ , \qquad b > 0 \tag{6}$$

$h : [0,\infty) \to [1,\infty)$, $h(0) = 1$, $h(\cdot)$ is continuously differentiable and $\tilde{h}(x) = \frac{xh'(x)}{h(x)}$ is nondecreasing.

This family of maps exhibits behavior similar to that of the logistic fmaily such as pitchfork bifurcation of periodic behavior, chaotic behaivor as the parameter value is increased etc.

The random logistic case was first introduced by R.N. Bhattacharya and B.V. Rao (1993). Contributions to it include Bhattacharya and Majumdar (2004), Bhattacharya and Waymire (1999), Athreya and Dai (2000, 2002), Athreya and Schuh (2002), Dai (2002), Athreya (2003), Athreya (2004a, b).

Deterministic interval maps have been studied a great deal in the dynamical systems literature. Random perturbations of such system have been investigated in the book of Y. Kifer. Useful references for the deterministic case are the books by Devaney (1989), de Melo and van Strien (1993).

3. Random dynamical systems. The stochastic recursive time series defined by (1) is an example of a random dynamical system obtained by iteration of random jointly measurable maps. This set up will be described now.

Let (S, s) and (K, κ) be two measurable spaces and $f : K \times s \to S$ be jointly measurable, i.e. $(s \times \kappa, s)$ measurable. Let $\{\theta_i(\omega)\}_{i \geq 1}$ be a sequence of K

valued random variables on a probability space (Ω, B, P). Let $X_0 : \Omega \to S$ be an S-valued r.v. Let

$$X_{n+1}(\omega) = f(\theta_{n+1}(\omega), X_n(\omega)), \qquad n \geq 0. \tag{7}$$

Then for each n, $X_n : \Omega \to S$ is a random variable and hence $\{X_{n+1}(\omega)\}_{n\geq 0}$ is a well defined S-valued stochastic process on (Ω, B, P). When $\{\theta_i\}_{i\geq 1}$ are i.i.d. r.v. independent of X_0 then $\{X_n\}_{n\geq 0}$ is an S-valued Markov chain on (Ω, B, P) with transition function

$$P(x, A) = P\{\omega : f(\theta(\omega), x) \leftarrow A\}. \tag{8}$$

It turns out that if S is a polish space then for every probability transition kernel $P(\cdot,\cdot)$, i.e., a map from $S \times s \to [0,1]$ such that for each x, $P(x,\cdot)$ is a probability measure on (S, s) and for each A in s, $P(\cdot, A) : S \to [0,1]$ is s measurable, there exists a random dynamical system of i.i.d. random maps $\{f_i(x,\omega)\}_{i\geq 1}$ from $S \times \Omega \to S$ that is jointly measurable for each i and $\{f_i(\cdot,\omega)\}_{i\geq 1}$ are i.i.d. stochastic processes such that the Markov chain generated by the recursive equation

$$X_{n+1}(\omega) = f_{n+1}(X_n(\omega), \omega) \tag{9}$$

has transition function $P(\cdot,\cdot)$, i.e.

$$P(x, A) = P\{\omega : f(x,\omega) \in A\}.$$

See Kifer (1986) and Athreya and Stenflo (2000). As simple examples of this consider the following.

1. *The vacillating probabilist.*

$$S = [0,1],$$
$$X_{n+1} = \frac{X_n}{2} + \frac{\epsilon_{n+1}}{2}$$

$\{\epsilon_n\}_{n\geq 1}$ are i.i.d. Bernouilli $(\frac{1}{2})$ r.v. Athreya (1996).

2. *Sierpinski Gasket.* Let S be an equilateral triangle with vertices v_1, v_2, v_3 and $\{X_n\}_{n\geq 0}$ be define by

$$X_{n+1} = \frac{X_n + \epsilon_{n+1}}{2}$$

where $\{\epsilon_n\}_{n\geq 1}$ are i.i.d. with distribution

$$P(\epsilon_1 = V_i) = \frac{1}{3} \qquad i = 1,2,3.$$

3. Let $\{A_n, b_n\}_{n\geq 1}$ be i.i.d r.v. such that for each n, A_n is $K \times K$ real matrix and b_n is a $K \times 1$ vector. Let

$$X_{n+1} = A_{n+1}X_n + b_{n+1}.$$

Suppose $E \log ||A_1|| < 0$ and $E(\log ||b_1||)^+ < \infty$ where $||A_1||$ is the matrix norm and $||b_1||$ is the Euclidean norm. Then it can be shown that X_n converges in distribution and the limit π is nonatomic (provided the distribution of (A_1, b_1) is not degenerate). Note that this example includes the previous two. Further, it can be shown that w.p.1 the limit point set of $\{X_n\}_{n\geq 0}$ coincides with the support k of the limit distribution π. This result has been used to solve the inverse problem of generating k by running an appropriate Markov chain $\{X_n\}_{n\geq 0}$ and looking at the limit point set of its sample path. For this the book by Barnsley (1993) may be consulted. When S is Polish and the $\{f_i\}_{i\geq 1}$ are i.i.d. Lifschitz maps several sufficient conditions are known for the existence of a stationary distribution, its uniqueness and convergence to it. Two are given below.

THEOREM 3.1. *Let (S, d) be Polish and (Ω, B, P) be a probability space. Let $\{f_i(x, \omega\}_{i\geq 1}$ be i.i.d. maps form $S \times \Omega \to S$ such that for each i f_i is jointly measurable. Let $X_{n+1}(\omega) = f_{n+1}(X_n(\omega), \omega),\ n \geq 0$*
a) Let $f_i(\cdot, \omega)$ be Lifschitz w.p.1 and let

$$s(f_1) \equiv \sup_{x \neq y} \frac{d(f_1(x, \omega), f_1(y, \omega)}{d(x, y)}$$

Assume $E(\log s(f_1)) < 0$ and $E(\log d(f_1(x_0, \omega), x_0))^+ < \infty$ for some x_0 in S.
Then, for any initial distribution, the sequence $\{X_n\}$ converges in distribution to a limit π that is unique and stationary for the Markov chain $\{X_n\}$.
b) Let for some $p > 0$

$$\sup_{x \neq y} E\frac{(d(f_1(x, \omega), f_1(y, \omega)))^p}{d(x, y)} < 1$$

and for some x_0

$$E(\log d(f_1(x_0, \omega), x_0))^+ < \infty$$

Then the conclusion of (a) holds.

For a proof of (a) see Diaconis & Freedman (1991). For a proof of (b) see Athreya (2004b). The main tool is to show that the dual sequence $\hat{X}_n = f_1(f_2 \ldots (f_n(\cdot)))$ converges w.p.1 and that X_n and $\hat{X}_n$ have the same distribution. For related results see N. Carlson (2004) and Wu (2002).

For Feller Markov chains on Polish spaces one of the methods of finding stationary distributions is to use the weak compactness of the occupation measures and the Foster-Lyaponov criterion.
More specifically, define the *occupation measures* by

$$\Gamma_{n,x}(A) \equiv \frac{1}{n} \sum_{0}^{n-1} P(x_j \in A)\ , \qquad n \geq 1 \tag{10}$$

THEOREM 3.2. *Let Γ be a vague limit point of $\{\Gamma_{n,x}(\cdot)\}$, that is, Γ is a measure such that $\Gamma(S) \le 1$ and for some subsequence $n_k \to \infty$, $\int_S g\, d\Gamma_{n,x} \to \int g\, d\Gamma$ for all continuous functions g with compact support. Suppose S admits an "approximate identity" i.e. $\exists \{g_k\}_{k\ge 1}$ such that for each k, g_k is a continuous function with compact support and for all x in S, $0 \le g_k(x) \uparrow 1$. Then, Γ is stationary for P, i.e. $\Gamma(A) = \int_S P(x, A)\Gamma(dx)$, $\forall A \in s$.*

The Foster-Lyaponov condition ensures that any vague limit Γ is nontrivial.

THEOREM 3.3. *Suppose there exists a function $V : S \to [0, \infty)$, a set $K \subset S$ and constants $\alpha > 0$, $M < \infty$ such that*

i) $\forall x \notin k, \quad E(V(X_1)\ |X_0 = x) - V(x) \le -\alpha.$

ii) $\forall x \in S, \quad E(V(X_1)\ |X_0 = x) - V(x) \le M.$

Then $\underline{\lim}\, \Gamma_{n,x_0}(k) \ge \frac{\alpha}{\alpha+M} > 0.$

In ecological and economic applications when $S = [0, \infty)$, the above condition is verified for a compact set $k \subset (0, \infty)$ so that Γ is different from the delta measure at 0.

For proofs the above two results see Athreya (2004a, b).

4. Stationary distributions for Markov chains satisfying (1). Let $\{X_n\}_{n\ge 0}$ be a Markov chain defined by (1). A necessary condition for the existence of a stationary distribution π such that $\pi(0, \infty) > 0$ is provided below.

THEOREM 4.1. *Let $E(\ln c_1)^+ < \infty$. Suppose there exists a probability distribution π on $[0, \infty)$ that is stationary for $\{X_n\}_{n\ge 0}$ and $\pi(0, \infty) > 0$. Then,*

i) $E(\ln c_1)^- < \infty,$

ii) $\int |\ln g(x)|\, \pi(dx) < \infty,$

iii) $E \ln c_1 = -\int \ln g(x)\, \pi(dx)$ *and hence strictly positive.*

COROLLARY 4.1. *If $E \ln c_1 \le 0$ then $\pi \equiv \delta_0$, the delta measure at 0 is the only stationary distribution for $\{X_n\}_{n\ge 0}$. Further, X_n converges to 0 w.p.1 if $E \ln c_1 < 0$ and in probability if $E \ln c_1 = 0$.*

A sufficient condition is given below.

THEOREM 4.2. *Let $D \equiv \sup x\, g(x) < \infty$. Let*

i) $E|\ln C_1| < \infty, \qquad E \ln C_1 > 0,$

ii) $E|\ln g(C_1, D)| < \infty.$

Then, there exists a stationary distribution π for $\{X_n\}$ such that $\pi(0, \infty) = 1$.

For the logistic case this reduces to $E \ln C_1 > 0$ and $E|\ln(4 - C_1)| < \infty$ and for the Ricker case to $E \ln C_1 > 0$ and $EC_1 < \infty$.

For proofs of these and more results see Athreya (2004). The stationary distribution is not unique, in general. For an example in the logistic case see Athreya and Dai (2002). Under some smoothness hypothesis on the distribution of c_1 uniqueness does hold as will be shown in the next section. For some convergence results see Athreya (2004a,b).

5. Harris irreducibility.

DEFINITION 5.1. *A Markov chain $\{X_n\}_{n\geq 0}$ with state space (S, s) and transition function $P(\cdot,\cdot)$ is Harris irreducible with reference measure φ on (S, s) if*

i) *φ in σ-finite and*

ii) *$\varphi(A) > 0 \Longrightarrow P(X_n \in A$ for some $n \geq 1 | X_0 = x)$ is > 0 for every x in S.*

(Equivalently if there exists a σ-finite measure φ on (S, s) such that for each x in S, the Green's measure $G(x, A) \equiv \sum_{n=0}^{\infty} P(X_n \in A | X_0 = x)$ dominates φ.)

If $S = N$, the set of natural numbers and $P \equiv ((p_{ij}))$ is a transition probability matrix and if $\forall i, j\ \exists n_{ij} \in P_{ij}^{n_{ij}} > 0$ then $\{X_n\}$ is Harris irreducible with the counting measure on N as the reference measure. An important consequence of Harris irreducibility is the following

THEOREM 5.1. *Let $\{X_n\}_{n\geq 0}$ be Harris irreducible with state space (S, s), transition function $P(\cdot,\cdot)$ and reference measure φ. Suppose there exists a probability measure π on (S, s) that is stationary for P. Then*

i) *π is unique.*

ii) *For any x in S, the* occupation measures $\Gamma_{n,x}(A) \equiv \frac{1}{n}\sum_0^{n-1} P(x_j \in A | X_0 = x)$ *converge to $\pi(\cdot)$ in total variation.*

iii) *For any x in S, the empirical distribution $L_n(A) \equiv \frac{1}{n}\sum_0^{n-1} I_A(x_j) \to \pi(A)$ w.p.1 (P_x) (when $X_0 = x$) for each A in s.*

iv) *$\{X_n\}_{n\geq 0}$ is Harris recurrent i.e. $\varphi(A) > 0 \Rightarrow P(X_n) \in A$ for some $n \geq 1 | X_0 = x) = 1$ for all x in S.*

The Markov chain vacillating probabilist (Example 3.1) is not Harris irreducible but will be if ϵ_i has a distribution that has an absolutely continuous compnent.

It is also known that if s is countably generated then every Harris recurrent Markov chain with state space (S, s) is *regenerative* in the sense its sample paths could be broken up into a sequence of i.i.d. cycles as in the discrete state space case. For a proof of this and Theorem 5.1 see Athreya and Ney (1978), Nummelin (1984), Meyn and Tweedie (1993).

In the rest of this section conditions will be found for Harris irreducibility of $\{X_n\}_{n\geq 0}$ defined by (1).

Assume that $\{C_n\}_{n\geq 1}$ are i.i.d. with values in $(0, L)$, $L \leq \infty$ and for each $c \in (0, L)$, $f_c(x) \equiv cxg(x)$ maps $S = (0, k)$, $k \leq \infty$ to itself. For any function $f : S \to S$ the iterates of f are defined by

$$f^{(0)}(x) \equiv x, \quad f^{(m+1)}(x) = f(f^{(m)}(x)), \quad m \geq 0.$$

The first step is a local irreducibility result.

THEOREM 5.2. *Suppose*

i) *$\exists\, 0 < \alpha < \infty$, $\delta > 0$, a Borel function $\Psi : J \equiv (\alpha - \delta, \alpha + \delta) \to (0, \infty) \to P(C_1 \in B) \geq \int_{B\cap J} \Psi(\theta)d\theta$ for all Borel sets B.*

ii) $\exists\, 0 < p < \infty,\ m \geq 1$ *such that for the function* $f_\alpha(x) \equiv \alpha x\, g(x)$, $f_\alpha^{(m)}(p) = p$.

Then, $\exists \eta > 0 \rightarrow \forall x \in I \equiv (p-\eta, p+\eta),\ P_x(X_m \in A) > 0$ *for all Borel sets* A *such that* $\varphi(A) \equiv \lambda(A \cap I) > 0$ *where* λ *is Lebesgne measure.*

COROLLARY 5.1. *Suppose in addition to the hypotheses of Theorem 5.1,* $P_x(X_n \in I$ *for some* $n \geq 1)$ *is* > 0 *for all* x *in* $(0,k)$. *Then* $\{X_n\}_{n\geq 0}$ *is Harris irreducible with state space* $S = (0,k)$.

Using a deep result of Guckenheimer (1979) on S-unimodal maps a sufficient condition for the hypotheses of Corollary 5.1 can be found.

DEFINITION 5.2. *A map* $h:\ [0,1] \rightarrow [0,1]$ *is* S*-unimodal if*

i) $h(\cdot) \in C^3$, *i.e. 3 times continuously differentiable,*

ii) $h(0) = h(1) = 0$,

iii) $\exists\ 0 < c < 1 \ni h''(c) < 0$, h *is increasing in* $(0,c)$ *and decreasing in* $(c,1)$ *and*

iv) $(Sf)(x) \equiv \frac{h'''(x)}{h''(x)} - \frac{3}{2}\left(\frac{h''(x)}{h'(x)}\right)^2$ *if* $h'(x) > 0$ *and* $-\infty$ *if* $h'(x) = 0$ *is* < 0 *for all* $0 < x < 1$.

EXAMPLES. $h(x) =\equiv cx(1-x),\quad 0 < c \leq 4,\quad h(x) = x^2 \sin \pi x$.

DEFINITION 5.3. *A number p in (0,1) is a* stable periodic point *for* h *if for some* $m \geq 1$, $h^{(m)}(p) = p$ *and* $|h^{(m)}(p)| < 1$.

DEFINITION 5.4. *For* x *in (0,1) the* orbit O_x *is the set* $\{h^{(m)}(x)\}_{m\geq 0}$ *and* $w(x)$ *is the* limit point set of O_x.

THEOREM 5.3 (Guckenheimer (1979)). *Let* h *be* S*-unimodal with a stable periodic point* p. *Let* $K = \{x:\ 0 < x < 1,\ \omega(x) = \omega(p)\}$. *Then,* $\lambda(K) = 1$ *where* $\lambda(\cdot)$ *is the Lebesgue measure.*

Combining Theorem 5.2, 5.3 and Corollary 5.1 leads to

THEOREM 5.4. *Let* $S = [0,1]$. *Assume*

i) $\forall\ 0 < c < k,\ h_c(x) \equiv cxg(x)$ *is* S*-unimodal.*

ii) $\exists\ 0 < p < 1,\ 0 < \alpha < L\ \ni p$ *is a stable periodic point for* $h_\alpha(x) \equiv \alpha x g(x)$.

iii) $\exists\ \delta > 0$, *a Borel function* $\Psi:\ J \equiv (\alpha-\delta, \alpha+\delta) \rightarrow (0,\infty) \ni P(C_1 \in B) \geq \int_{B\cap J} \Psi(\theta)d\theta$ *for all Borel sets* B. *Then, the Markov chain* $\{X_n\}_{n\geq 0}$ *defined by*

$$X_{n+1} = C_{n+1} X_n\, g(X_n), \qquad n = 0,1,2,\ldots$$

where $\{C_n\}_{n\geq 1}$ *are i.i.d. is Harris irreducible with state space (0,1) reference measure* $\phi(\cdot) = \lambda(\cdot \cap I)$ *where* $I = (p-\eta, p+\eta)$ *for some appropriate* $\eta > 0$.

As a special case applied to random logistic maps one gets

THEOREM 5.5. *Let* $S = [0,1]$, *let* $\{C_n\}_{n\geq 1}$ *i.i.d. [0,4] valued r.v. and* $\{X_n\}_{n\geq 0}$ *be the the Markov chain defined by*

$$X_{n+1} = C_{n+1}\, X_n(1 - X_n), \qquad n \geq 0.$$

Suppose $\exists$ *an open interval* $J \subset (0,4)$ *and a function* $\Psi:\ J \rightarrow (0,\infty) \rightarrow P(C_1 \in B) \geq \int_{B\cap J} \Psi(\theta)d\theta$ *for all Borel sets* B.

If $J \cap (1,4) = \varphi$, assume in addition, that $\exists\ \beta > 1$ in the support of $C_1 \ni f_\beta(x) \equiv \beta x(1-x)$ admits a stable periodic point p in (0,1). Then $\{X_n\}_{n\geq 0}$ is Harris irreducible.

COROLLARY 5.2. *Suppose, in addition to the hypotheses of Theorem 5.5, that $\exists\ \ln C_1 > 0$ and $E|\ln(4-C_1)| < \infty$. Then, $\exists$ a unique stationary measure π for $\{X_n\}$ such that*

i) $\pi(0,1) = 1$,

ii) π *is absolutely continuous,*

iii) $\forall\ 0 < x < 1, \qquad P_x(X_n \in \cdot) \to \pi(\cdot)$ *in total variation.*

For proofs of all the results in this section except Theorem 5.1 see the Athreya (2003). It has been pointed out by one of the referees that the above Corollary has been obtained independently by R.N. Bhattacharya and M. Majumdar in a paper entitled "Stability in distribution of randomly perturbed quadratic maps as Markov processes", CAE working paper 02-03, Department of economics, Cornell University.

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PHASE CHANGES WITH TIME AND MULTI-SCALE HOMOGENIZATIONS OF A CLASS OF ANOMALOUS DIFFUSIONS*

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Abstract. Composite media often exhibit multiple spatial scales of heterogeneity. When the spatial scales are widely separated, transport through such media go through distinct phase changes as time progresses. In the presence of two such widely separated scales, one local and one large scale, the time scale for the appearance of the effects due to the large scale fluctuations is determined. In the case of transport in periodic media with such slowly evolving heterogeneity and divergence-free velocity fields, there is a first Gaussian phase which breaks down at the above time scale, and a second Gaussian phase occurs at a later time scale which is also precisely determined. In between there may be non-Gaussian phases, as shown by examples. Depending on the structure of the large scale fluctuations, the diffusion is either super-diffusive, with the effective diffusivity increasing to infinity, or it exhibits normal diffusivity which increases to a finite limit as time increases. Sub-diffusivity, with the effective diffusion coefficient tending to zero in time, is shown to arise in a certain class of velocity fields which are not divergence-free.

1. Introduction.

Electric and thermal conduction in composite media as well as diffusion of matter through them are problems of much significance in applications (see, [5–7, 16, 21]). Examples of such composite media are natural heterogeneous material such as soils, polycrystals, wood, animal and plant tissue, cell aggregates and tumors, and synthetic products such as fiber composites, cellular solids, gels, foams, colloids, concrete, etc. The evolution equation that arises in such contexts is generally a Fokker-Planck equation of the form

$$\frac{\partial c(t,y)}{\partial t} = \frac{1}{2}\nabla \cdot (D(y)\nabla c) - \nabla \cdot (v(y)c), \qquad c(0,.) = \delta_x \tag{1.1}$$

where $D(\cdot)$ is a $k \times k$ positive definite matrix-valued function depending on local properties of the medium, and its eigenvalues are assumed bounded away from zero and infinity; $v(\cdot)$ is a vector field which arises from other sources. To fix ideas one may think of $v(\cdot)$ as the velocity of a fluid (say, water) in a porous medium (such as a saturated aquifer) in which $c(t, y)$ is the concentration of a solute (e.g., a chemical pollutant) injected at a point in the medium ([12, 16, 21, 25, 31, 36, 38]). One may also think of (1.1) as the equation of transport, or diffusion, of a substance in a turbulent fluid ([1, 3, 35]).

One of the main aims of the study of transport in disordered media is to derive from the local, or microscopic, Equation (1.1) a macroscopic equation with constant coefficients governing c over much larger space/time

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scales, under appropriate assumptions. Such a derivation is known as *homogenization* in partial differential equations. The macroscopic equation is then of the form

$$\frac{\partial c(t,y)}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{k} D_{i,j}\,\frac{\partial^2 c}{\partial y_i\,\partial y_j} - \sum_{i=1}^{k} \bar{v}_i\,\frac{\partial c}{\partial y_i}\,, \tag{1.2}$$

where $D = (D_{i,j})$ is the *effective dispersion* or, *diffusivity.* This program has been carried out in complete generality for *periodic* $D(\cdot)$, $v(\cdot)$ in Bensoussan et al. (1978) (also see [1, 2, 8, 23, 30, 38]). Another popular model assumes $D(\cdot)$, $v(\cdot)$ are *stationary ergodic random fields* ([1, 2, 7, 23, 38]). Papanicolaou and Varadhan (1980) and Kozlov (1979) independently derived homogenizations when (1.1) is in divergence form (i.e., $v(\cdot) = 0$ in (1.1)). For a class of two-dimensional problems in such random media with $D(\cdot) = D$ constant and $v(\cdot)$ a (divergence free) shearing motion, a derivation of homogenization and analysis of asymptotics is carried out in Avellaneda and Majda (1990), (1992) (also see [1]).

From a *probabilistic point of view*, homogenization of (1.1) in the form (1.2) means that a diffusion (Markov process) $X(\cdot)$ generated by $A = \frac{1}{2}\nabla\cdot(D(\cdot)\nabla) + v(\cdot)\cdot\nabla$ converges in law, under a scaling of time and space with properly large units, to a Brownian motion $W(\cdot)$ with (constant) diffusion matrix D and (constant) drift velocity vector $\bar{v}$:

$$\varepsilon X\Big(\frac{t}{\varepsilon^2}\Big) - \frac{t}{\varepsilon}\,\bar{v} \longrightarrow W(t), \qquad (t \geq 0), \quad \text{as } \varepsilon \downarrow 0. \tag{1.3}$$

It is known that if the coefficients are periodic, or stationary ergodic random fields, and $v(\cdot)$ is divergence free, the effective diffusivity is larger than the average of the local diffusivity $D(\cdot)$.

We have so far considered homogenization under a single scale of heterogeneity. Natural composite media generally exhibit *multiple scales of heterogeneity*, i.e., heterogeneity that evolves with distance. It has been observed in many instances, and sometimes verified theoretically, that this often leads to increase in the effective dispersivity D with the spatial scale, say, L. For the case of solute dispersion in porous media, such as saturated aquifers, one may see this by introducing a scale parameter in $v(\cdot)$, or by relating D to the correlation length, and still using a single large scale ([13, 23, 38]).

Our objective in the present survey is to introduce different widely separated spatial scales of heterogeneity explicitly in the model and study (i) the effective diffusivity as a function of the spatial scale, and (ii) the time scales for the different (Gaussian and non-Gaussian) phases the diffusion passes through as time progresses. In the next section we give a fairly complete description of this for the case of periodic coefficients and a divergence free velocity field $v(\cdot)$ with two widely separated scales—a local scale and a large scale. The case of additional appropriately widely separated scales may be understood from this. Examples in Section 4 illustrate the emergence of non-Gaussian phases in between Gaussian ones.

Before concluding this introduction, let us mention the classical work of Richardson (1926) who looked at already existing data on diffusion in air over 12 or so different orders of spatial scale, and conjectured that the diffusivity D_L at the spatial scale L satisfies

$$D_L \propto L^{4/3}. \tag{1.4}$$

This was related later by Batchelor (1952) to the turbulence spectrum $v \propto L^{1/3}$ derived by Kolmogorov (1941). The length scale $L(t)$ and the diffusion coefficient $D_{L(t)}$, as functions of time t, are now related using $L(t)$ as the root mean squared distance from the mean flow (see Ben Arous and Owhadi (2002)): $L^2(t) \propto D_{L(t)}t \propto L^{4/3}(t)t$, leading to $L(t) \propto t^{3/2}$ and $D_{L(t)} \propto t^2$. This was also derived by Obhukov (1941) by a dimensional argument similar to that of Kolmogorov (1941). In particular, $D_{L(t)} \to \infty$ as $t \to \infty$, that is, this is a case of *super-diffusivity.* For a precise analysis of a two-dimensional model with constant $D(\cdot) = D$ and a stationary ergodic $v(\cdot)$, we refer to Avellaneda and Majda (1990), (1992).

2. A general model with two spatial scales: The first phase of asymptotics and the time scale for its breakdown. Consider the general model (1.1) with $v(\cdot)$ of the form

$$v(y) = b(y) + \beta\Big(\frac{y}{a}\Big), \tag{2.1}$$

where a is a large parameter, $b(\cdot)$, and $\beta(\cdot/a)$ represent the local and large scale velocities, respectively. The solution to (1.1) is the fundamental solution $p(t; x, y)$. Consider a diffusion $X(t)$, $t \geq 0$, on R^k with transition probability density p, starting at $x = X(0)$. To avoid the artificial importance of the origin, take the initial point x to be

$$x = ax_0 \tag{2.2}$$

where x_0 is a given point in R^k, so that the initial value of $\beta(\cdot/a)$ is $\beta(x_0)$. One may represent such a diffusion as the solution to the stochastic integral equation

$$\begin{aligned} X(t) &= ax_0 + \int_0^t \Big\{ b(X(s)) + d(X(s)) + \beta\Big(\frac{X(s)}{a}\Big)\Big\} ds \\ &\quad + \int_0^t \sigma(X(s))dB(s), \end{aligned} \tag{2.3}$$

where $\sigma(x) = \sqrt{D(x)}$, $d(x) = (d_1(x), \ldots, d_k(x))'$, $d_j(x) = \sum_i (\partial/\partial x_i) D_{ij}(x)$, and $B(\cdot)$ is a standard k-dimensional Brownian motion. Since $\beta(\cdot/a)$ changes slowly, at the rate of $1/a$, one expects that for an initial period of time the process $X(\cdot)$ will behave like the diffusion $Y(\cdot)$ governed by

$$
\begin{aligned}
Y(t) &= ax_0 + \int_0^t \{b(Y(s)) + d(Y(s))\}ds + t\beta(x_0) \\
&\quad + \int_0^t \sigma(Y(s))dB(s).
\end{aligned} \tag{2.4}
$$

Indeed, the L^1-distance between $p(t;x,y)$ and the transition density $q(t;x,y)$ of $Y(t)$ is negligible for the times $t \ll a^{2/3}$. Actually, the *total variation distance* $||P_{0,t} - Q_{0,t}||_v$ between the distributions $P_{0,t}$ of the process $\{X(s) : 0 \le s \le t\}$ and the distribution $Q_{0,t}$ of the process $\{Y(s) : 0 \le s \le t\}$ goes to zero in this range. More precisely, one has the following result obtained in [12] (also see [9]).

THEOREM 2.1. *Assume $b(\cdot)$ and its first order derivatives are bounded, as are $D(\cdot)$, $\beta(\cdot)$ and their first and second order derivatives. Assume also that the eigenvalues of $D(\cdot)$ are bounded away from zero and infinity. Then*

$$
||P_{0,t} - Q_{0,t}||_v \longrightarrow 0 \qquad \text{as} \quad \frac{t}{a^{2/3}} \longrightarrow 0. \tag{2.5}
$$

Proof. By the Cameron-Martin-Girsanov Theorem (Ikeda and Watanabe (1981), pp. 176–181),

$$
\begin{aligned}
Z(t) &:= \int_0^t \sigma^{-1}(Y(s))\left\{\beta\left(\frac{Y(s)}{a}\right) - \beta\left(\frac{Y(0)}{a}\right)\right\}dB(s) \\
&\quad - \frac{1}{2}\int_0^t \left|\sigma^{-1}(Y(s))\left\{\beta\left(\frac{Y(s)}{a}\right) - \beta\left(\frac{Y(0)}{a}\right)\right\}\right|^2 ds.
\end{aligned} \tag{2.6}
$$

Since $E\exp\{Z(t)\} = 1$, $E|1 - \exp\{Z(t)\}| = 2E(1 - \exp\{Z(t)\})^+ \le 2[E|Z(t)| \wedge 1]$. Now the expected value of the second integral in (2.6) can be shown, using Ito's Lemma ([26]), to be bounded by $[c_1t^2/a^2 + c_2t^3/a^2 + c_3t^3/a^4]/\lambda$ where λ is the infimum of all eigenvalues of $D(\cdot)$, and c_1, c_2, c_3, depend only on the upper bounds of the components of $b(\cdot), \beta(\cdot), D(\cdot)$ and of their first order derivatives, and also of the second order derivatives of $\beta(\cdot)$. Since the expected value of the square of the norm of the stochastic integral equals the expected value of the Reimann integral of the squared norm of the integrand, one has

$$
||P_{0,t} - Q_{0,t}||_v \le \theta + \frac{1}{2}\theta^{1/2}
$$

where $\theta = [c_1t^2/a^2 + c_2t^3/a^2 + c_3t^3/a^4]/\lambda$. □

One may show by examples (see Section 4) that the large scale fluctuations (namely, fluctuations of $\beta(\cdot/a)$) can not be ignored in general for times t of the order $a^{2/3}$ or larger, i.e., the time scale in (2.5) is precise.

Theorem 2.1 implies that a *first homogenization* occurs for times $1 \ll t \ll a^{2/3}$, provided $Y(\cdot)$ defined by (2.4) is asymptotically Gaussian. This is the case, e.g., if $b(\cdot), D(\cdot)$ are periodic, or are ergodic random fields satisfying some additional conditions ([1–3, 14, 34, 38]). *No assumption is needed on* $\beta(\cdot)$, except the smoothness and boundedness conditions imposed in Theorem 2.1. To illustrate this, let $b(\cdot)$ and $D(\cdot)$ be periodic with the same period lattice, say, $\mathbb{Z}^k$, and assume for simplicity that

$$\operatorname{div} b(\cdot) = 0. \tag{2.7}$$

Then, by Bensoussan et al. (1978) (or, Bhattacharya (1985)), and Theorem 2.1, one has

$$\lim_{\varepsilon\downarrow 0,\ \varepsilon a^{1/3}\to\infty} \varepsilon X\left(\frac{t}{\varepsilon^2}\right) - \frac{t}{\varepsilon}\left[\bar{b} + \beta(x_0)\right] \xrightarrow{\mathcal{L}} \mathrm{BM}(0, K), \tag{2.8}$$

where the right side is a Brownian motion with 0 drift and a (constant) diffusion matrix K:

$$\bar{b} = \int_{[0,1]^k} b(x)dx, \qquad \bar{D} = \int_{[0,1]^k} D(x)dx,$$

$$\begin{aligned} K &= \bar{D} + \int_{[0,1]^k} \operatorname{Grad}\Psi(x)D(x)(\operatorname{Grad}\Psi(x))'dx \\ &\quad - \int_{[0,1]^k} \{\operatorname{Grad}\Psi(x)D(x) + D(x)(\operatorname{Grad}\Psi(x))'\}dx, \end{aligned} \tag{2.9}$$

$$\Psi(x) = (\psi_1, \psi_2, \ldots, \psi_k)', \qquad \operatorname{Grad}\Psi(x) = \begin{pmatrix} \operatorname{grad}\psi_1(x) \\ \operatorname{grad}\psi_2(x) \\ \vdots \\ \operatorname{grad}\psi_k(x) \end{pmatrix}$$

where ψ_r is the mean-zero periodic solution of

$$\begin{aligned} L_0\psi_r(x) &= b_r(x) - \bar{b}_r + d_r(x), \qquad (1 \le r \le k), \\ L_0 := &\ \frac{1}{2}\nabla\cdot D(\cdot)\nabla + (b(\cdot) + d(\cdot) + \beta(x_0))\cdot\nabla. \end{aligned} \tag{2.10}$$

Note that (2.8), (2.9) imply that with appropriately large units of space and time (of the orders of $1/\varepsilon$ and $1/\varepsilon^2$, respectively, with $\varepsilon \downarrow 0$, $\varepsilon \gg a^{-1/3}$, the diffusion $X(\cdot)$ governed by $L = \frac{1}{2}\nabla\cdot D(\cdot)\nabla + (b(\cdot) + \beta(\cdot/a))\cdot\nabla$ may be approximated by a Brownian motion generated by $L = \frac{1}{2}\nabla\cdot K\nabla + (\bar{b} + \beta(x_0))\cdot\nabla$.

3. The second Gaussian phase and its time scale, examples of non-Gaussian intermediate phases. We now consider the model (1.1) with a constant $D(\cdot) = D$ for simplicity and with $v(\cdot) = b(\cdot) + \beta(\cdot/a)$, as in (2.1), satisfying the following assumptions:

$$\begin{aligned} &\text{A1:} \quad b(\cdot), \beta(\cdot) \text{ are periodic with period lattice } \mathbb{Z}^k; \\ &\text{A2:} \quad \operatorname{div} b(\cdot) = 0 = \operatorname{div} \beta(\cdot); \\ &\text{A3:} \quad a \text{ is a positive integer.} \end{aligned} \tag{3.1}$$

One may take a (in A3) to be rational, and an arbitrary period lattice rather than $\mathbb{Z}^k$ (in A1).

The diffusion $X(\cdot)$ is now governed by

$$X(t) = x + \int_0^t \left\{ b(X(s)) + \beta\left(\frac{X(s)}{a}\right) \right\} ds + \sigma B(t) \qquad t \geq 0, \tag{3.2}$$

where $\sigma = \sqrt{D}$. Then $\dot{X}(t) := X(t) \bmod a \equiv (X_1(t) \bmod a, \ldots, X_k(t) \bmod a)$, $t \geq 0$, is a diffusion on the *big torus* $\mathcal{T}_a = \{x \bmod a : x \in \mathbb{R}^k\}$. As in (2.8), (2.9) the asymptotic distribution of $X(t)$, $t \geq 0$, is given by ([6, 8])

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \varepsilon X\left(\frac{t}{\varepsilon^2}\right) - \frac{t}{\varepsilon}(\bar{b} + \bar{\beta}) \xrightarrow{\mathcal{L}} \mathrm{BM}(0, K), \\ &K = D + \int_{[0,a]^k} \operatorname{Grad} \Psi(x) D (\operatorname{Grad} \Psi(x))' \frac{1}{a^k}\, dx, \end{aligned} \tag{3.3}$$

where $\Psi = (\psi_1, \psi_2, \ldots, \psi_k)'$ and ψ_r is the mean zero periodic (with period a) solution of

$$\begin{aligned} L_a \psi_r(x) &= b_r(x) + \beta_r\left(\frac{x}{a}\right) - \bar{b}_r - \bar{\beta}_r, \qquad 1 \leq r \leq k, \\ L_a &:= \frac{1}{2} \sum_{i,j=1}^k D_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + (b(\cdot) + \beta(\cdot/a)) \cdot \nabla. \end{aligned} \tag{3.4}$$

For large a, that is, as $a \to \infty$, we need to determine how large must t be for $X(t)$ to have the Brownian limit (3.3) or, equivalently, for the homogenization of L_a in the form

$$\bar{L}_a = \frac{1}{2} \sum_{i,j=1}^k K_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + (\bar{b} + \bar{\beta}) \cdot \nabla. \tag{3.5}$$

A related, and at least equally important, problem is to analyze the asymptotic dispersion K as a function of a.

Two crucial ingredients for this final phase analysis are 1) the speed at which $\dot{X}(t)$ approaches the uniform (equilibrium) distribution on $\mathcal{T}_a$ (as $a \to \infty$), and 2) the asymptotic relation between a and the dispersion matrix of the limiting Gaussian in the final phase. By spectral methods analogous to those of Diaconis and Stroock (1991) and Fill (1991), the L^1-distance between the distributions of $\dot{X}(t)$, with arbitrary $\dot{X}(0)$, and the equilibrium distribution is bounded above by $ca^{k/2}\exp\{-c't/a^2]\}$ for some positive constants c and c' (see [9, 12]). Unfortunately, the presence of the factor $a^{k/2}$ leads to the relaxation time to equilibrium as $t \gg a^2 \log a$, appearing with a logarithmic factor. A fascinating result of Franke (2004), on the other hand, provides the correct rate $t \gg a^2$. To state this let us scale $X(t)$ as follows

$$\begin{aligned} Y(t) &:= \frac{X(a^2t)}{a}, \qquad \dot{Y}(t) = Y(t) \bmod a, \\ dY(t) &= a\{b(aY(t)) + \beta(Y(t))\}dt + \sigma d\bar{B}(t), \\ \bar{B}(t) &:= \frac{B(a^2t)}{a}. \end{aligned} \tag{3.6}$$

Then one unit of time of $Y(\cdot)$ equals a^2 units of time of $X(\cdot)$ and one spatial unit of $Y(\cdot)$ equals a spatial units of $X(\cdot)$. Note that $\dot{Y}(t)$, $t \gg 0$, is a diffusion on the *unit torus* $\mathcal{T}_1$. The generator of $Y(\cdot)$ (which is also the generator of $\dot{Y}(\cdot)$ when the domain is restricted to periodic functions) is

$$A_a := \frac{1}{2} \sum_{1 \le i,j \le k} D_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + (ab(a\cdot) + a\beta(\cdot)) \cdot \nabla. \tag{3.7}$$

In order to determine the speed of convergence of $\dot{Y}(\cdot)$ to equilibrium we will make use of the main result in Franke (2004), which says the following: Suppose $q_{v,D}(t; x, y)$ is the transition probability density of a diffusion on the torus $\mathcal{T}_1$ having a constant nonsingular diffusion matrix D and a divergence-free drift velocity $v(\cdot)$. Then

$$\inf_{v(\cdot)} \Big\{ \min[q_{v,D}(1; x, y) : \; x, y \in \mathcal{T}_1] \Big\} \equiv \delta > 0. \tag{3.8}$$

Thus the transition density $q_a(t; x, y)$, say, of $\dot{Y}(\cdot)$ satisfies *Doeblin's condition* with the same lower bound $\delta > 0$ (at $t = 1$) and, therefore, one has (see, e.g., [15, pp. 214, 215])

$$\sup_{a \in \mathbb{Z}\backslash\{0\},\; x \in \mathcal{T}_1} \int_{\mathcal{T}_1} |q_a(t; x, y) - 1| dx \le c' e^{-\delta t}, \qquad t \ge 0, \tag{3.9}$$

for some positive constants c', δ. For the transition density p_a of the diffusion $\dot{X}(\cdot)$ on the *big torus* $\mathcal{T}_a$, (3.9) implies

$$\sup_{x \in \mathcal{T}_a} \int_{\mathcal{T}_a} \left| p_a(t; x, y) - \frac{1}{a^k} \right| dy \le c' e^{-\delta t / a^2}, \tag{3.10}$$