Harald Held

Shape Optimization under Uncertainty from a Stochastic Programming Point of View

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Stochastic Programming

Editor:

Prof. Dr. Rüdiger Schultz

Uncertainty is a prevailing issue in a growing number of optimization problems in science, engineering, and economics. Stochastic programming offers a flexible methodology for mathematical optimization problems involving uncertain parameters for which probabilistic information is available. This covers model formulation, model analysis, numerical solution methods, and practical implementations. The series ”Stochastic Programming“ presents original research from this range of topics.

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With a foreword by Prof. Dr. Rüdiger Schultz

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Foreword

Optimization problems whose constraints involve partial differential equations (PDEs) are relevant in many areas of technical, industrial, and economic applications. At the same time, they pose challenging mathematical research problems in numerical analysis and optimization.

The present text is among the first in the research literature addressing stochastic uncertainty in the context of PDE constrained optimization. The focus is on shape optimization for elastic bodies under stochastic loading. Analogies to finite dimensional two-stage stochastic programming drive the treatment, with shapes taking the role of nonanticipative decisions. The main results concern level set-based stochastic shape optimization with gradient methods involving shape and topological derivatives. The special structure of the elasticity PDE enables the numerical solution of stochastic shape optimization problems with an arbitrary number of scenarios without increasing the computational effort significantly. Both risk neutral and risk averse models are investigated.

This monograph is based on a doctoral dissertation prepared during 2004-2008 at the Chair of Discrete Mathematics and Optimization in the Department of Mathematics of the University of Duisburg-Essen. The work was supported by the Deutsche Forschungsgemeinschaft (DFG) within the Priority Program "Optimization with Partial Differential Equations".

Rüdiger Schultz

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Harald Held

Abstract

We consider an elastic body subjected to internal and external forces which are uncertain. Simply averaging the possible loadings will result in a structure that might not be robust for the individual loadings at all. Instead, we apply techniques from level set-based shape optimization and two-stage stochastic programming: In the first stage, the non-anticipative decision on the shape has to be taken. Afterwards, the realizations of the random forces are observed, and the variational formulation of the elasticity system takes the role of the second-stage problem. Taking advantage of the PDE's linearity, we are able to compute solutions for an arbitrary number of scenarios without increasing the computational effort significantly. The deformations are described by PDEs that are solved efficiently by Composite Finite Elements. The objective is, e.g., to minimize the compliance. A gradient method using the shape derivative is used to solve the problem. Results for 2D instances are shown. The obtained solutions strongly depend on the initial guess, in particular its topology. To overcome this issue, we included the topological derivative into our algorithm as well.

The stochastic programming perspective also allows us to incorporate risk measures into our model which might be a more appropriate objective in many practical applications.

Parts of this work have been published in [CHP$^+$09].

Contents

Symbol Index

$\mathscr{O}$	The elastic body
Γ_0	Part of the boundary that is to be optimized
Γ_D	The fixed Dirichlet boundary
Γ_N	Neumann boundary where the surface loads act on
λ, μ	Lamé coefficients
ϕ	Level set function
π	Vector of probabilities
$\mathbb{R}^n$	n-dimensional Euclidean space
ω	A scenario
$\mathbf{J}(\mathscr{O}) = J(\mathscr{O}, u(\mathscr{O}))$	Shape objective functional
A	Elasticity tensor
$e(u)$	Linearized strain tensor
$f_{,i}$	i^{th} partial derivative of a scalar function f, see A.4 on page 122
V	Function space $H^1_{\Gamma_D}(\mathscr{O}; \mathbb{R}^2)$
D	Working domain that contains all admissible shapes

A more detailed overview of the notations we used can be found in the Appendix A on page 121.

1 Introduction

Shape optimization problems arise in various practical applications. As stated in [DJPZ01], the object that is to be optimized is the *geometry* as a variable. Shape optimization is closely related to topology optimization, where not only the shape and sizing of a structure has to be found, but also the topology, i.e. the location and shape of holes (see e.g. [BS03]).

In this work, we consider an elastic body represented by an open bounded domain[1] $\mathscr{O} \subset \mathbb{R}^2$. This elastic body is subjected to volume forces and surface loads which are unknown in advance and vary stochastically over time. The objective is to find a shape that minimizes a certain functional under the given loading conditions. Of course, since the acting forces are uncertain and therefore not known in advance, one has to decide on the shape before one can observe the actual forces. This resembles the ideas and structure of linear two-stage stochastic programming problems. This work works out this analogy in the case of shape optimization for linear elastic material laws and stochastic volume and surface loadings.

The motivation behind the stochastic approach becomes evident when looking at the following particular situation, which is also described in [CC03]: Suppose, our task is to find a design for some elastic mechanical device that is as stiff as possible. The stiffness that is to be maximized in this context is an elastic energy as the result of applying forces acting on the design. Under the assumption that the loading is fixed and known, the optimization process yields a structure which resists that one particular given force as good as possible. It is not difficult to imagine situations where the optimal design is unstable with respect to variations of the forces. See for example instance Fig. 5.2 on page 103 in Chapter 5. There we have a square supported on its bottom edge and a homogeneous vertical surface load is acting on its upper edge. The resulting optimal structure consists of vertical pillars (see Fig. 5.2 (left)), which is clearly not optimal any more for any other but the given vertical loading. Note that the instability is not a malfunction in the optimization procedure but the model itself. One can only hope to find more stable and robust solutions if the model somehow incorporates uncertain loadings.

[1] Note that all results described here also hold for $\mathscr{O} \subset \mathbb{R}^3$. However, the computational results are all obtained for the 2-dimensional case, so for the ease of presentation we restrict ourselves to $\mathbb{R}^2$.

One way to achieve this is the stochastic programming approach to this kind of problem, which is the main contribution of this work.

Another possibility to avoid the vulnerability of the optimal designs with respect to variations of loadings, is the *robust optimization* approach. For details about robust optimization we refer to Ben-Tal et al. [BTN02] and references therein, here we only state the basic idea. Robust optimization aims to solve optimization problems in which some data are uncertain and is only known to belong to some uncertainty set $\mathscr{U}$. The following general (finite dimensional) optimization problem is considered in [BTN02]:

$$\min_{x_0\in\mathbb{R},x\in\mathbb{R}^n} \{x_0 : f_0(x,\zeta) - x_0 \leq 0, \quad f_i(x,\zeta) \leq 0, \quad i = 1,\dots,m\} \tag{1.1}$$

with the design vector x, the objective function f_0, constraints $f_1,\dots,f_m$, and uncertain data $\zeta \in \mathscr{U}$. Then, one associates with the uncertain problem (1.1) its so-called *robust counterpart* which is the (semi-infinite) optimization problem

$$\min_{x_0,x} \{x_0 : f_0(x,\zeta) \leq x_0, \quad f_i(x,\zeta) \leq 0, \quad i = 1,\dots,m \quad \forall\zeta \in \mathscr{U}\}. \tag{1.2}$$

Note that in particular any feasible x and x_0 in (1.2) have to satisfy the constraint $f_0(x,\zeta) \leq x_0, \forall\zeta \in \mathscr{U}$, which can be stated equivalently as $\max_{\zeta\in\mathscr{U}} f_0(x,\zeta) \leq x_0$. The right-hand side x_0 is the objective function in (1.2) which is to be minimized. Consequently, for an optimal design vector $\bar{x}$ we have $x_0 = \max_{\zeta\in\mathscr{U}} f_0(\bar{x},\zeta)$. In this sense, the robust counterpart (1.2) overcomes the issue of instability due to uncertain data by minimizing the worst possible case in the given range of data.

The idea of robust optimization has been applied to practical shape and topology optimization applications, such as airfoil shape optimization for example, where the forces are not always known in advance and may vary intensely. This is carried out for example in [Huy01]. Other applications and model formulations for robust shape optimization problems can be found e.g. in [CC99, CC03, dGAJ06]. To our knowledge, the ideas of stochastic (two-stage) programming, which also take the distribution of the random data into account, have not been applied to shape optimization problems under uncertainty yet.

In Section 1.2 we give an introduction to deterministic shape optimization problems. Section 1.1 deals with the formulation and properties of the underlying elasticity PDE[2]. The introduction closes with the ideas and important concepts of two-stage stochastic programming in Section 1.3.

Chapter 2 describes in detail the finite element method we used — the so-called Composite Finite Elements — to solve the elasticity PDE, including some implementational details.

[2] Partial Differential Equation

In Chapter 3 we show how some ideas from finite dimensional two-stage stochastic programming can be applied to the infinite dimensional setting of our stochastic shape optimization problems. It turns out that for this purpose duality plays an important role for an efficient way to compute solutions. This is worked out in Section 3.1. A reformulation of the stochastic shape optimization problem which suggests an immediate way to evaluate the objective function is obtained in Section 3.2. Based on this formulation of the problem, risk averse objective functionals are quite easy to be included, which can be found in Sections 3.3 and 3.4.

Of course, after having formulated appropriate stochastic shape optimization problems, one is also interested in solving them numerically. Along with this work, we developed a program which does that for the 2-dimensional case. The algorithm we implemented is essentially a steepest descent algorithm combined with a level set method. We mainly follow [AJT04] in that respect. In Section 4.1 we describe how we represent domains via level set functions, and what properties and advantages level set methods have. As mentioned before, we want to apply a steepest descent algorithm, so we need to know how to evaluate the objective function, and how to compute a descent direction. The former becomes clear in Chapter 3, and the latter is dealt with in Chapter 4. In particular, in Section 4.2 the notion of *shape derivative* is introduced which is essential for computing a descent direction.

One drawback of a steepest descent algorithm for our problem is that it requires an initial guess. In other words, one has to decide on a certain topology[3]. It turns out that this has a great influence on the outcome of the optimization algorithm (see e.g. [AJT04, AdGJT05, BS03]). The notion of convexity does not apply for functionals depending on domains. Hence there is no guarantee that a steepest descent algorithm finds an optimal solution. In general, one can only say that it terminates in a critical point (cf. for example [BGLS03, NW99, Rus06]). Moreover, the used level set method is in general not able to create new holes (see [AJT04]) but might be able to join several holes together. One attempt to overcome those problems is to embed the *topological derivative* as e.g. in [AdGJT05, BHR04]. More on the topological derivative and topology optimization in general can be found for instance in [AdGJT05, BS03, BHR04, BO05, GGM01, HL07, SZ99, SZ01] and references therein. We also included the topological derivative in our implementation which is described in Section 4.3. Finally, the complete algorithm is summarized and presented in Section 4.4.

Numerical results for the 2-dimensional case are presented in Chapter 5. For convenience, we summarized all the notations we used in the Appendix A.1.

[3] Here we mean the number of holes and their size and location.

1.1 The Elasticity PDE

As mentioned before, we seek to optimize the shape of an elastic body $\mathscr{O} \subseteq \mathbb{R}^2$ subjected to internal and external forces. Here we only want to give a brief introduction to elasticity and the PDE which serves as the state equation for the shape optimization problems that are considered in this work. More on elasticity theory can be found in [Cia88] and [Bra03]. The latter also addresses computational aspects using finite element methods.

Due to the forces acting on the body $\mathscr{O}$, the body is deformed and a point $x \in \mathscr{O}$ becomes the point x' of the deformed body as illustrated in Figure 1.1. Then we can express x' as $x' = x + u(x)$, where $u\colon \mathbb{R}^2 \to \mathbb{R}^2$ denotes the vector of displacement and is assumed to be sufficiently smooth. Those displacements are often assumed to be small and thus higher order terms in u are neglected. This leads to the theory of linearized elasticity which we consider in this work for isotropic elastic materials. One of the most important notions in elasticity theory is the *strain tensor*

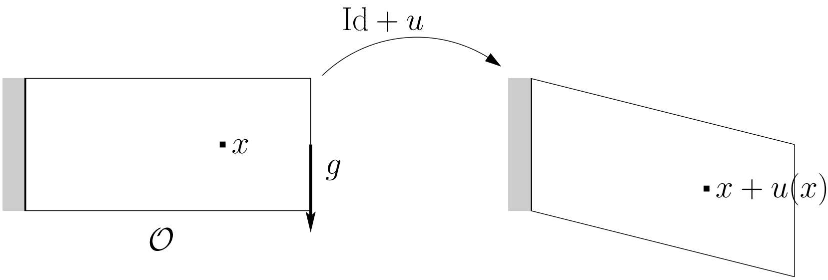

Fig. 1.1: Sketch of an elastic body $\mathscr{O}$ which is fixed on its left edge. Due to the surface load g the body deforms, and a point $x \in \mathscr{O}$ becomes $x + u(x)$.

which reads in the linearized theory as[4]

$$e_{ij}(u) := \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{1.3}$$

The 2×2 matrix $e(u) = (e_{ij}(u))$ is obviously symmetric, and the mapping $u \mapsto e(u)$ linear.

We distinguish between *volume forces* f and *surface loads* g. A typical example for a volume force is gravity, whereas an imposed load on a bridge would be a surface load. The resulting deformation due to those forces obviously depends on

[4] For the notation we used here for derivatives, see A.1, in particular A.4 (ii)