Frontiers in Mathematics

Jens Flemming

Variational Source Conditions, Quadratic Inverse Problems, Sparsity Promoting Regularization

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Variational Source Conditions, Quadratic Inverse Problems, Sparsity Promoting Regularization

New Results in Modern Theory of Inverse Problems and an Application in Laser Optics

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ISSN 1660-8046 ISSN 1660-8054 (electronic)
Frontiers in Mathematics
ISBN 978-3-319-95263-5 ISBN 978-3-319-95264-2 (eBook)
https://doi.org/10.1007/978-3-319-95264-2

Library of Congress Control Number: 2018950663

Mathematics Subject Classification (2010): 65J22, 65J20, 47A52, 47J06

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The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

This book is dedicated to Bernd Hofmann *on the occasion of his retirement.*

Preface

This book grew out of the author's habilitation thesis, which has been completed in January 2018. Parts II and III cover and slightly extend the material of the thesis. Part I, on the one hand, provides an introduction to the other parts and, on the other hand, contains new results on variational source conditions in the context of convergence rates theory for ill-posed inverse problems.

The intention of writing this book was to demonstrate new and to some extent nonorthodox ideas for handling ill-posed inverse problems. This book is not a comprehensive introduction to inverse problems. Instead, it focuses on few research topics and handles them in depth.

The three topics of the book, variational source conditions, quadratic inverse problems, and ℓ^1-regularization, seem to be quite different. The first one is of great generality and establishes the basis for several more concrete results in the book. The second one is concerned with nonlinear mappings in a classical Hilbert space setting, whereas the third deals with linear mappings in non-reflexive Banach spaces.

At the second sight, quadratic inverse problems and linear inverse problems with sparsity context have similar structures and their handling shows several parallels. Nevertheless, I decided to divide the book into three more or less independent parts and to give hints on cross connections from time to time. The advantage of this decision is that the reader may study the three parts in arbitrary order.

Finishing this book would not have been possible without constant support and advice by Prof. Bernd Hofmann (TU Chemnitz). I thank him a lot for his efforts in several regards during all the years I have been working in his research group. I also want to thank my colleagues and coauthors, especially Steven Bürger and Daniel Gerth, for interesting and fruitful discussions. Last but not least I have to express my thanks to the Faculty of Mathematics at TU Chemnitz as a whole for the cordial and cooperative working atmosphere.

Chemnitz, Germany — Jens Flemming
May 2018

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Part I

Variational Source Conditions

Inverse Problems, Ill-Posedness, Regularization

1

Abstract
We introduce the mathematical setting as well as basic notation used throughout the book. Different notions of ill-posedness in the context of inverse problems are discussed and the need for regularization leads us to Tikhonov-type methods and their behavior in Banach spaces.

1.1 Setting

Let X and Y be Banach spaces over $\mathbb{R}$ or $\mathbb{C}$ and let $F : X \supseteq \mathscr{D}(F) \to Y$ be a mapping between them with domain $\mathscr{D}(F)$. We aim to solve equations

$$F(x) = y^\dagger, \quad x \in \mathscr{D}(F), \tag{1.1}$$

with *exact and attainable* data $y^\dagger$ in Y. Solving such equations requires, in some sense, inversion of F. Hence the term *inverse problem*.

The mathematical field of inverse problems is not concerned with Eq. (1.1) in general but only with equations that are *ill-posed*. Loosely speaking, an equation is ill-posed if the inversion process is very sensitive to perturbations in the right-hand side $y^\dagger$. Such perturbations cannot be avoided in practice because $y^\dagger$ represents some measured quantity and measurements always are corrupted by noise. We provide and discuss different precise definitions of ill-posedness in the next section.

J. Flemming, *Variational Source Conditions, Quadratic Inverse Problems, Sparsity Promoting Regularization*, Frontiers in Mathematics,
https://doi.org/10.1007/978-3-319-95264-2_1

To analyze and overcome ill-posedness noise has to be taken into account. In other words, the exact right-hand side $y^\dagger$ is not available for the inversion process. Instead, we only have some noisy measurement y^δ at hand, which is assumed to belong to Y, too, and to satisfy

$$\left\| y^\delta - y^\dagger \right\| \leq \delta \tag{1.2}$$

with nonnegative noise level δ.

For later reference we list the following restrictions on our setting.

Assumption 1.1 We assume that

(i) equation (1.1) has a solution,
(ii) the domain $\mathscr{D}(F)$ is weakly sequentially closed,
(iii) the mapping F is weakly sequentially continuous.

Items (ii) and (iii) are satisfied if and only if for each sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{D}(F)$ and each x in X we have

$$x_n \rightharpoonup x \qquad \Rightarrow \qquad x \in \mathscr{D}(F), \quad F(x_n) \rightharpoonup F(x).$$

1.2 Ill-Posedness

1.2.1 Global Definitions by Hadamard and Nashed

The classical definition of ill-posedness was introduced by Hadamard.

Definition 1.2 The mapping F in Eq. (1.1) is *well-posed in the sense of Hadamard* if

(i) for each $y^\dagger$ in Y there exists a solution,
(ii) for each fixed right-hand side $y^\dagger$ there is at most one solution,
(iii) solutions depend continuously on the data.

Else F is *ill-posed in the sense of Hadamard*.

Items (i) and (ii) of the definition require that F is bijective and item (iii) says that the inverse mapping has to be continuous with respect to the norm or some other topology. Due to its restrictive nature Hadamard's definition only plays a minor role in modern theory of inverse problems. Existence of solutions usually is formulated as an assumption, cf. item (i) in Assumption 1.1, and uniqueness is not required because the developed theory will cover the case of multiple solutions.

In [1] Nashed proposed a definition of ill-posedness for bounded linear mappings F between Banach spaces X and Y with domain $\mathscr{D}(F) = X$.

Definition 1.3 Let F in Eq. (1.1) be linear and bounded. Then F is *well-posed in the sense of Nashed* if the range of F is closed in Y and *ill-posed in the sense of Nashed* if the range of F is not closed in Y.

Nashed's definition does not consider existence and uniqueness of solutions, but focusses on continuous (generalized) invertibility. If a generalized inverse exists, then it is continuous if and only if F is well-posed in the sense of Nashed, see [2, Theorem 5.6(b)]. But one should be aware of the fact, that in general Banach spaces generalized inverses are not always available, because the null space of F or the closure of the range may be uncomplemented, see Proposition 1.10 and Sect. 1.2.4 below. An important example for this situation is the setting used for analyzing ℓ^1-regularization in Part III.

If F is injective, then the inverse $F^{-1} : Y \supseteq \mathscr{R}(F) \to X$ is continuous on $\mathscr{R}(F)$ if and only if $\mathscr{R}(F)$ is closed. If X and Y are Hilbert spaces, then the Moore–Penrose inverse is a generalized inverse which always exists. Thus, in Hilbert spaces well-posedness in the sense of Nashed is equivalent to continuity of the Moore–Penrose inverse.

Nashed distinguished two types of ill-posedness in [1]. In Chap. 10 we have a closer look at this distinction in the context of ℓ^1-regularization.

1.2.2 Local Definitions by Hofmann and Ivanov

Hadamard's and Nashed's definitions of ill-posedness are of global nature. For nonlinear mappings F properties may vary from point to point and ill-posedness has to be understood in a local manner. Following the ideas in [3] we have to distinguish between local ill-posedness at a point x in X and local ill-posedness at a point y in Y.

The aim of defining precisely what is meant by ill-posedness is to describe the following situation mathematically: Given a sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathscr{R}(F)$ approximating the unknown exact data $y^\dagger$ in (1.1), a sequence $(x_n)_{n\in\mathbb{N}}$ of corresponding solutions to $F(x) = y_n$, $x \in \mathscr{D}(F)$, does not converge to a solution of (1.1). The difficulties are to choose concrete types of approximation and convergence and to handle the case of multiple solutions.

One possibility for defining ill-posedness locally at a point of the domain $\mathscr{D}(F)$ has been suggested in [4] by Hofmann, see also [5].

Definition 1.4 The mapping F is *locally well-posed in the sense of Hofmann* at a point x_0 in $\mathscr{D}(F)$ if there is some positive ε such that for each sequence $(x_n)_{n\in\mathbb{N}}$ in $B_\varepsilon(x_0)\cap\mathscr{D}(F)$ the implication

$$F(x_n) \to F(x_0) \quad \Rightarrow \quad x_n \to x_0$$

is true. Otherwise, F is *locally ill-posed in the sense of Hofmann* at the point x_0.

Local well-posedness in the sense of Hofmann implies that x_0 has to be an isolated solution to $F(x) = F(x_0)$, $x \in \mathscr{D}(F)$. In this sense, local uniqueness is part of this type of local well-posedness.

Ivanov introduced a similar concept in [6], but locally in Y. Thus, he gets around the question of uniqueness. See also [3, Definition 1].

Definition 1.5 The mapping F is *locally well-posed in the sense of Ivanov* at a point y_0 in $\mathscr{R}(F)$ if for each sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathscr{R}(F)$ the implication

$$y_n \to y_0 \quad \Rightarrow \quad \sup_{\tilde{x}\in F^{-1}(y_n)} \; \inf_{x\in F^{-1}(y_0)} \|\tilde{x}-x\| \to 0$$

is true. Otherwise, F is *locally ill-posed in the sense of Ivanov* at the point y_0.

The set-to-set distance

$$\sup_{\tilde{x}\in\tilde{M}} \inf_{x\in M} \|\tilde{x}-x\|$$

between two subsets $\tilde{M}$ and M of X used in the Definition 1.5 is not symmetric. It expresses the maximum distance of elements in $\tilde{M}$ to the set M. Since we cannot control which of possibly many approximate solutions is chosen by an inversion method, this type of distance is the right choice.

The only drawback of Definition 1.5 is that norm convergence cannot be replaced easily by other types of convergence to define ill-posedness with respect to the weak topology, for example. The following proposition provides an equivalent reformulation which avoids explicit use of norms. The proposition was already mentioned briefly in [3, Remark 1].

Proposition 1.6 *The mapping F is well-posed in the sense of Ivanov at a point y_0 in $\mathscr{R}(F)$ if and only if for each sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathscr{R}(F)$ converging to y_0 and for each sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ of preimages $\tilde{x}_n$ from $F^{-1}(y_n)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $F^{-1}(y_0)$ with $\tilde{x}_n - x_n \to 0$.*

Proof Let F be well-posed in the sense of Ivanov at the point y_0 and let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{R}(F)$ converging to y_0. Given a sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ with $\tilde{x}_n \in F^{-1}(y_n)$ we immediately see

$$\inf_{x\in F^{-1}(y_0)} \|\tilde{x}_n - x\| \to 0.$$

Fixing ε, for each n we find x_n in $F^{-1}(y_0)$ with

$$\|\tilde{x}_n - x_n\| \leq \inf_{x\in F^{-1}(y_0)} \|\tilde{x}_n - x\| + \varepsilon.$$

Thus, we obtain $\|\tilde{x}_n - x_n\| \leq 2\,\varepsilon$ for all sufficiently large n, which implies convergence $\tilde{x}_n - x_n \to 0$.

Now let y_0 be in $\mathscr{R}(F)$ and let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{R}(F)$ converging to y_0. Further, assume that for each sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ of preimages $\tilde{x}_n$ from $F^{-1}(y_n)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $F^{-1}(y_0)$ with $\tilde{x}_n - x_n \to 0$. If there would be some positive fixed ε with

$$\sup_{\tilde{x}\in F^{-1}(y_n)} \inf_{x\in F^{-1}(y_0)} \|\tilde{x} - x\| > \varepsilon,$$

we would find a sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ with

$$\inf_{x\in F^{-1}(y_0)} \|\tilde{x}_n - x\| > \varepsilon$$

for all n. Thus, there would be a sequence $(x_n)_{n\in\mathbb{N}}$ with

$$\varepsilon < \inf_{x\in F^{-1}(y_0)} \|\tilde{x}_n - x\| \leq \|\tilde{x}_n - x_n\| \to 0,$$

which contradicts $\varepsilon > 0$. This shows

$$\sup_{\tilde{x}\in F^{-1}(y_n)} \inf_{x\in F^{-1}(y_0)} \|\tilde{x} - x\| \to 0.$$

□

Remark 1.7 From Proposition 1.6 we easily see that the following condition is sufficient for local well-posedness in the sense of Ivanov at y_0: Each sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{D}(F)$ with $F(x_n) \to y_0$ contains a convergent subsequence and the limits of all convergent subsequences are solutions corresponding to the right-hand side y_0.

Throughout this book ill-posedness is to be understood in the sense of Ivanov if not otherwise stated.

1.2.3 Interrelations

The definitions of Hofmann and Ivanov are closely connected, but differ in two aspects. On the one hand, Hofmann's definition works in X and Ivanov's definition works in Y. On the other hand, and also as a consequence of the first difference, in Hofmann's definition well-posedness is restricted to isolated solutions whereas Ivanov's definition works for arbitrary solution sets.

Both views have their advantages. Hofmann's definition allows for a deeper analysis of ill-posedness phenomena. Due to its locality in X at each element of a set of isolated solutions we can distinguish between well-posedness and ill-posedness. That is, for one fixed data element at the same time there might exist solutions at which the mapping is well-posed and solutions at which the mapping is ill-posed in the sense of Hofmann. Analyzing an inverse problem with Hofmann's definition allows to identify regions of well-posedness and regions of ill-posedness. Thus, restricting the domain of the mapping F with the help of Hofmann's definition could make the inverse problem well-posed.

Ivanov's definition does not allow for such a detailed analysis. But its advantage is that it is closer to the issue of numerical instability. Given a data element, we want to know whether a sequence of approximate solutions based on noisy data becomes arbitrarily close to the set of exact solutions if the noise is reduced until it vanishes. This is exactly what Ivanov's definition expresses.

The interrelations between Hofmann's definition and Ivanov's definition are made precise by the following two propositions. The first proposition is a slightly extended version of [3, Proposition 2] and the second stems from oral communication with Bernd Hofmann (Chemnitz).

Proposition 1.8 *If the mapping F is locally well-posed in the sense of Ivanov at some point y_0 in $\mathscr{R}(F)$, then F is locally well-posed in the sense of Hofmann at each isolated solution corresponding to the data y_0.*

Proof Let F be locally well-posed in the sense of Ivanov at y_0 and let x_0 be an isolated solution to data y_0. Take a positive radius ε such that x_0 is the only solution to data y_0 in $B_{2\varepsilon}(x_0)$. For each sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ in $B_\varepsilon(x_0) \cap \mathscr{D}(F)$ and for the corresponding sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n := F(\tilde{x}_n)$ Proposition 1.6 yields a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{D}(F)$ with $F(x_n) = y_0$ and $\tilde{x}_n - x_n \to 0$. Since $(\tilde{x}_n)_{n\in\mathbb{N}}$ lies in $B_\varepsilon(x_0)$ and x_0 is the only solution in $B_{2\varepsilon}(x_0)$, we obtain $x_n = x_0$ for all n. Consequently, $\tilde{x}_n \to x_0$, which proves local well-posedness in the sense of Hofmann at x_0. □

Proposition 1.9 *There exist mappings F and points x_0 in $\mathscr{D}(F)$ such that F is locally well-posed in the sense of Hofmann at x_0 but locally ill-posed in the sense of Ivanov at $F(x_0)$.*

Proof Choose $X := \mathbb{R}$, $Y := \mathbb{R}$ and $F(x) := \frac{x^2}{1+x^4}$ with $\mathscr{D}(F) = X$. Then $x_0 := 0$ is the only solution to $F(x) = 0$, $x \in X$, and continuous invertibility of F near zero immediately implies local well-posedness in the sense of Hofmann.

On the other hand, we may consider a sequence $(y_n)_{n\in\mathbb{N}}$ with elements $y_n := F(x_n)$ such that $x_n \to \infty$. Then $y_n \to 0$, but

$$\sup_{\tilde{x}\in F^{-1}(y_n)} \inf_{x\in F^{-1}(0)} \|\tilde{x} - x\| \geq \inf_{x\in F^{-1}(0)} \|x_n - x\| = \|x_n\| \not\to 0.$$

Thus, F is locally ill-posed in the sense of Ivanov at $F(0)$. □

Finally, we state the interrelation between Nashed's definition and Ivanov's definition. The special case of Hilbert spaces, where each closed subspace is complemented, can be found in [3, Proposition 1].

Proposition 1.10 *Let F be a bounded linear operator with domain $\mathscr{D}(F) = X$ between the Banach spaces X and Y and let the null space $\mathscr{N}(F)$ be complemented in X. Then F is well-posed in the sense of Nashed if and only if F is locally well-posed in the sense of Ivanov at every point of $\mathscr{R}(F)$ and F is ill-posed in the sense of Nashed if and only if F is locally ill-posed in the sense of Ivanov at every point of $\mathscr{R}(F)$.*

Proof Let $\mathscr{N}(F)$ be complemented by U, that is, U is a closed linear subspace of X and $X = \mathscr{N}(F) \oplus U$. One easily shows, that the restriction $F|_U$ of F to U is bijective between U and $\mathscr{R}(F)$. Thus, the inverse $(F|_U)^{-1}$ is a well-defined linear operator, which due to $\mathscr{R}(F) = \mathscr{R}(F|_U)$ is bounded if and only if $\mathscr{R}(F)$ is closed. We see that F is well-posed in the sense of Nashed if and only if $(F|_U)^{-1}$ is bounded.

Let $(F|_U)^{-1}$ be bounded. To show local well-posedness in the sense of Ivanov at an arbitrary point y_0 in $\mathscr{R}(F)$ we choose sequences $(y_n)_{n\in\mathbb{N}}$ with $y_n \to y$ and $(\tilde{x}_n)_{n\in\mathbb{N}}$ in X with $F(\tilde{x}_n) = y_n$. By Proposition 1.6 we have to show that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $F(x_n) = y_0$ and $\tilde{x}_n - x_n \to 0$. Such a choice is given by

$$x_n := \tilde{x}_n - (F|_U)^{-1}\big(F(\tilde{x}_n)\big) + (F|_U)^{-1}(y_0),$$

because

$$F(x_n) = F(\tilde{x}_n) - F(\tilde{x}_n) + y_0 = y_0$$