

Jens Flemming

Variational **Source**
Conditions,
Quadratic
Inverse Problems,
Sparsity Promoting
Regularization

Frontiers in Mathematics

Advisory Editorial Board

Leonid Bunimovich (Georgia Institute of Technology, Atlanta)

William Y. C. Chen (Nankai University, Tianjin, China)

Benoît Perthame (Université Pierre et Marie Curie, Paris)

Laurent Saloff-Coste (Cornell University, Ithaca)

Igor Shparlinski (Macquarie University, New South Wales)

Wolfgang Spröβig (TU Bergakademie Freiberg)

Cédric Villani (Institut Henri Poincaré, Paris)

More information about this series at <http://www.springer.com/series/5388>

Jens Flemming

Variational Source Conditions, Quadratic Inverse Problems, Sparsity Promoting Regularization

New Results in Modern Theory of Inverse
Problems and an Application in Laser Optics

Jens Flemming
Fakultät für Mathematik
Technische Universität Chemnitz
Chemnitz, Germany

ISSN 1660-8046 ISSN 1660-8054 (electronic)
Frontiers in Mathematics
ISBN 978-3-319-95263-5 ISBN 978-3-319-95264-2 (eBook)
<https://doi.org/10.1007/978-3-319-95264-2>

Library of Congress Control Number: 2018950663

Mathematics Subject Classification (2010): 65J22, 65J20, 47A52, 47J06

© Springer Nature Switzerland AG 2018

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors, and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This book is published under the imprint Birkhäuser, www.birkhauser-science.com by the registered company Springer Nature Switzerland AG part of Springer Nature.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

This book is dedicated to Bernd Hofmann on the occasion of his retirement.

Preface

This book grew out of the author's habilitation thesis, which has been completed in January 2018. Parts **II** and **III** cover and slightly extend the material of the thesis. Part **I**, on the one hand, provides an introduction to the other parts and, on the other hand, contains new results on variational source conditions in the context of convergence rates theory for ill-posed inverse problems.

The intention of writing this book was to demonstrate new and to some extent nonorthodox ideas for handling ill-posed inverse problems. This book is not a comprehensive introduction to inverse problems. Instead, it focuses on few research topics and handles them in depth.

The three topics of the book, variational source conditions, quadratic inverse problems, and ℓ^1 -regularization, seem to be quite different. The first one is of great generality and establishes the basis for several more concrete results in the book. The second one is concerned with nonlinear mappings in a classical Hilbert space setting, whereas the third deals with linear mappings in non-reflexive Banach spaces.

At the second sight, quadratic inverse problems and linear inverse problems with sparsity context have similar structures and their handling shows several parallels. Nevertheless, I decided to divide the book into three more or less independent parts and to give hints on cross connections from time to time. The advantage of this decision is that the reader may study the three parts in arbitrary order.

Finishing this book would not have been possible without constant support and advice by Prof. Bernd Hofmann (TU Chemnitz). I thank him a lot for his efforts in several regards during all the years I have been working in his research group. I also want to thank my colleagues and coauthors, especially Steven Bürger and Daniel Gerth, for interesting and fruitful discussions. Last but not least I have to express my thanks to the Faculty of Mathematics at TU Chemnitz as a whole for the cordial and cooperative working atmosphere.

Chemnitz, Germany
May 2018

Jens Flemming

Contents

Part I Variational Source Conditions

1	Inverse Problems, Ill-Posedness, Regularization	3
1.1	Setting	3
1.2	Ill-Posedness	4
1.2.1	Global Definitions by Hadamard and Nashed	4
1.2.2	Local Definitions by Hofmann and Ivanov	5
1.2.3	Interrelations	8
1.2.4	Nashed's Definition in Case of Uncomplemented Null Spaces	10
1.3	Tikhonov Regularization	12
2	Variational Source Conditions Yield Convergence Rates	15
2.1	Evolution of Variational Source Conditions	15
2.2	Convergence Rates	17
3	Existence of Variational Source Conditions	21
3.1	Main Theorem	21
3.2	Special Cases	25
3.2.1	Linear Equations in Hilbert Spaces	25
3.2.2	Bregman Distance in Banach Spaces	26
3.2.3	Vanishing Error Functional	28

Part II Quadratic Inverse Problems

4	What Are Quadratic Inverse Problems?	31
4.1	Definition and Basic Properties	31
4.2	Examples	35
4.2.1	Autoconvolutions	35
4.2.2	Kernel-Based Autoconvolution in Laser Optics	40
4.2.3	Schlieren Tomography	45
4.3	Local Versus Global Ill-Posedness	48

4.4	Geometric Properties of Quadratic Mappings' Ranges	49
4.5	Literature on Quadratic Mappings	52
5	Tikhonov Regularization	55
6	Regularization by Decomposition	59
6.1	Quadratic Isometries	59
6.2	Decomposition of Quadratic Mappings	63
6.3	Inversion of Quadratic Isometries	67
6.4	A Regularization Method	75
6.5	Numerical Example	78
7	Variational Source Conditions	97
7.1	About Variational Source Conditions	97
7.2	Nonlinearity Conditions	98
7.3	Classical Source Conditions	99
7.4	Variational Source Conditions Are the Right Tool	100
7.5	Sparsity Yields Variational Source Conditions	101
Part III Sparsity Promoting Regularization		
8	Aren't All Questions Answered?	109
9	Sparsity and ℓ^1-Regularization	111
9.1	Sparse Signals	111
9.2	ℓ^1 -Regularization	113
9.3	Other Sparsity Promoting Regularization Methods	118
9.4	Examples	119
9.4.1	Denoising	119
9.4.2	Bidiagonal Operator	119
9.4.3	Simple Integration and Haar Wavelets	120
9.4.4	Simple Integration and Fourier Basis	121
10	Ill-Posedness in the ℓ^1-Setting	125
11	Convergence Rates	129
11.1	Results in the Literature	129
11.2	Classical Techniques Do Not Work	131
11.3	Smooth Bases	132
11.4	Non-smooth Bases	138
11.5	Convergence Rates Without Source-Type Assumptions	145
11.6	Convergence Rates Without Injectivity-Type Assumptions	148
11.6.1	Distance to Norm Minimizing Solutions	148
11.6.2	Sparse Solutions	152
11.6.3	Sparse Unique Norm Minimizing Solution	155

11.6.4	Non-sparse Solutions	158
11.6.5	Examples	160
A	Topology, Functional Analysis, Convex Analysis	165
A.1	Topological Spaces and Nets	165
A.2	Reflexivity, Weak and Weak* Topologies	166
A.3	Subdifferentials and Bregman Distances	168
B	Verification of Assumption 11.13 for Example 11.18	171
	References	177
	Index	181

Part I

Variational Source Conditions



Abstract

We introduce the mathematical setting as well as basic notation used throughout the book. Different notions of ill-posedness in the context of inverse problems are discussed and the need for regularization leads us to Tikhonov-type methods and their behavior in Banach spaces.

1.1 Setting

Let X and Y be Banach spaces over \mathbb{R} or \mathbb{C} and let $F : X \supseteq \mathcal{D}(F) \rightarrow Y$ be a mapping between them with domain $\mathcal{D}(F)$. We aim to solve equations

$$F(x) = y^\dagger, \quad x \in \mathcal{D}(F), \quad (1.1)$$

with *exact and attainable* data y^\dagger in Y . Solving such equations requires, in some sense, inversion of F . Hence the term *inverse problem*.

The mathematical field of inverse problems is not concerned with Eq. (1.1) in general but only with equations that are *ill-posed*. Loosely speaking, an equation is ill-posed if the inversion process is very sensitive to perturbations in the right-hand side y^\dagger . Such perturbations cannot be avoided in practice because y^\dagger represents some measured quantity and measurements always are corrupted by noise. We provide and discuss different precise definitions of ill-posedness in the next section.

To analyze and overcome ill-posedness noise has to be taken into account. In other words, the exact right-hand side y^\dagger is not available for the inversion process. Instead, we only have some noisy measurement y^δ at hand, which is assumed to belong to Y , too, and to satisfy

$$\|y^\delta - y^\dagger\| \leq \delta \quad (1.2)$$

with nonnegative noise level δ .

For later reference we list the following restrictions on our setting.

Assumption 1.1 We assume that

- (i) equation (1.1) has a solution,
- (ii) the domain $\mathcal{D}(F)$ is weakly sequentially closed,
- (iii) the mapping F is weakly sequentially continuous.

Items (ii) and (iii) are satisfied if and only if for each sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{D}(F)$ and each x in X we have

$$x_n \rightharpoonup x \quad \Rightarrow \quad x \in \mathcal{D}(F), \quad F(x_n) \rightharpoonup F(x).$$

1.2 Ill-Posedness

1.2.1 Global Definitions by Hadamard and Nashed

The classical definition of ill-posedness was introduced by Hadamard.

Definition 1.2 The mapping F in Eq. (1.1) is *well-posed in the sense of Hadamard* if

- (i) for each y^\dagger in Y there exists a solution,
- (ii) for each fixed right-hand side y^\dagger there is at most one solution,
- (iii) solutions depend continuously on the data.

Else F is *ill-posed in the sense of Hadamard*.

Items (i) and (ii) of the definition require that F is bijective and item (iii) says that the inverse mapping has to be continuous with respect to the norm or some other topology. Due to its restrictive nature Hadamard's definition only plays a minor role in modern theory of

inverse problems. Existence of solutions usually is formulated as an assumption, cf. item (i) in Assumption 1.1, and uniqueness is not required because the developed theory will cover the case of multiple solutions.

In [1] Nashed proposed a definition of ill-posedness for bounded linear mappings F between Banach spaces X and Y with domain $\mathcal{D}(F) = X$.

Definition 1.3 Let F in Eq. (1.1) be linear and bounded. Then F is *well-posed in the sense of Nashed* if the range of F is closed in Y and *ill-posed in the sense of Nashed* if the range of F is not closed in Y .

Nashed's definition does not consider existence and uniqueness of solutions, but focusses on continuous (generalized) invertibility. If a generalized inverse exists, then it is continuous if and only if F is well-posed in the sense of Nashed, see [2, Theorem 5.6(b)]. But one should be aware of the fact, that in general Banach spaces generalized inverses are not always available, because the null space of F or the closure of the range may be uncomplemented, see Proposition 1.10 and Sect. 1.2.4 below. An important example for this situation is the setting used for analyzing ℓ^1 -regularization in Part III.

If F is injective, then the inverse $F^{-1} : Y \supseteq \mathcal{R}(F) \rightarrow X$ is continuous on $\mathcal{R}(F)$ if and only if $\mathcal{R}(F)$ is closed. If X and Y are Hilbert spaces, then the Moore–Penrose inverse is a generalized inverse which always exists. Thus, in Hilbert spaces well-posedness in the sense of Nashed is equivalent to continuity of the Moore–Penrose inverse.

Nashed distinguished two types of ill-posedness in [1]. In Chap. 10 we have a closer look at this distinction in the context of ℓ^1 -regularization.

1.2.2 Local Definitions by Hofmann and Ivanov

Hadamard's and Nashed's definitions of ill-posedness are of global nature. For nonlinear mappings F properties may vary from point to point and ill-posedness has to be understood in a local manner. Following the ideas in [3] we have to distinguish between local ill-posedness at a point x in X and local ill-posedness at a point y in Y .

The aim of defining precisely what is meant by ill-posedness is to describe the following situation mathematically: Given a sequence $(y_n)_{n \in \mathbb{N}}$ in $\mathcal{R}(F)$ approximating the unknown exact data y^\dagger in (1.1), a sequence $(x_n)_{n \in \mathbb{N}}$ of corresponding solutions to $F(x) = y_n$, $x \in \mathcal{D}(F)$, does not converge to a solution of (1.1). The difficulties are to choose concrete types of approximation and convergence and to handle the case of multiple solutions.

One possibility for defining ill-posedness locally at a point of the domain $\mathcal{D}(F)$ has been suggested in [4] by Hofmann, see also [5].

Definition 1.4 The mapping F is *locally well-posed in the sense of Hofmann* at a point x_0 in $\mathcal{D}(F)$ if there is some positive ε such that for each sequence $(x_n)_{n \in \mathbb{N}}$ in $B_\varepsilon(x_0) \cap \mathcal{D}(F)$ the implication

$$F(x_n) \rightarrow F(x_0) \quad \Rightarrow \quad x_n \rightarrow x_0$$

is true. Otherwise, F is *locally ill-posed in the sense of Hofmann* at the point x_0 .

Local well-posedness in the sense of Hofmann implies that x_0 has to be an isolated solution to $F(x) = F(x_0)$, $x \in \mathcal{D}(F)$. In this sense, local uniqueness is part of this type of local well-posedness.

Ivanov introduced a similar concept in [6], but locally in Y . Thus, he gets around the question of uniqueness. See also [3, Definition 1].

Definition 1.5 The mapping F is *locally well-posed in the sense of Ivanov* at a point y_0 in $\mathcal{R}(F)$ if for each sequence $(y_n)_{n \in \mathbb{N}}$ in $\mathcal{R}(F)$ the implication

$$y_n \rightarrow y_0 \quad \Rightarrow \quad \sup_{\tilde{x} \in F^{-1}(y_n)} \inf_{x \in F^{-1}(y_0)} \|\tilde{x} - x\| \rightarrow 0$$

is true. Otherwise, F is *locally ill-posed in the sense of Ivanov* at the point y_0 .

The set-to-set distance

$$\sup_{\tilde{x} \in \tilde{M}} \inf_{x \in M} \|\tilde{x} - x\|$$

between two subsets \tilde{M} and M of X used in the Definition 1.5 is not symmetric. It expresses the maximum distance of elements in \tilde{M} to the set M . Since we cannot control which of possibly many approximate solutions is chosen by an inversion method, this type of distance is the right choice.

The only drawback of Definition 1.5 is that norm convergence cannot be replaced easily by other types of convergence to define ill-posedness with respect to the weak topology, for example. The following proposition provides an equivalent reformulation which avoids explicit use of norms. The proposition was already mentioned briefly in [3, Remark 1].

Proposition 1.6 *The mapping F is well-posed in the sense of Ivanov at a point y_0 in $\mathcal{R}(F)$ if and only if for each sequence $(y_n)_{n \in \mathbb{N}}$ in $\mathcal{R}(F)$ converging to y_0 and for each sequence $(\tilde{x}_n)_{n \in \mathbb{N}}$ of preimages \tilde{x}_n from $F^{-1}(y_n)$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $F^{-1}(y_0)$ with $\tilde{x}_n - x_n \rightarrow 0$.*

Proof Let F be well-posed in the sense of Ivanov at the point y_0 and let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{R}(F)$ converging to y_0 . Given a sequence $(\tilde{x}_n)_{n \in \mathbb{N}}$ with $\tilde{x}_n \in F^{-1}(y_n)$ we immediately see

$$\inf_{x \in F^{-1}(y_0)} \|\tilde{x}_n - x\| \rightarrow 0.$$

Fixing ε , for each n we find x_n in $F^{-1}(y_0)$ with

$$\|\tilde{x}_n - x_n\| \leq \inf_{x \in F^{-1}(y_0)} \|\tilde{x}_n - x\| + \varepsilon.$$

Thus, we obtain $\|\tilde{x}_n - x_n\| \leq 2\varepsilon$ for all sufficiently large n , which implies convergence $\tilde{x}_n - x_n \rightarrow 0$.

Now let y_0 be in $\mathcal{R}(F)$ and let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{R}(F)$ converging to y_0 . Further, assume that for each sequence $(\tilde{x}_n)_{n \in \mathbb{N}}$ of preimages \tilde{x}_n from $F^{-1}(y_n)$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $F^{-1}(y_0)$ with $\tilde{x}_n - x_n \rightarrow 0$. If there would be some positive fixed ε with

$$\sup_{\tilde{x} \in F^{-1}(y_n)} \inf_{x \in F^{-1}(y_0)} \|\tilde{x} - x\| > \varepsilon,$$

we would find a sequence $(\tilde{x}_n)_{n \in \mathbb{N}}$ with

$$\inf_{x \in F^{-1}(y_0)} \|\tilde{x}_n - x\| > \varepsilon$$

for all n . Thus, there would be a sequence $(x_n)_{n \in \mathbb{N}}$ with

$$\varepsilon < \inf_{x \in F^{-1}(y_0)} \|\tilde{x}_n - x\| \leq \|\tilde{x}_n - x_n\| \rightarrow 0,$$

which contradicts $\varepsilon > 0$. This shows

$$\sup_{\tilde{x} \in F^{-1}(y_n)} \inf_{x \in F^{-1}(y_0)} \|\tilde{x} - x\| \rightarrow 0.$$

□

Remark 1.7 From Proposition 1.6 we easily see that the following condition is sufficient for local well-posedness in the sense of Ivanov at y_0 : Each sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{D}(F)$ with $F(x_n) \rightarrow y_0$ contains a convergent subsequence and the limits of all convergent subsequences are solutions corresponding to the right-hand side y_0 .

Throughout this book ill-posedness is to be understood in the sense of Ivanov if not otherwise stated.

1.2.3 Interrelations

The definitions of Hofmann and Ivanov are closely connected, but differ in two aspects. On the one hand, Hofmann's definition works in X and Ivanov's definition works in Y . On the other hand, and also as a consequence of the first difference, in Hofmann's definition well-posedness is restricted to isolated solutions whereas Ivanov's definition works for arbitrary solution sets.

Both views have their advantages. Hofmann's definition allows for a deeper analysis of ill-posedness phenomena. Due to its locality in X at each element of a set of isolated solutions we can distinguish between well-posedness and ill-posedness. That is, for one fixed data element at the same time there might exist solutions at which the mapping is well-posed and solutions at which the mapping is ill-posed in the sense of Hofmann. Analyzing an inverse problem with Hofmann's definition allows to identify regions of well-posedness and regions of ill-posedness. Thus, restricting the domain of the mapping F with the help of Hofmann's definition could make the inverse problem well-posed.

Ivanov's definition does not allow for such a detailed analysis. But its advantage is that it is closer to the issue of numerical instability. Given a data element, we want to know whether a sequence of approximate solutions based on noisy data becomes arbitrarily close to the set of exact solutions if the noise is reduced until it vanishes. This is exactly what Ivanov's definition expresses.

The interrelations between Hofmann's definition and Ivanov's definition are made precise by the following two propositions. The first proposition is a slightly extended version of [3, Proposition 2] and the second stems from oral communication with Bernd Hofmann (Chemnitz).

Proposition 1.8 *If the mapping F is locally well-posed in the sense of Ivanov at some point y_0 in $\mathcal{R}(F)$, then F is locally well-posed in the sense of Hofmann at each isolated solution corresponding to the data y_0 .*

Proof Let F be locally well-posed in the sense of Ivanov at y_0 and let x_0 be an isolated solution to data y_0 . Take a positive radius ε such that x_0 is the only solution to data y_0 in $B_{2\varepsilon}(x_0)$. For each sequence $(\tilde{x}_n)_{n \in \mathbb{N}}$ in $B_\varepsilon(x_0) \cap \mathcal{D}(F)$ and for the corresponding sequence $(y_n)_{n \in \mathbb{N}}$ with $y_n := F(\tilde{x}_n)$ Proposition 1.6 yields a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{D}(F)$ with $F(x_n) = y_0$ and $\tilde{x}_n - x_n \rightarrow 0$. Since $(\tilde{x}_n)_{n \in \mathbb{N}}$ lies in $B_\varepsilon(x_0)$ and x_0 is the only solution in $B_{2\varepsilon}(x_0)$, we obtain $x_n = x_0$ for all n . Consequently, $\tilde{x}_n \rightarrow x_0$, which proves local well-posedness in the sense of Hofmann at x_0 . \square

Proposition 1.9 *There exist mappings F and points x_0 in $\mathcal{D}(F)$ such that F is locally well-posed in the sense of Hofmann at x_0 but locally ill-posed in the sense of Ivanov at $F(x_0)$.*

Proof Choose $X := \mathbb{R}$, $Y := \mathbb{R}$ and $F(x) := \frac{x^2}{1+x^4}$ with $\mathcal{D}(F) = X$. Then $x_0 := 0$ is the only solution to $F(x) = 0$, $x \in X$, and continuous invertibility of F near zero immediately implies local well-posedness in the sense of Hofmann.

On the other hand, we may consider a sequence $(y_n)_{n \in \mathbb{N}}$ with elements $y_n := F(x_n)$ such that $x_n \rightarrow \infty$. Then $y_n \rightarrow 0$, but

$$\sup_{\tilde{x} \in F^{-1}(y_n)} \inf_{x \in F^{-1}(0)} \|\tilde{x} - x\| \geq \inf_{x \in F^{-1}(0)} \|x_n - x\| = \|x_n\| \not\rightarrow 0.$$

Thus, F is locally ill-posed in the sense of Ivanov at $F(0)$. □

Finally, we state the interrelation between Nashed's definition and Ivanov's definition. The special case of Hilbert spaces, where each closed subspace is complemented, can be found in [3, Proposition 1].

Proposition 1.10 *Let F be a bounded linear operator with domain $\mathcal{D}(F) = X$ between the Banach spaces X and Y and let the null space $\mathcal{N}(F)$ be complemented in X . Then F is well-posed in the sense of Nashed if and only if F is locally well-posed in the sense of Ivanov at every point of $\mathcal{R}(F)$ and F is ill-posed in the sense of Nashed if and only if F is locally ill-posed in the sense of Ivanov at every point of $\mathcal{R}(F)$.*

Proof Let $\mathcal{N}(F)$ be complemented by U , that is, U is a closed linear subspace of X and $X = \mathcal{N}(F) \oplus U$. One easily shows, that the restriction $F|_U$ of F to U is bijective between U and $\mathcal{R}(F)$. Thus, the inverse $(F|_U)^{-1}$ is a well-defined linear operator, which due to $\mathcal{R}(F) = \mathcal{R}(F|_U)$ is bounded if and only if $\mathcal{R}(F)$ is closed. We see that F is well-posed in the sense of Nashed if and only if $(F|_U)^{-1}$ is bounded.

Let $(F|_U)^{-1}$ be bounded. To show local well-posedness in the sense of Ivanov at an arbitrary point y_0 in $\mathcal{R}(F)$ we choose sequences $(y_n)_{n \in \mathbb{N}}$ with $y_n \rightarrow y_0$ and $(\tilde{x}_n)_{n \in \mathbb{N}}$ in X with $F(\tilde{x}_n) = y_n$. By Proposition 1.6 we have to show that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ with $F(x_n) = y_0$ and $\tilde{x}_n - x_n \rightarrow 0$. Such a choice is given by

$$x_n := \tilde{x}_n - (F|_U)^{-1}(F(\tilde{x}_n)) + (F|_U)^{-1}(y_0),$$

because

$$F(x_n) = F(\tilde{x}_n) - F(\tilde{x}_n) + y_0 = y_0$$