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Clifford Analysis and Related Topics

CART 2014, Tallahassee, Florida, December 15–17

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Clifford Analysis and Related Topics

In Honor of Paul A. M. Dirac, CART 2014, Tallahassee, Florida, December 15–17

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Preface

The present volume arises from the international conference *Clifford Analysis and Related Topics* held at the Florida State University, Tallahassee on December 2014. The conference was organized by Craig Nolder (Florida State University) and John Ryan (University of Arkansas) with the intent of celebrating the English theoretical physicist Paul Adrien Maurice Dirac, who died in Tallahassee in 1984, after spending his last decade of his life at Florida State University. P.A.M. Dirac made fundamental discoveries in the early formation of quantum mechanics. He shared the 1933 Nobel Prize in Physics with Erwin Schrödinger. He is the founder of the field of quantum electrodynamics. Notably, he developed a factorization of the Klein–Gordon equation which leads to the system of first-order Dirac equations which provided a relativistic wave equation for the electron. These equations provided a way to describe intrinsic spin and suggested the existence of antimatter, at first the positron which was discovered soon after the equations appeared. The equations turned out to describe all spin 1/2 particles, the fermions. The Dirac equations are based on a matrix representation of a Clifford algebra, now called Pauli matrices. Clifford algebras have found many applications in physics since this time including a role in the algebraic theory of the standard model of particle physics. Dirac was the Lucasian Professor of Mathematics at Cambridge from 1932 until 1969. He then came to Florida, working at Miami University, Coral Gables, and Florida State University, Tallahassee. He was a Visiting Professor at FSU during 1970–71 and accepted a Full Professorship in 1972. Dirac passed on August 8, 1984 and is buried in Roselawn Cemetery, Tallahassee, Fl.

Paul Dirac's work is at the very heart of Clifford Analysis, an active branch of mathematics that has grown significantly over the last 40 years and which covers both theoretical and applied physics. The field of Clifford Analysis began as a function theory for the solutions of the Dirac equation for spinor fields and, in such, can be regarded as a natural generalization to higher dimensions of the function theory of complex holomorphic functions.

The conference involved participants from Venezuela, Portugal, Brazil, Cape Verde, and USA, and this volume reflects not only the main contributions but also the stimulating and friendly atmosphere prevailing among the attendants.

Furthermore, the editors would like to express their gratitude to the anonymous referees without which this volume would have never seen the light.

We conclude with a statement of Dirac, published on *Scientific American*, May 1963:

> It seems to be one of the fundamental features of nature that fundamental physical laws are described in terms of a mathematical theory of great beauty and power, needing quite a high standard of mathematics for one to understand it.

It is our hope that the contributions on this volume make due honors to this statement.

Aveiro, Portugal — Paula Cerejeiras
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The original version of the book frontmatter was revised: The fourth editor's affiliation has been corrected. The correction to the book frontmatter is available at https://doi.org/10.1007/978-3-030-00049-3_9

Contents

Lambda-Harmonic Functions: An Expository Account

K. Ballenger-Fazzone and C. A. Nolder

Abstract In this paper, we compile a variety of results on the $\lambda-$Laplacian operator, denoted by Δ_λ, a generalization of the well-known Laplacian in $\mathbb{R}^n$. We have compiled a list of known properties for Δ_λ when $\lambda = \frac{n-2}{2}$ and present analogous properties for Δ_λ. We close by discussing the $\lambda-$Poisson kernel, the function that solves the Dirichlet problem on the closed ball in $\mathbb{R}^n$.

Keywords Clifford analysis · Dirichlet problem · Lambda-Harmonic Laplacian · Poisson kernel

1 Introduction

The purpose of this paper is to compile a variety of interesting results on the $\lambda-$Laplacian operator, a generalization of the Laplacian Δ in $\mathbb{R}^n$. In Sect. 2, we define what it means for a function to be $\lambda-$harmonic and discuss how this operator is related to the Laplacian. We look at the special case when $\lambda = \frac{n-2}{2}$, known as the Invariant Laplacian, in Sect. 3 and present some properties of this operator. Section 4 provides a great deal of set up to show how some properties of the Invariant Laplacian do not generalize when $\lambda \neq \frac{n-2}{2}$. Finally, in Sect. 5, we discuss $\lambda-$Poisson kernel, which turns out to be the solution to the Dirichlet problem for $\lambda-$harmonic functions on the unit ball, and prove some new results for this kernel. Section 5 helps us to set up our next paper where we will solve the Dirichlet problem for $\lambda-$harmonic functions on an annular domain.

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2 λ-Harmonic Functions

We denote by $\mathbb{B}^n$ the unit ball of $\mathbb{R}^n$ centered at the origin, where $n \geq 2$. A point $x \in \mathbb{B}^n$ is denoted $x = (x_1, x_2, \ldots, x_n)$. When the center or radius is important to us, we will denote by $B(a, r)$ an open ball centered at $a \in \mathbb{R}^n$ with radius $r > 0$. We denote the boundaries by S^{n-1} and $S(a, r)$ respectively.

Definition 1 Let $\lambda \in \mathbb{R}$. A function $u \in C^2(\mathbb{B}^n)$ is **λ-harmonic** if

$$\Delta_\lambda u = 0$$

in $\mathbb{B}^n$, where

$$\Delta_\lambda \overset{\text{DEF}}{=\!=} \left(1 - |x|^2\right) \left[\frac{1 - |x|^2}{4} \Delta + \lambda E + \lambda \left(\frac{n-2}{2} - \lambda \right) \right], \tag{1}$$

is the λ-Laplacian,

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

is the Laplacian on $\mathbb{R}^n$, and

$$E = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

is the Euler operator.

Therefore,

$$\Delta_\lambda u = \left(1 - |x|^2\right) \left[\frac{1 - |x|^2}{4} \Delta u + \lambda E u + \lambda \left(\frac{n-2}{2} - \lambda \right) u \right].$$

If $\Delta_\lambda u = 0$, then u is an eigenvector of the differential operator

$$\frac{1 - |x|^2}{4} \Delta + \lambda E.$$

That is,

$$\frac{1 - |x|^2}{4} \Delta u + \lambda E u = \lambda \left(\frac{2-n}{2} + \lambda \right) u.$$

The λ-Laplacian is a generalization of two well-known operators:

1. If $\lambda = 0$, then

$$\Delta_0 = \frac{(1 - |x|^2)^2}{4} \Delta.$$

Thus solutions to $\Delta_0 u = 0$ are called harmonic.

2. If $\lambda = \dfrac{n-2}{2}$, then

$$\Delta_{\frac{n-2}{2}} = \frac{1}{4}\tilde{\Delta},$$

where

$$\tilde{\Delta} = (1-|x|^2)^2\Delta + 2(n-2)(1-|x|^2)E$$

is the invariant Laplacian (or Laplace-Beltrami operator with respect to the Poincaré metric on $\mathbb{B}^n$). We call solutions to $\tilde{\Delta}u = 0$ **invariant harmonic** (or $\mathscr{M}$-harmonic) [18].

Before we discuss more about the λ-Laplacian, we first look at the invariant Laplacian in a different light.

3 The Invariant Laplacian $\left(\lambda = \frac{n-2}{2}\right)$

The invariant Laplacian $\tilde{\Delta}$ is the Laplace-Beltrami operator with respect to the Poincaré metric $ds = \dfrac{2|dx|}{1-|x|^2}$ on $\mathbb{B}^n$. We can also define $\tilde{\Delta}$ in a geometric way. We remark that the content from this subsection comes from [18].

Definition 2 Let Ω be an open subset of $\mathbb{B}^n$ with $f \in C^2(\mathbb{B}^n)$ and $a \in \mathbb{B}^n$. We define the **invariant Laplacian** by

$$(\tilde{\Delta} f)(a) = \Delta(f \circ \phi_a)(0),$$

where

$$x^* = \begin{cases} x/|x|^2, & \text{if } x \neq 0 \\ 0, & \text{if } x = \infty \\ \infty, & \text{if } x = 0, \end{cases}$$

$$\phi_a(x) = \psi_a(x)^* = \frac{\psi_a(x)}{|\psi_a(x)|^2},$$

and

$$\psi_a(x) = a + (1-|a|^2)(a-x)^*.$$

It is easy to see that ψ_a is a Möbius transformation mapping 0 to a^* and a to ∞.

Definition 3 Let Ω be an open subset of $\mathbb{B}^n$ with $f \in C^1(\mathbb{B}^n)$ and $a \in \mathbb{B}^n$. We define the **invariant gradient** by

$$(\tilde{\nabla} f)(a) = -\nabla(f \circ \phi_a)(0),$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ is the usual gradient.

Remark 1 The minus sign in the definition above ensures that both ∇u and $\tilde{\nabla} u$ point in the same direction.

Let $f \in C^2(\mathbb{B}^n)$ and let $y = \psi(x)$ be a C^2 map from $\mathbb{B}^n$ into $\mathbb{B}^n$. If $g = f \circ \psi$, then

$$\nabla g(x) = \psi^{'}(x) \nabla f(\psi(x))$$

and

$$\Delta g(x) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial y_i \partial y_j} \langle \nabla y_i, \nabla y_j \rangle + \sum_{j=1}^{n} \frac{\partial f}{\partial y_i} \Delta y_i,$$

where $\psi^{'}(x)$ is the Jacobian matrix of ψ and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. Setting $y = \phi_a(x)$, we have that

$$\tilde{\nabla} f(a) = (1 - |a|^2) \nabla f(a)$$

and

$$\tilde{\Delta} f(a) = (1 - |a|^2)^2 \Delta f(a) + 2(n-2)(1 - |a|^2) \langle a, \nabla f(a) \rangle$$

as before.

Solutions to the invariant Laplacian are invariant under Möbius transformations of $\mathbb{B}^n$. We denote the group of Möbius transformations that leave $\mathbb{B}^n$ invariant by $\mathscr{M}(\mathbb{B}^n)$.

Theorem 1 *Let $f \in C^2(\mathbb{B}^n)$ and $\psi \in \mathscr{M}(\mathbb{B}^n)$. Then*

$$\tilde{\Delta}(f \circ \psi) = (\tilde{\Delta} f) \circ \psi$$

and

$$|\tilde{\nabla}(f \circ \psi)| = |(\tilde{\nabla} f) \circ \psi|.$$

Proof The proof of the first equality can be found in [18]. For the second, we mimic the proof from [15]. Let $b \in \psi^{-1}(\Omega)$ and set $a = \psi(b)$. Then we see that $\phi_a \circ \psi \circ \phi_b$ is a Möbius transformation of $\mathbb{B}^n$ that fixes 0. Therefore $\psi \circ \phi_b = \phi_a \circ A$, where A is some orthogonal transformation. Thus

$$\begin{aligned}|\tilde{\nabla}(f \circ \psi)(b)| &= |-\nabla(f \circ \psi \circ \phi_b)(0)| \\ &= |-\nabla(f \circ \phi_a \circ A)(0)| \\ &= |-\nabla(f \circ \phi_a)(0)| \\ &= |\tilde{\nabla}(f)(a)| \\ &= |\tilde{\nabla}(f)(\psi(b))|.\end{aligned}$$

We now present a few nice facts about the invariant Laplacian.

Theorem 2 *Invariance Properties of* $\tilde{\Delta}$

1. $\tilde{\Delta}$ *is a linear operator mapping* $C^2(\mathbb{B}^n) \to C(\mathbb{B}^n)$
2. *translations of invariant harmonic functions are invariant harmonic*
3. $r-$*dilates of invariant harmonic functions are invariant harmonic*
4. $\tilde{\Delta}$ *commutes with orthogonal transformations*

Proof To prove the first property, it is enough to assume that $u, v \in C^2(\mathbb{B}^n)$ and $c, d \in \mathbb{R}$. Then

$$\begin{aligned}\tilde{\Delta}(cu + dv) &= \left(1 - |x|^2\right)\left[\frac{1-|x|^2}{4}\Delta(cu+dv) + \left(\frac{n-2}{2}\right)E(cu+dv)\right] \\ &= \left(1 - |x|^2\right)\left[\frac{1-|x|^2}{4}\left(\Delta(cu)+\Delta(dv)\right) + \left(\frac{n-2}{2}\right)E(cu) + \left(\frac{n-2}{2}\right)E(dv)\right] \\ &= c\tilde{\Delta}(u) + d\tilde{\Delta}(v).\end{aligned}$$

The proof of *2.* is clear.

To prove *3*, we first assume $r \in \mathbb{R}$ with $r > 0$ and define $u_r(x) = u(rx)$, for $x \in (1/r)\mathbb{B}^n$. Then direct calculations show that

$$\Delta(u_r) = r^2(\Delta u)_r$$

and

$$E(u_r) = r^2(Eu)_r.$$

It follows that, if $\tilde{\Delta}u = 0$, then

$$\begin{aligned}\tilde{\Delta}(u_r) &= \left(1 - |x|^2\right)\left[\frac{1-|x|^2}{4}\Delta(u_r) + \left(\frac{n-2}{2}\right)E(u_r)\right] \\ &= \left(1 - |x|^2\right)\left[r^2\frac{1-|x|^2}{4}(\Delta u)_r + r^2\left(\frac{n-2}{2}\right)(Eu)_r\right] \\ &= r^2(\tilde{\Delta}u)_r \\ &= 0.\end{aligned}$$

To prove *4*, we must show that if T is an orthogonal transformation and $u \in C^2(\mathbb{B}^n)$, then

$$\tilde{\Delta}(u \circ T) = (\tilde{\Delta} u) \circ T$$

on $T^{-1}(\mathbb{B}^n)$. We proceed following an argument from [2]. Let $[t_{jk}]$ denote the matrix for T relative to the standard basis in $\mathbb{R}^n$. Then

$$\frac{\partial}{\partial x_m}(u \circ T) = \sum_{j=1}^{n} t_{jm} \left(\frac{\partial}{\partial x_j} u \right) \circ T. \tag{2}$$

It is easy to see from (2) that

$$E(u \circ T) = (Eu) \circ T$$

and differentiating (2) shows

$$\Delta(u \circ T) = (\Delta u) \circ T.$$

Putting these two together then shows that

$$\tilde{\Delta}(u \circ T) = (\tilde{\Delta} u) \circ T.$$

We conclude this survey of the invariant Laplacian by listing the invariant analogous of some classical results from harmonic analysis.

Definition 4 Let Ω be an open subset of $\mathbb{B}^n$. A function $f \in C^2(\mathbb{B}^n)$ is **invariant subharmonic** (or $\mathcal{M}$-subharmonic) on Ω if $\tilde{\Delta} f(x) \geq 0$ for all $x \in \Omega$.

Remark 2 It is easy to prove that if f is invariant harmonic (invariant subharmonic) on $\mathbb{B}^n$, then $f \circ \psi$ is invariant harmonic (invariant subharmonic) on $\mathbb{B}^n$, for all $\psi \in \mathcal{M}(\mathbb{B}^n)$.

We can extend a mean-value property to invariant subharmonic functions.

Theorem 3 ***Invariant Subharmonic Mean-Value Property***
Let Ω be an open subset of $\mathbb{B}^n$ and let $f \in C^2(\Omega)$. Then f is invariant subharmonic on Ω if and only if for all $a \in \Omega$

$$f(a) \leq \int_{\mathbb{S}^{n-1}} f(\phi_a(rt)) d\sigma(t)$$

for all $r > 0$ such that $E(a, r) \subset \Omega$, where σ denotes the normalized surface measure on $\mathbb{S}^{n-1}$ and $E(a, r)$ is the Euclidean ball centered at a with radius r. f is invariant harmonic on Ω if and only if the equality holds.

The proof can be found in [18]. Invariant harmonic functions also satisfy a maximum principle.

Theorem 4 ***Invariant Subharmonic Harmonic Maximum Principle***
Let Ω be an open subset of $\mathbb{B}^n$ and let $f \in C^2(\Omega)$ such that f is invariant subharmonic in Ω and continuous on $\overline{\Omega}$. If $f \leq 0$ on $\partial\Omega$, then $f \leq 0$ in Ω.

The proof can be found in [18]. More information on the invariant Laplacian on $\mathbb{B}^n$ can be found in [6, 9], whereas [15] discusses the invariant Laplacian on the unit ball in $\mathbb{C}^n$.

4 λ-Harmonic Functions Continued

In order to continue our discussion on Δ_λ, we must first review some preliminaries: the hypergeometric function and spherical harmonics.

4.1 The Hypergeometric Function

We define the (rising) Pochhammer symbol $(a)_l$ for an arbitrary $a \in \mathbb{C}$ and $l = 0, 1, \ldots,$ by

$$(a)_l = \begin{cases} 1, & \text{if } l = 0 \\ a(a+1)\cdots(a+l-1), & \text{if } l > 0 \end{cases}$$

If a is not a negative integer, then

$$(a)_l = \frac{\Gamma(a+l)}{\Gamma(a)},$$

where Γ is the Gamma function defined on $\mathbb{C} \setminus \{-1, -2, \ldots\}$. Thus, for $x \in \mathbb{B}^n$, the **hypergeometric function** is defined to be

$${}_2F_1(a, b; c; x) \overset{\text{DEF}}{=\!=} \sum_{l=0}^{\infty} \frac{(a)_l (b)_l}{(c)_l} \frac{x^l}{l!},$$

and the series converges absolutely for all $x \in \mathbb{B}^n$ if $c - a - b > 0$. The function is undefined if c is a non-positive integer.

For convenience, we define

$$F_{\lambda,k}(x) \overset{\text{DEF}}{=\!=} {}_2F_1\left(-\lambda, k + \frac{n-2}{2} - \lambda; k + \frac{n}{2}; x\right).$$

We remark that many of the proofs involving the hypergeometric function rely on various formulas found in [3, 11].

4.2 Homogeneous Harmonic Polynomials and Spherical Harmonics

We begin this subsection by discussing some classical results concerning homogeneous polynomials from harmonic analysis. We direct the reader to [2, 7, 12, 13] for more information.

Definition 5 A polynomial Y_m is **homogeneous of degree** m if Y_m is of the form

$$Y_m(x) \stackrel{\text{DEF}}{=\!=} \sum_{|\alpha|=m} c_\alpha x^\alpha,$$

where

Alternatively, Y_m is homogeneous of degree m if, for all $t \in \mathbb{R}$,

$$Y_m(tx) = t^m Y_m(x).$$

It is well-known that every degree m polynomial Y on $\mathbb{R}^n$ can be written uniquely as

$$Y(x) = \sum_{j=0}^{m} Y_j(x),$$

where Y_j is homogeneous of degree j. It is then easy to see that Y is harmonic if and only if Y_j is harmonic for each $j = 0, 1, \dots, m$.

Notation 5 *We denote by* $\mathscr{P}_m(\mathbb{R}^n)$ *the set of all homogeneous polynomials on* $\mathbb{R}^n$ *of degree* m *and by* $\mathscr{H}_m(\mathbb{R}^n)$ *the set of all homogeneous harmonic polynomials on* $\mathbb{R}^n$ *of degree* m.

We are able to decompose $\mathscr{P}_m(\mathbb{R}^n)$ into the direct sum of two subspaces, which we present in the following theorem [2].

Theorem 6 *If* $m \geq 2$*, then we can write*

$$\mathscr{P}_m(\mathbb{R}^n) \equiv \mathscr{H}_m(\mathbb{R}^n) \oplus |x|^2 \mathscr{P}_{m-2}(\mathbb{R}^n).$$

The proof of Theorem 6 relies on the fact that no multiple of the polynomial $|x|^2$ is harmonic (see Corollary 5.3 [2]).

To proceed further, we must introduce hyperspherical coordinates on n-dimensional Euclidean space. Our coordinate system consists of one radial coordinate r and $n-1$ angular coordinates denoted by $\phi_1, \phi_2, \dots, \phi_{n-1}$, where $\phi_{n-1} \in [0, 2\pi]$ and $\phi_i \in [0, \pi]$, for $i = 1, 2, \dots, n-2$. Then the relationship between Euclidean coordinates $x_1, \dots x_n$ and hyperspherical coordinates is given by