George E. Andrews
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Ramanujan's Lost Notebook

Part V

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Some CEOs of Mock Theta Functions

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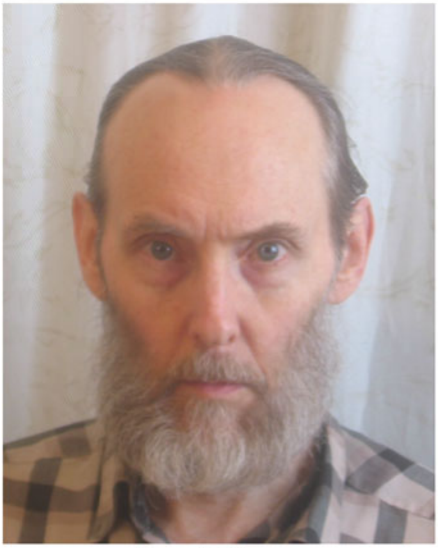
Sander Zwegers



Dean Hickerson

Eric Mortenson

We are especially grateful to these mathematicians whose work and guidance on mock theta functions made it possible to complete this volume.

I have shown you today the highest secret of my own realization. It is supreme and most mysterious indeed.

Verse 575 of Vivekachudamani,
by Adi Shankaracharya
Sixth Century, A.D.

Preface

This is the fifth and final volume that the authors have written in their examination of all the claims made by S. Ramanujan in *The Lost Notebook and Other Unpublished Papers*. Published by Narosa in 1988, the treatise contains the "Lost Notebook," which was discovered by the first author in the spring of 1976 at the library of Trinity College, Cambridge. Also included in this publication are partial manuscripts, fragments, and letters from Ramanujan to G.H. Hardy. In his last letter, Ramanujan introduced *mock theta functions* to the mathematical world for the first time. Most of this volume is devoted to Ramanujan's beautiful identities involving mock theta functions, which populate his "Lost Notebook." Also featured are Ramanujan's many elegant Euler products, found in scattered entries and in a manuscript published with the "Lost Notebook." A few continued fractions are also examined.

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Contents

1

Introduction

This is the fifth and final volume devoted to an explication of the content of *Ramanujan's Lost Notebook and Other Unpublished Papers* [232]. As the title indicates, [232] features the original "Lost Notebook," discovered by the first author in the library at Trinity College, Cambridge in the spring of 1976. However, [232] also contains several unpublished manuscripts by Ramanujan, letters that Ramanujan wrote to G.H. Hardy from nursing homes, Ramanujan's last letter to Hardy, and miscellaneous pages by Ramanujan. It has been our goal to cover all of this material.

After a respite from q-series in our fourth book [35], we return to q-series in this final volume. In particular, we examine the material on mock theta functions found in the lost notebook. Undoubtedly, the mock theta functions are among the most important of Ramanujan's contributions to mathematics. They are currently a prominent topic of contemporary research, and their influence is being felt in several areas of mathematics and physics. It is far too early to offer a definitive assessment of their value on the future of mathematics, but suffice it to say, it will be substantial. Readers may wish to consult one of the several surveys on mock theta functions, in particular surveys by the first author [26], [29, pp. 247–267], the treatise of K. Bringmann, A. Folsom, K. Ono, and L. Rolen [75], the lectures of K. Ono [223] and D. Zagier [280], W. Duke's brief paper [129], Folsom's excellent surveys [134], [136], and Ono's engaging article [224].

Having already emphasized the prominence of "q" in this volume, it seems appropriate here to introduce the q-notation that will be used in the remainder of the sequel. *Always*, it is to be assumed that q is a complex number with $|q| < 1$. First, define, for any complex number a and each non-negative integer n,

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad n \geq 1, \tag{1.0.1}$$

and

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n. \tag{1.0.2}$$

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If the *base* q is "constant" throughout a section, then we may delete it from our notation and write

$$(a)_n := (a;q)_n, \quad n \geq 0, \quad \text{and} \quad (a)_\infty := (a;q)_\infty. \tag{1.0.3}$$

Occasionally, we encounter products of several products. In such instances, it is convenient to use the notation

$$(a_1, a_2, \ldots, a_m; q)_n := (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n, \quad n, m \geq 1, \tag{1.0.4}$$

and

$$(a_1, a_2, \ldots, a_m; q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty, \quad m \geq 1. \tag{1.0.5}$$

In some instances, we may abbreviate the notation by writing $(a_1, a_2, \ldots, a_m)_n$ and $(a_1, a_2, \ldots, a_m)_\infty$, respectively. Second set, for each non-negative integer n and complex number $a \neq 0$,

$$[a;q]_n := (a;q)_n (q/a;q)_n \quad \text{and} \quad [a;q]_\infty := (a;q)_\infty (q/a;q)_\infty. \tag{1.0.6}$$

For every pair of non-negative integers n, m, define

$$[a_1, a_2, \ldots, a_m; q]_n := [a_1;q]_n [a_2;q]_n \cdots [a_m;q]_n, \tag{1.0.7}$$

and

$$[a_1, a_2, \ldots, a_m; q]_\infty := [a_1;q]_\infty [a_2;q]_\infty \cdots [a_m;q]_\infty, \quad m \geq 1. \tag{1.0.8}$$

Furthermore, set

$$[a_1, a_2, \ldots, a_m]_n := [a_1, a_2, \ldots, a_m; q]_n, \tag{1.0.9}$$

and

$$[a_1, a_2, \ldots, a_m]_\infty := [a_1, a_2, \ldots, a_m; q]_\infty. \tag{1.0.10}$$

The q-analogue of the binomial coefficient $\binom{n}{m}$, $n, m \geq 0$, $m \leq n$, is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix} := \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}. \tag{1.0.11}$$

The term, "order," for mock theta functions is, at best, somewhat vague. Ramanujan, in his last letter to Hardy (Chapter 14), describes third order, fifth order, and seventh order mock theta functions. He gives no explanation for this characterization. We hazard a guess that the "order" is associated to the modulus of the related theta functions. The third order functions are related through identities to Euler's classical pentagonal number theorem. Fifth order functions are linked closely to the Rogers–Ramanujan identities (and thus the number 5). The seventh order functions (of which Ramanujan only says that "These are not related to each other.") must have been named owing to their natural similarity to identities (59)–(61) in [250] and which Ramanujan had access to in [238] and [239].

The discovery subsequently of unnamed mock theta functions in the Lost Notebook and elsewhere left many researchers with the vexing question of what order to give each of these new functions. Generally, a choice was made from the examination of terms in related Appell–Lerch series or Hecke-type series involving indefinite quadratic forms. We have retained the names from the literature even though we have no more justification than making our text compatible with what had gone before.

Third order mock theta functions are discussed in Chapter 2. Chapter 3 is analogous to Chapter 2 in that basic properties of fifth order mock theta functions are established. We return to the third order mock theta functions in Chapter 4 and derive partial fraction expansions that are intimately connected with the generating function for ranks of partitions. Returning to fifth order mock theta functions in Chapter 5, we prove the equivalence of identities involving fifth order mock theta functions in each of two sets of five identities; each set of identities came to be known as the *mock theta conjectures*. Chapter 6 is devoted to proofs of the mock theta conjectures. Sixth order mock theta functions are addressed in Chapter 7. The entries on tenth order mock theta functions are difficult, especially the fifth, sixth, seventh, and eighth, and consequently five chapters, Chapters 8–12, are devoted to the proofs of the eight entries on tenth order mock theta functions.

Most readers probably have some acquaintance with Ramanujan's arithmetical function $\tau(n)$, which is generated by a Dirichlet series possessing an Euler product. Scattered throughout the Lost Notebook are many further results providing Euler product representations for important Dirichlet series, and these are discussed and proved in Chapter 15, which is based on a paper that the second author coauthored with B. Kim and K.S. Williams [63].

Most of Ramanujan's claims on continued fractions in the Lost Notebook, especially the Rogers–Ramanujan continued fraction, are discussed in our first book [32]. A few scattered entries are examined in our following three books on the Lost Notebook [33], [34], and [35]. This volume contains our examination of the remaining entries on continued fractions, which were first examined in a paper that the second coauthor wrote with S.-Y. Kang and J. Sohn [62].

Although this is our final volume on the Lost Notebook, a plethora of questions need to be answered—in particular, questions about the paths and reasoning that Ramanujan took to his discoveries. Throughout the years, the authors have asked countless questions as they marvelled about Ramanujan's thinking and ingenuity. We have addressed only a small proportion of these in Chapter 19. Readers will undoubtedly have their own such questions.

Many mathematicians contributed proofs to this volume, and so we would like to personally thank them, for if it were not for their many beautiful and deep contributions, this volume would never have been written. To that end, we are extremely grateful to Song Heng Chan, Youn–Seo Choi, Frank

Garvan, Dean Hickerson, Soon-Yi Kang, Byungchan Kim, Eric Mortenson, Jaebum Sohn, Kenneth S. Williams, Hamza Yesilyurt, and Sander Zwegers.

We are particularly grateful to Shaun Cooper who offered many helpful comments on our manuscript, to Jaebum Sohn who read almost all of our manuscript in complete detail uncovering a plethora of misprints and offering numerous suggestions, and to Eric Mortenson who brought us up to date with several references and many additional helpful suggestions. Several useful suggestions and corrections were also supplied by S. Bhargava, Mike Hirschhorn, Michael Somos, and Youn-Seo Choi.

2

Third Order Mock Theta Functions: Elementary Identities

2.1 Introduction

In his last letter to G.H. Hardy (see Chapter 14), Ramanujan listed four third order mock theta functions, namely,

$$f_3(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, \tag{2.1.1}$$

$$\phi_3(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}, \tag{2.1.2}$$

$$\psi_3(q) := \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}, \tag{2.1.3}$$

and

$$\chi_3(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{j=1}^{n}(1-q^j+q^{2j})}. \tag{2.1.4}$$

(We use above the basic notation (1.0.1).) In addition to these, the following third order mock theta function appear in the Lost Notebook:

$$\omega_3(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}, \tag{2.1.5}$$

$$\nu_3(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}}, \tag{2.1.6}$$

and

$$\rho_3(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{\prod_{j=0}^{n}(1+q^{2j+1}+q^{4j+2})}. \tag{2.1.7}$$

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G.N. Watson [269, p. 62] believed that these last three mock theta functions were his discoveries, but he apparently only had Ramanujan's last letter available to him at the time he wrote. The methods that Watson introduced in [270] inspired the more general theorems in the next section as well as the theorems in [13] and [16].

In this chapter we consider the identities connecting these third order mock theta functions to each other and to various classical theta functions. The chapter concludes with Entry 2.3.9 in which Ramanujan generalized a couple of previous results by introducing a second variable. Two methods will be employed: (1) q-series manipulation (cf. our second book [33, Chapter 1]), (2) partial fractions (cf. our first book [32, Chapter 12]). Indeed, it would have been natural to include all of the entries in this chapter in one or the other of the two chapters just cited. We feel, however, that this final volume should contain everything from the Lost Notebook related to mock theta functions.

2.2 Basic Theorems

For the convenience of readers, we reproduce two theorems from [33, p. 6, Theorem 1.2.1; p. 7, Theorem 1.2.2]; see also [12]. (We employ the notation (1.0.2).)

Theorem 2.2.1. *If h is a positive integer, then, for $|t|, |b| < 1$,*

$$\sum_{m=0}^{\infty} \frac{(a;q^h)_m (b;q)_{hm}}{(q^h;q^h)_m (c;q)_{hm}} t^m = \frac{(b;q)_\infty (at;q^h)_\infty}{(c;q)_\infty (t;q^h)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m (t;q^h)_m}{(q;q)_m (at;q^h)_m} b^m. \tag{2.2.1}$$

Theorem 2.2.2. *For $|t|, |b| < 1$,*

$$\sum_{n=0}^{\infty} \frac{(a;q^2)_n (b;q)_n}{(q^2;q^2)_n (c;q)_n} t^n = \frac{(b;q)_\infty (at;q^2)_\infty}{(c;q)_\infty (t;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(c/b;q)_{2n} (t;q^2)_n}{(q;q)_{2n} (at;q^2)_n} b^{2n} + \frac{(b;q)_\infty (atq;q^2)_\infty}{(c;q)_\infty (tq;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(c/b;q)_{2n+1} (tq;q^2)_n}{(q;q)_{2n+1} (atq;q^2)_n} b^{2n+1}. \tag{2.2.2}$$

In [12], purely elementary identities were derived which implied many of the results in Ramanujan's last letter to Hardy. We shall prove limiting versions of Theorems 1 and 2 of [15].

Theorem 2.2.3. *For $b, c \in \mathbb{C}$, $b \neq 0$,*

$$\sum_{n=0}^{\infty} \frac{(b;q)_n (-1)^n (c/b)^n q^{n^2}}{(q^2;q^2)_n (c;q)_n} = \frac{(cq/b;q^2)_\infty (cb;q^2)_\infty}{(c;q)_\infty (q;q^2)_\infty (-b;q)_\infty} \sum_{n=0}^{\infty} \frac{(b^2;q^2)_n (-1)^n q^{n^2}}{(q^2;q^2)_n (cb;q^2)_n} + \frac{b(c/b;q^2)_\infty (cbq;q^2)_\infty}{(c;q)_\infty (q;q^2)_\infty (-b;q)_\infty} \sum_{n=0}^{\infty} \frac{(b^2;q^2)_n (-1)^n q^{n^2+2n}}{(q^2;q^2)_n (cbq;q^2)_n}.$$

Proof. In Theorem 2.2.2, replace a by $cq/(bt)$ and let $t \to 0$. This yields

$$\sum_{n=0}^{\infty} \frac{(b;q)_n(-1)^n(c/b)^n q^{n^2}}{(q^2;q^2)_n(c;q)_n} = \frac{(b;q)_\infty(cq/b;q^2)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(c/b;q)_{2n}b^{2n}}{(q;q)_{2n}(cq/b;q^2)_n}$$
$$+\frac{(b;q)_\infty(cq^2/b;q^2)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(c/b;q)_{2n+1}b^{2n+1}}{(q;q)_{2n+1}(cq^2/b;q^2)_n}$$
$$=\frac{(b;q)_\infty(cq/b;q^2)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(c/b;q^2)_n b^{2n}}{(q^2;q^2)_n(q;q^2)_n}$$
$$+\frac{b(b;q)_\infty(c/b;q^2)_\infty}{(1-q)(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(cq/b;q^2)_n b^{2n}}{(q^2;q^2)_n(q^3;q^2)_n}$$
$$=\frac{(b;q)_\infty(cq/b;q^2)_\infty}{(c;q)_\infty}\frac{(cb;q^2)_\infty}{(q;q^2)_\infty(b^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(b^2;q^2)_n(-1)^n q^{n^2}}{(q^2;q^2)_n(cb;q^2)_n}$$
$$+\frac{b(b;q)_\infty(c/b;q^2)_\infty}{(1-q)(c;q)_\infty}\frac{(cbq;q^2)_\infty}{(q^3;q^2)_\infty(b^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(b^2;q^2)_n(-1)^n q^{n^2+2n}}{(q^2;q^2)_n(cbq;q^2)_n},$$

where the last equality follows by two applications of Theorem 2.2.1 with $h=1$, $b=0$, $t=b^2$, and q replaced by q^2. In the first application, a is replaced by c/b and $c=q$; in the second, a is replaced by cq/b and $c=q^3$. Simplifying the last equality now yields the desired result. □

Theorem 2.2.4. *For $b, c \in \mathbb{C}$, $b \neq 0$,*

$$\sum_{n=0}^{\infty}\frac{(b;q)_n q^{n(n+1)/2}}{(q;q)_n(c;q)_n}$$
$$=\frac{(bq;q^2)_\infty(-q;q)_\infty(c^2/b;q^2)_\infty}{(c;q)_\infty(-c/b;q)_\infty}\sum_{n=0}^{\infty}\frac{(b;q^2)_n(c/b)^{2n}q^{2n^2-n}}{(q;q)_{2n}(c^2/b;q^2)_n}$$
$$+\frac{(b;q^2)_\infty(-q;q)_\infty(c^2q/b;q^2)_\infty}{(c;q)_\infty(-c/b;q)_\infty}\sum_{n=0}^{\infty}\frac{(bq;q^2)_n(c/b)^{2n+1}q^{2n^2+n}}{(q;q)_{2n+1}(c^2q/b;q^2)_n}.$$

Proof. In Theorem 2.2.1, set $h=1$, replace a by $-q/t$, and let $t \to 0$. Hence,

$$\sum_{n=0}^{\infty}\frac{(b;q)_n q^{n(n+1)/2}}{(q;q)_n(c;q)_n}=\frac{(b;q)_\infty(-q;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(c/b;q)_n b^n}{(q;q)_n(-q;q)_n}$$
$$=\frac{(b;q)_\infty(-q;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{((c/b)^2;q^2)_n b^n}{(q^2;q^2)_n(-c/b;q)_n}$$
$$=\frac{(b;q)_\infty(-q;q)_\infty}{(c;q)_\infty}\left\{\frac{(c^2/b;q^2)_\infty}{(-c/b;q)_\infty(b;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(b;q^2)_n(c/b)^{2n}q^{n(2n-1)}}{(q;q)_{2n}(c^2/b;q^2)_n}\right.$$
$$\left.+\frac{(c^2q/b;q^2)_\infty}{(-c/b;q)_\infty(bq;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(bq;q^2)_n(c/b)^{2n+1}q^{n(2n+1)}}{(q;q)_{2n+1}(c^2q/b;q^2)_n}\right\},$$

where we applied Theorem 2.2.2 with $t = b$, $a = (c/b)^2$, c replaced by $-c/b$, and $b \to 0$. Simplification completes the proof of Theorem 2.2.4. □

2.3 The Third Order Identities

First recall the definition and product representation for Ramanujan's theta function $\varphi(q)$, namely [33, p. 17, equation (1.4.3)], [55, pp. 36, 37; Entry 22(i), equation (22.4)]

$$\varphi(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}. \tag{2.3.1}$$

In his last letter to Hardy [230, p. 354], [67, p. 222], Ramanujan offered the following identity relating the three third order mock theta functions $f_3(q)$, $\phi_3(q)$, and $\psi_3(q)$, defined, respectively, in (2.1.1)–(2.1.3).

Entry 2.3.1 (p. 31, 2nd and 3rd equations). *With* $\varphi(q)$, $f_3(q)$, $\phi_3(q)$, *and* $\psi_3(q)$ *defined in* (2.3.1), (2.1.1), (2.1.2), *and* (2.1.3), *respectively,*

$$2\phi_3(-q) - f_3(q) = f_3(q) + 4\psi_3(-q) = \frac{\varphi^2(-q)}{(q;q)_\infty}. \tag{2.3.2}$$

Proof. In Theorem 2.2.3, set $b = q$ and $c = -q$. Using Euler's theorem and replacing n by $n-1$ in the second series on the right-hand side in Theorem 2.2.3, we find that

$$f_3(q) = \phi_3(-q) - 2\psi_3(-q). \tag{2.3.3}$$

Next, in Theorem 2.2.3, set $b = -q$ and $c = q$ to deduce that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{\varphi^2(-q)} \left\{\phi_3(-q) + 2\psi_3(-q)\right\}. \tag{2.3.4}$$

Once we recall that [17, p. 21, equation (2.2.9)]

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty}, \tag{2.3.5}$$

we see that (2.3.2) follows directly from (2.3.3) and (2.3.4). □

The next entry involves two further third order mock theta functions $\rho_3(q)$ and $\omega_3(q)$, defined, respectively, in (2.1.7) and (2.1.5). Also recall Ramanujan's theta function $\psi(q)$ and its product representation given by [33, p. 17, equation (1.4.10)], [55, p. 36, Entry 22(ii)]

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{2.3.6}$$

Entry 2.3.2 (p. 15, top equation). *With* $\rho_3(q)$, $\omega_3(q)$, *and* $\psi(q)$ *defined above,*

$$q^{1/2}\left\{\frac{2}{3}\rho_3(-q)+\frac{1}{3}\omega_3(-q)\right\}=q^{1/2}\frac{\psi^2(-q^3)}{(q^2;q^2)_\infty}. \tag{2.3.7}$$

G.N. Watson proved this result in [269, p. 63]; however, he states, "Rather strangely (particularly in view of his having discovered both sets of functions of order 5) he [Ramanujan] seems to have overlooked the existence of the set of functions which I call $\omega(q)$, $\nu(q)$, $\rho(q)$." This strongly indicates that Watson either did not possess or had totally ignored the Lost Notebook [232] in 1935 when he wrote [269].

Proof. We follow Watson [269]. Employing [32, p. 263, equation (12.2.5)], namely,

$$\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2}}{1-cq^{n+1/2}}=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(cq^{1/2};q)_{n+1}(q^{1/2}/c;q)_{n+1}}, \tag{2.3.8}$$

with q replaced by q^2 and $c=1$, we find that

$$\begin{aligned}\omega_3(q)&=\frac{1}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)}}{1-q^{2n+1}}\\&=\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n(n+1)}(1+q^{2n+1})}{1-q^{2n+1}}.\end{aligned} \tag{2.3.9}$$

Next, apply (2.3.8) with q replaced by q^2 and $c=e^{2\pi i/3}$ to deduce that

$$\rho_3(q)=\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n(n+1)}(1-q^{4n+2})}{(1-e^{2\pi i/3}q^{2n+1})(1-e^{-2\pi i/3}q^{2n+1})}. \tag{2.3.10}$$

Hence, combining (2.3.9) and (2.3.10) term by term, we find that

$$\begin{aligned}2\rho_3(q)+\omega_3(q)&=\frac{3}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n(n+1)}(1+q^{6n+3})}{1-q^{6n+3}}\\&=\frac{3}{(q^2;q^2)_\infty}\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty^2}=3\frac{\psi^2(q^3)}{(q^2;q^2)_\infty},\end{aligned} \tag{2.3.11}$$

where we have used [32, p. 264, equation (12.2.9)], i.e.,

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{1-cq^n}=\frac{(q;q)_\infty^2}{(c;q)_\infty(q/c;q)_\infty}, \tag{2.3.12}$$

with q replaced by q^6 and $c=q^3$, and where we also invoked (2.3.6). This last identity (2.3.11) is equivalent to (2.3.7), and so the proof is complete. ☐

The next entry involves another third order mock theta function $\chi_3(q)$, defined in (2.1.4).

Entry 2.3.3 (p. 15, 2nd equation). *If* $f_3(q)$ *is defined by* (2.1.1), $\varphi(-q)$ *is given by* (2.3.1), *and* $\chi_3(q)$ *is given by* (2.1.4), *then*

$$\chi_3(q) = \frac{1}{4} f_3(q) + \frac{3}{4} \frac{\varphi^2(-q^3)}{(q;q)_\infty}. \tag{2.3.13}$$

Proof. We begin by employing [32, p. 263, equation (12.2.3)], i.e.,

$$\frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-cq^n} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(c;q)_{n+1}(q/c;q)_n}, \tag{2.3.14}$$

twice, first with $c = e^{\pi i/3}$ to find that

$$\chi_3(q) = \frac{1}{(q;q)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^n) q^{n(3n+1)/2}}{1-q^n+q^{2n}} \right), \tag{2.3.15}$$

and second with $c = -1$ to find that

$$f_3(q) = \frac{1}{(q;q)_\infty} \left(1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n} \right). \tag{2.3.16}$$

Therefore, by (2.3.15) and (2.3.16),

$$\begin{aligned}
&4\chi_3(q) - f_3(q) \\
&\quad = \frac{1}{(q;q)_\infty} \left(3 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{3n}} \left((1+q^n)^2 - (1-q^n+q^{2n}) \right) \right) \\
&\quad = \frac{3}{(q;q)_\infty} \left(1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1+q^{3n}} \right) \\
&\quad = \frac{3}{(q;q)_\infty} \varphi^2(-q^3),
\end{aligned}$$

by an appeal to (2.3.12) with $c = -1$ and q replaced by q^3, and to (2.3.1). This last identity is equivalent to (2.3.13), and so the proof is complete. □

Entry 2.3.4 (p. 17, 3rd equation). *If* $f_3(q)$ *is defined by* (2.1.1) *and* $\varphi(-q)$ *by* (2.3.1), *then*

$$\sum_{n=0}^{\infty} \frac{q^{3n^2}}{(-q;q^3)_{n+1}(-q^2;q^3)_n} = 1 - \frac{1}{4} f_3(q^3) + \frac{1}{4} \frac{\varphi^2(-q)}{(q^3;q^3)_\infty}. \tag{2.3.17}$$

We have replaced q by q^3 in Ramanujan's original formulation.

Proof. We begin by employing (2.3.14) with q replaced by q^3 and $c=-q$ to deduce that

$$\sum_{n=0}^{\infty}\frac{q^{3n^2}}{(-q;q^3)_{n+1}(-q^2;q^3)_n}=\frac{1}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n(3n+1)/2}}{1+q^{3n+1}} \tag{2.3.18}$$
$$=\frac{1}{2(q^3;q^3)_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n(3n+1)/2}}{1+q^{3n+1}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n(3n+1)/2-1}}{1+q^{3n-1}}\right),$$

where we replaced n by $-n$ to achieve the second sum on the right-hand side of (2.3.18). Next, by (2.3.14) with q replaced by q^3 and $c=-1$,

$$f_3(q^3)=\frac{2}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n(3n+1)/2}}{1+q^{3n}}. \tag{2.3.19}$$

Hence, by (2.3.18) and (2.3.19),

$$4\sum_{n=0}^{\infty}\frac{q^{3n^2}}{(-q;q^3)_{n+1}(-q^2;q^3)_n}+f_3(q^3)$$
$$=\frac{2}{(q^3;q^3)_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n(3n+1)/2}}{1+q^{3n+1}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n(3n+1)/2-1}}{1+q^{3n-1}}\right.$$
$$\left.+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n(3n+1)/2}}{1+q^{3n}}\right)$$
$$=\frac{2}{(q^3;q^3)_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^{3n+1}q^{\binom{3n+2}{2}}}{1+q^{3n+1}}\left(-q^{-3n-1}-1+1\right)\right.$$
$$\left.+\sum_{n=-\infty}^{\infty}\frac{(-1)^{3n-1}q^{\binom{3n}{2}}}{1+q^{3n-1}}\left(-q^{3n-1}-1+1\right)+\sum_{n=-\infty}^{\infty}\frac{(-1)^{3n}q^{\binom{3n+1}{2}}}{1+q^{3n}}\right)$$
$$=\frac{2}{(q^3;q^3)_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(n+1)/2}}{1+q^n}+\sum_{n=-\infty}^{\infty}(-1)^nq^{\binom{3n+1}{2}}\right.$$
$$\left.+\sum_{n=-\infty}^{\infty}(-1)^nq^{\binom{3n}{2}}\right)$$
$$=\frac{2}{(q^3;q^3)_\infty}\left(\frac{1}{2}\varphi^2(-q)+2(q^3;q^3)_\infty\right), \tag{2.3.20}$$

by (2.3.12) with $c=-1$, (2.3.1), and the pentagonal number theorem [17, p. 11, Corollary 1.7]

$$\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n-1)/2}=(q;q)_\infty. \tag{2.3.21}$$

This last equality of (2.3.20) is equivalent to (2.3.17), and so the proof is complete. □

Entry 2.3.5 (p. 17, 4th equation). *For* $\omega_3(q)$ *defined by* (2.1.5) *and* $\psi(q)$ *defined by* (2.3.6),

$$\sum_{n=0}^{\infty} \frac{q^{6n^2}}{(q;q^6)_{n+1}(q^5;q^6)_n} = \frac{1}{2}\left(1 + q^2\omega_3(q^3) + \frac{\psi^2(q)}{(q^6;q^6)_\infty}\right). \tag{2.3.22}$$

We have replaced q by q^6 in Ramanujan's original formulation.

Proof. Employing (2.3.14) with q replaced by q^6 and $c = q$, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{6n^2}}{(q;q^6)_{n+1}(q^5;q^6)_n} &= \frac{1}{(q^6;q^6)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(3n+1)}}{1-q^{6n+1}} \qquad (2.3.23)\\ &= \frac{1}{2(q^6;q^6)_\infty}\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(3n+1)}}{1-q^{6n+1}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(3n+1)-1}}{1-q^{6n-1}}\right),\end{aligned}$$

where to obtain the second sum on the right-hand side above, we replaced n by $-n$ in the first sum on the right-hand side. Hence, by (2.3.9) and (2.3.23),

$$\begin{aligned}&2\sum_{n=0}^{\infty} \frac{q^{6n^2}}{(q;q^6)_{n+1}(q^5;q^6)_n} - 1 - q^2\omega_3(q^3)\\ &= \frac{1}{(q^6;q^6)_\infty}\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(3n+1)}}{1-q^{6n+1}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(3n+1)-1}}{1-q^{6n-1}}\right.\\ &\qquad \left. -(q^6;q^6)_\infty - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n(n+1)+2}}{1-q^{6n+3}}\right)\\ &= \frac{1}{(q^6;q^6)_\infty}\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(3n+1)}}{1-q^{6n+1}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2-3n}}{1-q^{6n-1}}(q^{6n-1} - 1 + 1)\right.\\ &\qquad \left. -(q^6;q^6)_\infty - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(3n+1)(3n+2)}}{1-q^{6n+3}}\right)\\ &= \frac{1}{(q^6;q^6)_\infty}\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}}{1-q^{2n+1}} - (q^6;q^6)_\infty + \sum_{n=-\infty}^{\infty} (-1)^n q^{9n^2-3n}\right)\\ &= \frac{1}{(q^6;q^6)_\infty}\left(\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} - (q^6;q^6)_\infty + (q^6;q^6)_\infty\right)\\ &= \frac{1}{(q^6;q^6)_\infty}\psi^2(q), \qquad (2.3.24)\end{aligned}$$

where we used (2.3.12) with q replaced by q^2 and $c = q$, (2.3.21), and (2.3.6). The last equality in (2.3.24) is equivalent to (2.3.22), and this completes the proof. □

Entry 2.3.6 (p. 29, 8th equation). *If $f_3(q)$ is given by (2.1.1) and $\varphi(-q)$ is given by (2.3.1), then*

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q)_{2n} q^{n^2}}{(q^6;q^6)_n} = \frac{3}{4} f_3(q^3) + \frac{1}{4} \frac{\varphi^2(-q)}{(q^3;q^3)_\infty}. \tag{2.3.25}$$

Proof. We initially apply [32, p. 273, Entry 12.4.2]

$$(-aq;q)_\infty(-q/a;q)_\infty(q;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(-aq^2;q^2)_n(-q^2/a;q^2)_n}$$
$$= 1 + \sum_{n=1}^{\infty} \left(2(-1)^n + a^n + a^{-n}\right) \frac{q^{n(n+1)/2}}{1+q^n}, \tag{2.3.26}$$

with $a = -\omega := -e^{2\pi i/3}$. In the third equality below we appeal to [55, p. 114, Entry 8(v)], to wit,

$$\varphi^2(-q) = 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n}.$$

Thus,

$$\begin{aligned}
(q^3;q^3)_\infty &\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(\omega q^2;q^2)_n(\omega^{-1}q^2;q^2)_n} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n (2 + \omega^n + \omega^{-n}) \frac{q^{n(n+1)/2}}{1+q^n} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n}(1 + \omega^n + \omega^{-n}) \\
&= 1 + \frac{1}{4}\left(\varphi^2(-q) - 1\right) + 3\sum_{n=1}^{\infty} \frac{(-1)^n q^{3n(3n+1)/2}}{1+q^{3n}} \\
&= \frac{3}{4} + \frac{1}{4}\varphi^2(-q) + \frac{3}{4}\left(f_3(q^3)(q^3;q^3)_\infty - 1\right) \\
&= \frac{3}{4} f_3(q^3)(q^3;q^3)_\infty + \frac{1}{4}\varphi^2(-q),
\end{aligned}$$

where in the penultimate line we employed (2.3.16). We see that this last equality is equivalent to (2.3.25), and so the proof is complete. □

Entry 2.3.7 (p. 29, 9th equation). *We have*

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q^2;q^2)_n q^{2n}}{(-q^6;q^6)_n} = \frac{3}{2}\sum_{n=0}^{\infty} (-1)^n q^{(3n+2)(3n+1)/2}$$
$$+ \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} + \frac{1}{2}\frac{(q;-q)_\infty}{(-q^6;q^6)_\infty} \sum_{n=0}^{\infty} q^{3n^2+2n}(1 - q^{2n+1}). \tag{2.3.27}$$

In our statement of Entry 2.3.7 we have replaced Ramanujan's x by q. Also, the three sums on the right-hand side agree with the terms listed by Ramanujan even though it appears he would have arranged the terms differently. Finally, we note that while there are only false theta functions (instead of mock theta functions) in (2.3.27), the result is sufficiently similar to the previous entry to merit inclusion in this chapter.

Proof. We begin by recording [33, p. 122, Entry 6.3.9], namely,

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n}}{(-aq^2;q^2)_n(-q^2/a;q^2)_n} = (1+a)\sum_{n=0}^{\infty}(-a)^n q^{n(n+1)/2} - \frac{a(q;q^2)_\infty}{(-aq^2;q^2)_\infty(-q^2/a;q^2)_\infty}\sum_{n=0}^{\infty} a^{3n}q^{3n^2+2n}(1-aq^{2n+1}).$$

Setting $a=\omega := e^{2\pi i/3}$, we deduce that, upon some algebraic simplification,

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q^2;q^2)_n q^{2n}}{(-q^6;q^6)_n} = (1+\omega)\sum_{n=0}^{\infty}(-\omega)^n q^{n(n+1)/2} - \frac{\omega(q;q^2)_\infty(-q^2;q^2)_\infty}{(-q^6;q^6)_\infty}\sum_{n=0}^{\infty} q^{3n^2+2n}(1-\omega q^{2n+1}). \quad (2.3.28)$$

If we add the complex conjugate of (2.3.28) to itself (assuming that q is real for the time being), we find that

$$2\sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q^2;q^2)_n q^{2n}}{(-q^6;q^6)_n} = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\left((1+\omega)\omega^n + (1+\omega^{-1})\omega^{-n}\right) + \frac{(q;q^2)_\infty(-q^2;q^2)_\infty}{(-q^6;q^6)_\infty}\sum_{n=0}^{\infty} q^{3n^2+2n}(1-q^{2n+1}). \quad (2.3.29)$$

Now,

$$(1+\omega)\omega^n + (1+\omega^{-1})\omega^{-n} = \begin{cases} -2, & \text{if } n \equiv 1 \pmod 3, \\ 1, & \text{otherwise.} \end{cases}$$

Hence, using the calculation above in (2.3.29), we find that

$$2\sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q^2;q^2)_n q^{2n}}{(-q^6;q^6)_n} = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2} + 3\sum_{n=0}^{\infty}(-1)^n q^{(3n+2)(3n+1)/2}$$

$$+\frac{(q;-q)_\infty}{(-q^6;q^6)_\infty}\sum_{n=0}^{\infty} q^{3n^2+2n}(1-q^{2n+1}),$$

and this is the desired result multiplied by 2. □

For the next entry, we need another third order mock theta function $\nu_3(q)$, which is defined in (2.1.6).

Entry 2.3.8 (p. 31, last equation). *If $\nu_3(q)$ is given by (2.1.6), $\omega_3(q)$ is given by (2.1.5), and $\psi(q)$ is given by (2.3.6), then*

$$\nu_3(-q) = q\omega_3(q^2) + \frac{\psi(q^2)}{(q^2;q^4)_\infty}. \tag{2.3.30}$$

This formula was given by Watson [269, p. 63], who clearly believed that Ramanujan did not have this result. See the remark following Equation (2.3.7).

Proof. In Theorem 2.2.4, replace q by q^2, then set $b = q^2$ and $c = q^3$, and lastly multiply both sides by $1/(1-q)$. Thus,

$$\begin{aligned}
\nu_3(-q) &= \frac{(q^4;q^4)_\infty(-q^2;q^2)_\infty(q^4;q^4)_\infty}{(q;q^2)_\infty(-q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{4n^2}}{(q^4;q^4)_n^2}\\
&\quad + \frac{(q^2;q^4)_\infty(-q^2;q^2)_\infty(q^2;q^4)_\infty}{(q;q^2)_\infty(-q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{4n^2+4n+1}}{(q^2;q^4)_{n+1}^2}\\
&= \frac{\psi(q^2)}{(q^2;q^4)_\infty} + q\omega_3(q^2),
\end{aligned}$$

by Euler's identity, (2.3.6), (2.3.5), and (2.1.5). □

Entry 2.3.9 (p. 31, 2nd and 3rd equations). *With $\varphi(q)$ defined by (2.3.1),*

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-aq^2;q^2)_n} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{a^n q^{n^2}}{(-q;q)_n(-aq;q)_n} + \frac{1}{2}\frac{\varphi(-q)}{(-aq;q)_\infty} \tag{2.3.31}$$

and

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-aq^2;q^2)_n} = (1+a)\sum_{n=1}^{\infty}\frac{(-1)^{n-1} q^{n^2}}{(-aq;q^2)_n} + \frac{\varphi(-q)}{(-aq;q)_\infty}. \tag{2.3.32}$$

If we use (2.3.1), the identities (2.3.31) and (2.3.32) reduce to the assertions in (2.3.2) when $a = 1$. They are also equivalent to equations (3a) and (3b) in [12]. Note also that we have replaced Ramanujan's x with aq.

Proof. If we put $b = q$ and $c = -aq$ in Theorem 2.2.3 and use Euler's theorem, we find that

$$\sum_{n=0}^{\infty}\frac{a^nq^{n^2}}{(-q;q)_n(-aq;q)_n}=\sum_{n=0}^{\infty}\frac{(-1)^nq^{n^2}}{(-aq^2;q^2)_n}+(1+a)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2}}{(-aq;q^2)_n}. \quad (2.3.33)$$

Next in Theorem 2.2.3, set $b=-q$ and $c=aq$ to deduce, with the help of Euler's theorem, that

$$\sum_{n=0}^{\infty}\frac{a^nq^{n^2}}{(q;q)_n(aq;q)_n}=\frac{(-aq;q)_\infty}{(aq;q)_\infty\varphi(-q)}\sum_{n=0}^{\infty}\frac{(-1)^nq^{n^2}}{(-aq^2;q^2)_n}-\frac{(1+a)(-aq;q)_\infty}{(aq;q)_\infty\varphi(-q)}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2}}{(-aq;q^2)_n}. \quad (2.3.34)$$

Recall that [17, p. 20, Corollary 2.6]

$$\sum_{n=0}^{\infty}\frac{a^nq^{n^2}}{(q;q)_n(aq;q)_n}=\frac{1}{(aq;q)_\infty}. \quad (2.3.35)$$

Put (2.3.35) into (2.3.34) and multiply both sides by $\varphi(-q)(aq;q)_\infty/(-aq;q)_\infty$. We then deduce (2.3.32). Now return to (2.3.33), replace the latter sum on the right-hand side by the expressions obtained from (2.3.32), and thereby obtain (2.3.31). □

3

Fifth Order Mock Theta Functions: Elementary Identities

3.1 Introduction

In Chapter 14, we reproduce Ramanujan's last letter to Hardy. In it, Ramanujan's ten fifth order mock theta functions are given in their original "three or four terms of the series" format. We repeat them here in standard notation. Because Ramanujan used the same notation for each of the two sets of five functions, to avoid ambiguity and to be consistent with the notation introduced by Watson [270], we have appended the subscript 0 to those members of the first family, and the subscript 1 to those members of the second family. First,

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n}, \tag{3.1.1}$$

$$\phi_0(q) := \sum_{n=0}^{\infty} \left(-q;q^2\right)_n q^{n^2}, \tag{3.1.2}$$

$$\psi_0(q) := \sum_{n=1}^{\infty} (-q;q)_{n-1}\, q^{n(n+1)/2}, \tag{3.1.3}$$

$$F_0(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{\left(q;q^2\right)_n}, \tag{3.1.4}$$

$$\chi_0(q) := \sum_{n=0}^{\infty} \frac{q^n}{\left(q^{n+1};q\right)_n}, \tag{3.1.5}$$

$$\widetilde{\chi}_0(q) := 1 + \sum_{n=0}^{\infty} \frac{q^{2n+1}}{\left(q^{n+1};q\right)_{n+1}}. \tag{3.1.6}$$

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We note here, as did Watson [270], upon making two applications of the following corollary of the q-binomial theorem,

$$\frac{1}{(z;q)_N} = \sum_{j=0}^{\infty} \begin{bmatrix} N+j-1 \\ j \end{bmatrix} z^j$$

[32, p. 200, equation (8.2.5)], where the q-binomial coefficient $\begin{bmatrix} n \\ m \end{bmatrix}$ is defined in (1.0.11), that

$$\chi_0(q) = 1 + \sum_{n=0}^{\infty} \frac{q^{n+1}}{\left(q^{n+2};q\right)_{n+1}} \tag{3.1.7}$$

$$= 1 + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q^{n+1+m(n+2)} \begin{bmatrix} n+m \\ m \end{bmatrix}$$

$$= 1 + \sum_{m=0}^{\infty} \frac{q^{2m+1}}{(q^{m+1};q)_{m+1}}$$

$$= \widetilde{\chi}_0(q). \tag{3.1.8}$$

Thus there are really only five different 5th order mock theta functions with the subscript 0. Second,

$$f_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n}, \tag{3.1.9}$$

$$\phi_1(q) := \sum_{n=1}^{\infty} \left(-q;q^2\right)_{n-1} q^{n^2}, \tag{3.1.10}$$

$$\psi_1(q) := \sum_{n=0}^{\infty} (-q;q)_n \, q^{n(n+1)/2}, \tag{3.1.11}$$

$$F_1(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{\left(q;q^2\right)_{n+1}}, \tag{3.1.12}$$

$$\chi_1(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_{n+1}}. \tag{3.1.13}$$

In addition, we need several other functions familiar to Ramanujan and appearing throughout these volumes. In particular [55, p. 36, Entries 22(i), (ii)], [33, p. 150],

$$\varphi(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}, \tag{3.1.14}$$

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{3.1.15}$$

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{3.1.16}$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{3.1.17}$$

The remainder of this chapter is based on the results in [15], which generalize the original work of Watson [270] on the fifth order mock theta functions. Our treatment differs from [15] in that it is in standard notation and is slightly less general, thus making it more easily read.

3.2 Basic Theorems

Theorem 3.2.1. *Let $s = 0$ or 1. Then*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a;q)_{2n+s}(b;q)_n t^{2n+s}}{(q;q)_{2n+s}(c;q)_n} \\
&= \frac{1}{2} \frac{(b;q)_\infty (at;q)_\infty}{(c;q)_\infty (t;q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_{2m}(t;q)_m (b^2 q^{-s})^m}{(q;q)_{2m}(at;q)_m} \\
&\quad + \frac{1}{2}(-1)^s \frac{(b;q)_\infty (-at;q)_\infty}{(c;q)_\infty (-t;q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_{2m}(-t;q)_m (b^2 q^{-s})^m}{(q;q)_{2m}(-at;q)_m} \\
&\quad + \frac{1}{2} \frac{(b;q)_\infty (atq^{1/2};q)_\infty}{(c;q)_\infty (tq^{1/2};q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_{2m+1}(tq^{1/2};q)_m (b^2 q^{-s})^{m+1/2}}{(q;q)_{2m+1}(atq^{1/2};q)_m} \\
&\quad + \frac{1}{2}(-1)^s \frac{(b;q)_\infty (-atq^{1/2};q)_\infty}{(c;q)_\infty (-tq^{1/2};q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_{2m+1}(-tq^{1/2};q)_m (b^2 q^{-s})^{m+1/2}}{(q;q)_{2m+1}(-atq^{1/2};q)_m}.
\end{aligned}$$

Proof. By two applications of the q-binomial theorem [33, p. 6, equation (1.2.2)],

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a;q)_{2n+s}(b;q)_n t^{2n+s}}{(q;q)_{2n+s}(c;q)_n} \\
&\qquad = \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_{2n+s} t^{2n+s}}{(q;q)_{2n+s}} \frac{(cq^n;q)_\infty}{(bq^n;q)_\infty} \\
&\qquad = \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_{2n+s} t^{2n+s}}{(q;q)_{2n+s}} \sum_{m=0}^{\infty} \frac{(c/b;q)_m (bq^n)^m}{(q;q)_m} \\
&\qquad = \frac{1}{2} \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m b^m}{(q;q)_m} \sum_{n=0}^{\infty} \frac{(a;q)_n t^n q^{m(n-s)/2}}{(q;q)_n} (1 + (-1)^{n+s}) \\
&\qquad = \frac{1}{2} \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m b^m q^{-ms/2}}{(q;q)_m}
\end{aligned}$$