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Geometric Phase and Virtual Magnetic Monopole

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Preface

Adiabatic theory of both classical and quantum systems plays an important role in addressing various problems with multi-time-scale characteristics, ranging from atomic and molecular processes to the evolution of the universe. In the classical case, the well-known adiabatic theorem states, in terms of action-angle variables, that the action is the adiabatic invariant and that if the Hamiltonian is taken around a given cycle in parameter space, then the angle variable conjugate to the action acquires a purely geometrical quantity, which is termed the Hannay angle. The adiabatic theorem of quantum systems, however, becomes much more intricate due to the involvement of the complex-valued wave function/probability amplitude.

A complete theory for the adiabatic evolution of quantum systems rests on three pillars. First, Born and Fock proved the quantum adiabatic theorem shortly after the discovery of the Schrödinger equation. The adiabatic theorem states that the probability on each instantaneous (nondegenerate) eigenstate remains constant when the external condition changes slowly in time. Second, in addition to the typical dynamical phase, given by the time integral of the eigenenergy, the phase of an evolving eigenstate has a geometric part, called the Berry phase, that depends only on the geometric path in the parameter space. Third, this geometric phase can be interpreted as the flux of a virtual magnetic monopole field through the surface enclosed by the closed circuit in the parameter space. The adiabatic theory has played a crucial role in the preparation and control of quantum states. The Berry phase and related geometric phases have important applications in modern physics, such as in high-precision quantum measurement, quantum information processing, quantum computing, and condensed-matter physics.

In this book, we generalize the adiabatic theory to the nonlinear evolution of quantum systems. In physics, the nonlinearity has been introduced as possible modifications of quantum mechanics on the fundamental level. However, our motivation derives mainly from the practical applications of adiabatic control of Bose-Einstein condensates (BECs), which can often be accurately described by the nonlinear Schrödinger equation. Here, the nonlinearity stems from a mean-field treatment of the interactions between coherent atoms. The appearance of nonlinearity leads not only to the lack of unitarity but also to the absence of the

superposition principle. We overcome these challenges by combining ideas from classical adiabatic dynamics and quantum geometric phases. The developed theory of nonlinear quantum adiabatic evolution is expected to be useful in guiding adiabatic manipulation of the condensate atoms and other nonlinear systems.

The book is organized as follows. In Chap. 1, we introduce the basic concepts of adiabatic theory, such as the adiabatic invariant, the Hannay angle, the adiabatic theorem, the Berry phase, and the virtual magnetic monopole. Some typical examples of adiabatic evolution are presented. In Chap. 2, we discuss the physical origins of the nonlinearity in quantum many-body systems. The nonlinear adiabatic theory, including the adiabatic evolution of the quantum states and the nonlinear geometric phase, is introduced. In Chap. 3, we discuss the commutability between the adiabatic limit and the semiclassical limit. We show the relationship between the quantum Berry phase, the classical Hannay angle, and the mean-field geometric phase of an interacting bosonic many-body system. In Chap. 4, we introduce exotic virtual magnetic monopoles and fields such as the disk-shaped virtual magnetic field, fractional virtual magnetic monopole, and virtual magnetic monopole chain. In Chap. 5, we describe selected important applications of nonlinear adiabatic evolution in the geometric phase, in tunneling dynamics, and in quantum interference. We anticipate that readers will find this book useful in providing basic concepts and important applications on nonlinear adiabatic evolution of quantum systems.

I am deeply indebted to my beloved family for their continued support. I am also grateful to my students Li-Da Zhang, Fu-Quan Dou, Hui Cao, Qiang Wang, and Wen-Yuan Wang for reading parts of the manuscript and contributing useful remarks. In particular, I thank Profs. Q. Niu, B. Wu, B. B. Hu, and B. W. Li for long-term fruitful collaborations. Some of our previous collaborating works are included in this book.

Beijing, China Jie Liu
January 2018

Contents

Chapter 1
Introduction to Adiabatic Evolution

Abstract In this chapter, we introduce the basic concepts of adiabatic theory in both classical and quantum systems. We discuss classical adiabatic motion, introduce the concepts of the classical adiabatic invariant and the Hannay angle, and give three examples: the one-dimensional harmonic oscillator, the celestial two-body problem, and the Foucault pendulum. We describe quantum adiabatic evolution, present the quantum adiabatic theorem, and describe the adiabatic geometric phase (specifically, the Berry phase) and the virtual magnetic monopole. Five examples of quantum adiabatic evolution are shown. We also discuss classical-quantum correspondence.

1.1 Classical Adiabatic Motion

1.1.1 Classical Adiabatic Invariant

We introduce the adiabatic invariant, which is the conserved quantity in adiabatic evolution. For convenience, we consider one-dimensional finite motion of a mechanical system and use the parameter R to describe the properties of the system or of the external field in which it is placed [1]. We assume that the parameter $R(t)$ slowly varies with time because of the external field influence. In other words, the change of the parameter R is very small during one motion period of the system T, i.e.,

$$\frac{dR}{dt} \ll 1. \tag{1.1}$$

Clearly, if the parameter R is time independent, then the energy of the system E is conserved, and the system executes periodic motion. If the parameter R is time dependent, then the energy of the system is not conserved. However, because the parameter changes very slowly with time, the rate of the energy change dE/dt should also be very small. Averaging this rate of change over the motion period and eliminating the fast oscillation part, one can obtain the stable value of dE/dt denoting the slow change of the system energy; this value is proportional to the rate of the parameter dR/dt. In fact, the slowly varying quantity E is a function of the parameter

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R, and the dependence of E on R can be expressed in terms of their combination equaling a constant quantity. The quantity that remains invariant during the evolution of a system with a slowly varying parameter is called the "adiabatic invariant".

The Hamiltonian of the system is $H(p, q; R)$ (here, p and q are a pair of canonical variables corresponding to the generalized momentum and the generalized coordinate, respectively), and the derivative of energy versus time is

$$\frac{dE}{dt} = \frac{\partial H}{\partial R}\frac{dR}{dt}. \tag{1.2}$$

The right-hand side of the equation depends not only on the slow variable R but also on the fast variables p and q. To find the stable variation rule for the system energy, one must apply averaging to (1.2) over the entire motion period T. The parameter R changes very slowly; therefore, the change of dR/dt is also slow. As a result, one can bring dR/dt out of the averaging operation, i.e.,

$$\overline{\frac{dE}{dt}} = \overline{\frac{\partial H}{\partial R}}\frac{dR}{dt}. \tag{1.3}$$

Note that when one applies averaging to the function $\partial H/\partial R$, one considers only p and q as variables. In other words, this averaging operation is used when the parameter R remains constant. In explicit form, one has

$$\overline{\frac{\partial H}{\partial R}} = \frac{1}{T}\int_0^T \frac{\partial H}{\partial R}dt. \tag{1.4}$$

From the Hamilton equation $dq/dt = \partial H/\partial p$, one has

$$dt = \frac{dq}{\partial H/\partial p}. \tag{1.5}$$

Applying this equation, the integration (1.4) with respect to the time t can be replaced with integration with respect to the generalized coordinate q. Furthermore, one can rewrite the motion period T in the following integral form:

$$T = \int_0^T dt = \oint \frac{dq}{\partial H/\partial p}, \tag{1.6}$$

where $\oint$ denotes the integral over the whole region of the change of the generalized coordinate during one period of motion. (For rotation, the coordinate q becomes the rotation angle ϕ, and the integral is over a cycle, i.e., from 0 to 2π). Based on this transformation, the Eq. (1.3) can be rewritten as

$$\overline{\frac{dE}{dt}} = \frac{dR}{dt}\frac{\oint \frac{\partial H/\partial R}{\partial H/\partial p}dq}{\oint \frac{1}{\partial H/\partial p}dq}. \tag{1.7}$$

Note that the integral in this equation is applied along the trajectory of motion for a fixed parameter R. Clearly, when the motion follows such a trajectory, the Hamiltonian retains a constant E. As a result, the generalized momentum p can be expressed as a given function of the generalized coordinate q and the independent parameters E and R, i.e., $p(q; E, R)$. Then, by computing the derivative of $H(p, q; R) = E$ versus the parameter R, one has

$$\frac{\partial H}{\partial R} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial R} = \frac{dE}{dR} = 0, \tag{1.8}$$

i.e.,

$$\frac{\partial H/\partial R}{\partial H/\partial p} = -\frac{\partial p}{\partial R}. \tag{1.9}$$

Substituting this equation back into the integral in the numerator on the right-hand side of Eq. (1.7) and expressing the core of the denominator as $\partial p(q; E, R)/\partial E$, one obtains

$$\overline{\frac{dE}{dt}} = -\frac{dR}{dt}\frac{\oint \frac{\partial p}{\partial R} dq}{\oint \frac{\partial p}{\partial E} dq}, \tag{1.10}$$

i.e.,

$$\oint \left(\frac{\partial p}{\partial E}\overline{\frac{dE}{dt}} + \frac{\partial p}{\partial R}\frac{dR}{dt} \right) dq = 0. \tag{1.11}$$

One introduces the integral along the motion trajectory for the given parameters E and R:

$$I = \frac{1}{2\pi}\oint p dq. \tag{1.12}$$

Equation (1.11) can be expressed as

$$\overline{\frac{dI}{dt}} = 0. \tag{1.13}$$

This result implies that if the parameter R slowly varies with time, then I remains constant, i.e., I is the adiabatic invariant of the system.

We now employ the concept of phase space to show the geometric meaning of the integral in Eq. (1.12). For a system with one degree of freedom, the phase space simplifies to the phase plane spanned by the generalized coordinate p and the generalized momentum q, and the phase trajectory for the periodic motion is a closed orbit in this phase plane. The integral (1.12) along this orbit gives the area enclosed by the closed trajectory. Thus, the adiabatic invariant can be expressed in the following integral form:

$$I = \frac{1}{2\pi} \int \int dpdq. \tag{1.14}$$

We now discuss a near-integrable Hamiltonian for small perturbations and for slow (or "adiabatic") perturbations [2]. For small perturbations, the Hamiltonian has the general form

$$H = H_0(\boldsymbol{I}, t) + \varepsilon H_1(\boldsymbol{I}, \boldsymbol{\theta}, t) + \cdots, \tag{1.15}$$

where H_0 describes completely integrable motion, $\boldsymbol{I}$ and $\boldsymbol{\theta}$ are the N-dimensional actions and angles, and ε is a small parameter characterizing the magnitude of the nonintegrable part of H. For small perturbations, the derivatives of H_0 and H_1 are assumed to be of the same order as H_0 and H_1 themselves, i.e.,

$$\left|\frac{\partial H_0}{\partial t}\right| \sim |H_0|, \quad \left|\frac{\partial H_1}{\partial t}\right| \sim |H_1|, \quad \text{etc.} \tag{1.16}$$

For slow perturbations, the terms produced by differentiation are assumed to be smaller by order ε than the terms from which they are derived, e.g., for slow time variation,

$$\left|\frac{\partial H_0}{\partial t}\right| \sim \varepsilon |H_0|, \quad \text{etc.} \tag{1.17}$$

To keep track of this ordering, one can often insert the small parameter ε and write

$$H_0 = H_0(\varepsilon t) \tag{1.18}$$

such that

$$\frac{\partial H_0}{\partial t} = \varepsilon H_0', \tag{1.19}$$

where the prime notation denotes differentiation with respect to the argument $\tau = \varepsilon t$.

In this section, we are interested in systems for which the variation in all but one of the degrees of freedom, as well as in time, is slow [2]. Accordingly, one can write the Hamiltonian in the form

$$H = H_0(I, \varepsilon\boldsymbol{\eta}, \varepsilon t) + \varepsilon H_1(I, \theta, \varepsilon\boldsymbol{\eta}, \varepsilon t) + \cdots, \tag{1.20}$$

where I and θ are the action-angle variables for the unperturbed ($\varepsilon = 0$) motion in the single fast degree of freedom and $\boldsymbol{\eta} = (\boldsymbol{p}, \boldsymbol{q})$ are the "slow" canonical variables, not necessarily in action-angle form, for the remaining degrees of freedom. Since the system is effectively one-dimensional when $\varepsilon = 0$, this system is integrable, and I and θ can always be found. The small parameter ε in (1.20) "automatically" keeps track of the ordering when one differentiates H to construct the perturbation series; this parameter can be set to unity at the end of the calculation.

One can construct to first order the classical adiabatic invariant for the Hamiltonian (1.20). In zero order, the invariant is the action I associated with the fast degree of

freedom. To calculate the effect of the perturbation εH_1, one can find a transformation from $(I, \theta, \boldsymbol{\eta})$ to $(\bar{I}, \bar{\theta}, \bar{\boldsymbol{\eta}})$ such that the new Hamiltonian

$$\bar{H} = \bar{H}_0 + \varepsilon \bar{H}_1 + \cdots \tag{1.21}$$

is independent of the "fast" phase variable $\bar{\theta}$. Introducing the near-identity generating function

$$S = \bar{I}\theta + \bar{\boldsymbol{p}} \cdot \boldsymbol{q} + \varepsilon S_1(\bar{I}, \theta, \bar{\boldsymbol{p}}, \boldsymbol{q}, t) + \cdots, \tag{1.22}$$

one has, to first order, the transformations

$$I = \bar{I} + \varepsilon \frac{\partial S_1}{\partial \bar{\theta}}, \tag{1.23}$$

$$\theta = \bar{\theta} - \varepsilon \frac{\partial S_1}{\partial \bar{I}}, \tag{1.24}$$

$$\boldsymbol{p} = \bar{\boldsymbol{p}} + \varepsilon \frac{\partial S_1}{\partial \bar{\boldsymbol{q}}}, \tag{1.25}$$

$$\boldsymbol{q} = \bar{\boldsymbol{q}} - \varepsilon \frac{\partial S_1}{\partial \bar{\boldsymbol{p}}}. \tag{1.26}$$

Inserting these into H_0 and expanding to first order in ε, one has

$$H_0(I, \varepsilon\boldsymbol{\eta}, \varepsilon t) = H_0(\bar{I}, \varepsilon\bar{\boldsymbol{\eta}}, \varepsilon t) + \varepsilon\omega \frac{\partial S_1}{\partial \bar{\theta}}, \tag{1.27}$$

where $\omega = \partial H_0 / \partial \bar{I}$ is the fast frequency. Note that the terms in

$$-\frac{\partial H_0}{\partial \bar{\boldsymbol{q}}} \cdot \frac{\partial S_1}{\partial \bar{\boldsymbol{p}}}, \quad \frac{\partial H_0}{\partial \bar{\boldsymbol{p}}} \cdot \frac{\partial S_1}{\partial \bar{\boldsymbol{q}}} \tag{1.28}$$

are second order in ε and can be neglected. The canonical transformation equation is

$$\bar{H}(\bar{I}, \bar{\theta}, \varepsilon\bar{\boldsymbol{\eta}}, \varepsilon t) = H(I, \theta, \varepsilon\boldsymbol{\eta}, \varepsilon t) + \varepsilon \frac{\partial S(\bar{I}, \theta, \varepsilon\bar{\boldsymbol{p}}, \varepsilon\boldsymbol{q}, \varepsilon t)}{\partial(\varepsilon t)}. \tag{1.29}$$

Expanding $\bar{H}$, H, and S using the above transformations and equating like powers of ε, to zero order, one obtains

$$\bar{H}_0(\bar{I}, \varepsilon\bar{\boldsymbol{\eta}}, \varepsilon t) = H_0(\bar{I}, \varepsilon\bar{\boldsymbol{\eta}}, \varepsilon t), \tag{1.30}$$

and to first order, one has

$$\bar{H}_1(\bar{I}, \bar{\theta}, \varepsilon\bar{\boldsymbol{\eta}}, \varepsilon t) = \omega \frac{\partial S_1}{\partial \bar{\theta}} + H_1(\bar{I}, \bar{\theta}, \varepsilon\bar{\boldsymbol{\eta}}, \varepsilon t), \tag{1.31}$$

where $S_1 = S_1(\bar{I}, \bar{\theta}, \varepsilon\bar{\boldsymbol{\eta}}, \varepsilon t)$. Again, the term $\partial S_1/\partial t$ in (1.29) is second order and has been omitted from (1.31).

To make $\bar{H}_1$ independent of $\bar{\theta}$, one can choose S_1 to eliminate the oscillating part (in $\bar{\theta}$) of H_1. Holding the slow angle variables fixed, one can define the average over $\bar{\theta}$ alone as

$$\langle \bar{H}_1 \rangle_{\bar{\theta}} = \frac{1}{2\pi} \int_0^{2\pi} H_1 d\bar{\theta} \tag{1.32}$$

and the oscillating part over $\bar{\theta}$ as

$$\{H_1\}_{\bar{\theta}} = H_1 - \langle \bar{H}_1 \rangle_{\bar{\theta}}. \tag{1.33}$$

Separating (1.31) into its average and oscillating parts yields, for $\bar{H}$ to first order,

$$\bar{H}(\bar{I}, \varepsilon\bar{\boldsymbol{\eta}}, \varepsilon t) = H_0 + \varepsilon \langle \bar{H}_1 \rangle_{\bar{\theta}} \tag{1.34}$$

and, for S_1,

$$\omega \frac{\partial S_1}{\partial \bar{\theta}} = -\{H_1\}_{\bar{\theta}}, \tag{1.35}$$

which is easily integrated. To zero order, the adiabatic invariant is I. To first order, the new invariant is $\bar{I}$, which is given in terms of the old variables as

$$\bar{I}(I, \varepsilon\boldsymbol{\eta}, \varepsilon t) = I - \varepsilon \frac{\partial S_1}{\partial \theta}. \tag{1.36}$$

Substituting (1.35) into (1.36) and writing θ for the dummy variable $\bar{\theta}$, one obtains

$$\bar{I} = I + \frac{\varepsilon \{H_1\}_{\theta}}{\omega}. \tag{1.37}$$

In fact, any function of $\bar{I}$ can be chosen as the adiabatic invariant.

1.1.2 Adiabatic Geometric Angle—Hannay Angle

In the phase space (p, q), the particle races around a track (i.e., a contour of the instantaneous Hamiltonian $H(p, q; t)$) of fixed area $2\pi I$ (with I being the action variable) but slowly changing shape. Given the rule of conservation of action I for the contour that the particle lies on, it seems natural to explore the development of the complementary variable, the angle variable, which describes the location of the particle on the contour; that is, one might ask how many circuits the particle has made [3].

When the Hamiltonian $H(p, q; t)$ is "frozen", the instantaneous frequency of motion of the particle that can be obtained can be expressed by the derivative $(2\pi)^{-1} dH/dI$. Thus, it is tempting to write the total angle traversed over time T as simply

$$\int_0^T \frac{dH(p(t), q(t); t)}{dI} dt = \int_0^T \frac{dH(I; t)}{dI} dt, \tag{1.38}$$

where, in the last form, H is considered a function $H(I; t)$ of the area of its contours and the adiabatic invariant and $I(t) = I$ (constant) is invoked. Since the angle variable can be changed by virtue of the changing (I, θ) coordinate system in phase space, this framework (1.38) is obviously incomplete. To reveal the true structure of the situation, it is necessary to interpret the time dependence of the Hamiltonian function (and the (I, θ) coordinate system) as being produced by carrying them along a path $\boldsymbol{R}(t)$ in a parameter space $\boldsymbol{R} \equiv (R_1, R_2, \ldots)$ of two or more dimensions in which the functions $H(p, q; \boldsymbol{R})$, $I(p, q; \boldsymbol{R})$, and $\theta(p, q; \boldsymbol{R})$ are uniquely defined. The point of making $\boldsymbol{R}$ more than one dimensional is that one wishes to consider closed evolutions $\boldsymbol{R}(T) = \boldsymbol{R}(0)$ in which $\boldsymbol{R}(t)$ forms a loop. With just one parameter (the length of the shortening pendulum, for instance), the only way to restore the original length is to reverse the shortening, in which case the holonomy effect is not realized.

The exact rates of change of a particle's action and angle in this framework are as follows:

$$\dot{I} = -\frac{\partial H}{\partial \theta} + \dot{\boldsymbol{R}} \cdot \frac{\partial I}{\partial \boldsymbol{R}} = \dot{\boldsymbol{R}} \cdot \frac{\partial I}{\partial \boldsymbol{R}}, \tag{1.39}$$

$$\dot{\theta} = \frac{\partial H}{\partial I} + \dot{\boldsymbol{R}} \cdot \frac{\partial \theta}{\partial \boldsymbol{R}}, \tag{1.40}$$

where the overdot denotes the time derivative. The last terms in both (1.39) and (1.40) are the rates of change of action and angle coordinates at a fixed point (p, q) in phase space. These equations for nonadiabatic evolution of the Hamiltonian lead to changes in both I and θ that depend on the trajectory selected, i.e., on the initial values of both I and θ. For adiabatic evolution, the equations become

$$\dot{I} = 0 + \dot{\boldsymbol{R}} \cdot \left\langle \frac{\partial I}{\partial \boldsymbol{R}} \right\rangle = 0, \tag{1.41}$$

$$\dot{\theta} = \frac{\partial H}{\partial I} + \dot{\boldsymbol{R}} \cdot \left\langle \frac{\partial \theta}{\partial \boldsymbol{R}} \right\rangle, \tag{1.42}$$

where the average brackets denote the average around the Hamiltonian contour on which the particle lies. For any function $f(p, q)$, one can define a function $\langle f \rangle$ of action I by

$$\langle f \rangle = \frac{1}{2\pi} \oint_{\text{contour through}(p,q)} f d\theta \equiv \frac{1}{2\pi} \int f(p, q) \delta(I(p, q) - I) dp dq. \tag{1.43}$$

The average in (1.41) vanishes identically by Liouville's theorem and yields $\dot{I} = 0$ as required. There is no reason, however, why the average in (1.42) should vanish; therefore, the integration of this equation gives the dynamical angle change anticipated in (1.38) plus the additional angular change (namely, $\Delta\theta$) that we are interested in:

$$\Delta\theta = \int \dot{\boldsymbol{R}} \cdot \left\langle \frac{\partial\theta(p(t), q(t); \boldsymbol{R}(t))}{\partial \boldsymbol{R}} \right\rangle dt = \int \left\langle \frac{\partial\theta}{\partial \boldsymbol{R}} \right\rangle \cdot d\boldsymbol{R}. \tag{1.44}$$

In the last expression, time t has been completely eliminated because by definition (1.43), the average is a function of a conserved parameter (i.e., the initial action I). A different field $\langle \partial\theta/\partial \boldsymbol{R} \rangle$ exists for each I, on which $\Delta\theta$ therefore depends. This parameter does not depend on the initial angle.

For fixed I, the field $\langle \partial\theta/\partial \boldsymbol{R} \rangle$ depends on the angle variable coordination $\theta(p, q; \boldsymbol{R})$, which is to some extent arbitrary. Unlike the lines of constant action $I(p, q; \boldsymbol{R})$, which, for fixed $\boldsymbol{R}$, are fully determined as the contours of the Hamiltonian $H(p, q; \boldsymbol{R})$, the lines of constant angle are specified only after one of them (say $\theta = 0$) is chosen. This one, and thus all the others, can be arbitrarily twisted into a spiral, for example. Thus, the angle variable change $\Delta\theta$ inevitably depends on the angle coordinates chosen for the initial and final parameters $\boldsymbol{R}(0)$ and $\boldsymbol{R}(T)$. Only if these coordinate systems are identical, which in turn requires $\boldsymbol{R}(0) = \boldsymbol{R}(T)$ (barring especially favorable circumstances), can one expect to make coordinate-independent statements about $\Delta\theta$. The evolutions must be closed loops.

1.1.3 Example I: One-Dimensional Harmonic Oscillator

As the first example, we introduce the one-dimensional harmonic oscillator. To show the general method, we calculate to first order the adiabatic invariant for the slowly varying linear oscillator [2], whose Hamiltonian is

$$H_{ho} = \frac{1}{2} g(\tau) p^2 + \frac{1}{2} f(\tau) q^2, \tag{1.45}$$

where the small parameter ε has been inserted using $\tau = \varepsilon t$ to order the perturbation series. To prepare the system, one can transform to the action-angle variables I and θ of $H_0 = H_{ho}(\varepsilon = 0)$. In treating the harmonic oscillator, we adopt the generating function $F(q, \theta, \tau)$ given by

$$F = \frac{1}{2} R q^2 \cot\theta, \tag{1.46}$$

where $R(\tau) = \sqrt{f/g}$. Using $p = \partial F/\partial q$, $I = -\partial F/\partial\theta$, and $H(I, \theta, \tau) = H_{ho}(p, q, \tau) + \partial F(q, \theta, \tau)/\partial t$, one obtains $q = \sqrt{2I/R}\sin\theta$ and $p = \sqrt{2I/R}\cos\theta$. As a result, the transformed Hamiltonian is

$$H = \omega_0 I + \varepsilon \frac{1}{2} \frac{R'}{R} I \sin 2\theta, \tag{1.47}$$

where $\omega_0(\tau) = \sqrt{fg}$. The prime notation denotes differentiation with respect to τ. To zero order, the adiabatic invariant is just

$$I = \frac{H_0}{\omega_0} = \text{const.}. \tag{1.48}$$

This result implies that the number of quanta $\hbar\omega_0$ is conserved as the frequency of oscillation slowly varies. To find the first-order invariant, we apply (1.37) to (1.47) and obtain

$$\bar{I} = I(1 + \varepsilon P \sin 2\theta) = \text{const.}, \tag{1.49}$$

with $P(\varepsilon t) = R'/(2\omega_0 R)$. This expression shows that to first order, I contains a small component oscillating at twice the frequency of the fast variable. One can verify the constancy of the quantity $\bar{I}$ by taking the time derivative of (1.49),

$$\dot{\bar{I}} = \dot{I} + \varepsilon \dot{P} I \sin 2\theta + 2\varepsilon P I \cos 2\theta + \mathcal{O}(\varepsilon^2), \tag{1.50}$$

where the overdot denotes d/dt. If one applies Hamilton's equations to (1.47), the first and third terms on the right cancel, leaving to first order in ε

$$\dot{\bar{I}} = \varepsilon \dot{P} I \sin 2\theta. \tag{1.51}$$

When the standard slow perturbation ordering $\dot{P} \sim \varepsilon P$ is used, $\dot{\bar{I}}$ is of order ε^2. Therefore, $\bar{I}$ is a first-order invariant.

1.1.4 Example II: Celestial Two-Body Problem

The two-body problem is a special case of the motion of a particle in a central force field. Because of the conservation of angular momentum, the motion occurs on an invariant plane [4]. In the plane polar coordinates, the Hamiltonian for the motion of the particle is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + U(r), \tag{1.52}$$

where m is the mass of the particle, the momentum components are $p_r = m\dot{r}$ and $p_\theta = mr^2\dot{\theta}$ (with the overdot denoting the time derivative), and $U(r)$ is the potential function for the central force field. The Hamilton-Jacobi equation reads

$$\frac{1}{2m}\left[\left(\frac{\partial S_r}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial S_\theta}{\partial \theta}\right)^2\right]+U(r)=E, \tag{1.53}$$

where $S(r,\theta)=S_r(r)+S_\theta(\theta)$ is a variable-separated generating function. One can rewrite the above Hamilton-Jacobi equation as follows:

$$\left(\frac{\partial S_\theta}{\partial \theta}\right)^2=2mr^2\left[E-\frac{1}{2m}\left(\frac{\partial S_r}{\partial r}\right)^2-U(r)\right]. \tag{1.54}$$

The arbitrary choices of both θ and r require

$$\left(\frac{\partial S_r}{\partial r}\right)^2=2m\left[E-U(r)\right]-\frac{l^2}{r^2}, \tag{1.55}$$

$$\left(\frac{\partial S_\theta}{\partial \theta}\right)^2=l^2. \tag{1.56}$$

Applying the definitions of action-angle variables, i.e.,

$$I_r=\frac{1}{2\pi}\oint p_r dr=\oint \frac{\partial S_r}{\partial r}dr, \tag{1.57}$$

$$I_\theta=\frac{1}{2\pi}\oint p_\theta d\theta=\oint \frac{\partial S_\theta}{\partial \theta}d\theta, \tag{1.58}$$

one has

$$I_r=\frac{1}{2\pi}\int_0^{2\pi}\left[2m(E-U(r))-\frac{l^2}{r^2}\right]^{1/2}dr, \tag{1.59}$$

$$I_\theta=\frac{1}{2\pi}\int_0^{2\pi} l d\theta=l. \tag{1.60}$$

If the central force field takes the form $U(r)=-\mu/r$ with a constant quantity μ, then the action variable becomes

$$I_r=-l+\frac{\mu}{2}\sqrt{\frac{2m}{-E}}. \tag{1.61}$$

The corresponding Hamiltonian is given by

$$H=E=-\frac{m\mu^2}{2(I_r+I_\theta)^2}. \tag{1.62}$$

The frequency of particle motion in both the r and θ directions is

$$\omega = \frac{\partial H}{\partial I_r} = \frac{\partial H}{\partial I_\theta} = \frac{m\mu^2}{(I_r + I_\theta)^3}. \tag{1.63}$$

One finds that the motion of the particle in an inverse-square force field is simply due to the identical motion frequency in two directions. Thus, we can view the two-body motion with a Newton inverse-square gravity as a reduced motion. Substituting the familiar two-body elliptic motion energy $E' = E/m = -\mu/(2a)$ (with a being the semimajor axis) back into the equation for the action variable I_r, one has

$$I_r + I_\theta = \sqrt{\mu a} = L, \tag{1.64}$$

which implies conservation of the angular momentum L. Combining this variable with the frequency equation and Kepler's third law $n^2 a^3 = \mu$, one has

$$\omega = n = \frac{2\pi}{T}, \tag{1.65}$$

where T is the period of the elliptic motion; thus, n is the angular speed. For typical celestial motions, T is often large, and thus, ω is very small. This fact implies that the motions are nearly adiabatic.

1.1.5 Example III: Foucault Pendulum

The Foucault pendulum provides a simple and effective example of the anholonomy present in an adiabatically cycled system because the parameter space used to describe its motion is the physical space in which it moves. The Foucault pendulum is commonly considered from a rectangular coordinate system (x, y, z) fixed to the rotating Earth with its origin at the pendulum bob in its rest position and its z-axis pointing outward from the Earth along the axis or rest orientation of the pendulum. The x- and y-axes point south and east, respectively. The pendulum is treated in the small-oscillation limit, and the fictitious centrifugal force proportional to the square of the angular frequency of the Earth (and hence very small) can be neglected. The Foucault pendulum is then characterized as a simple two-dimensional harmonic pendulum with an added Coriolis force. The Lagrangian for this system reads [5]

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{m\Omega^2}{2}(x^2 + y^2) + m\omega_z(x\dot{y} - y\dot{x}), \tag{1.66}$$

where the overdot denotes the time derivative, m is the mass of the pendulum bob, $\Omega = \sqrt{g/l}$ is the angular frequency of the unperturbed pendulum, with g being the acceleration of gravity and l being the length of the pendulum, and $\omega_z = \omega \cos\theta$ is the z-component of the angular frequency of the Earth (i.e., ω) at colatitude θ. Since