Mathematical and Analytical Techniques with Applications to Engineering

Il Han Park

Design Sensitivity Analysis and Optimization of Electromagnetic Systems

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Dedicated to
Miree, Seoyeon and Hajin

Preface

Design is the process of properly placing materials in a space to obtain a desired performance. The placement of materials sets a device's shape, which determines its performance. The performance of the electromagnetic system is also determined by its shape. But, the performance of the electromagnetic system is expressed with the electromagnetic field; its performance is indirectly related to the shape. This book presents the design sensitivity analysis for the electromagnetic system, which is on the relation between the performance and the geometric design variables. The design sensitivity, which is the variation rate of the system performance with respect to the design variables, provides information on how the design variables affect the performance.

The electromagnetic systems are diverse in type and size, ranging from micro-electronic devices to large power apparatus. For analysis of such various electromagnetic systems, the finite element method is popular among the engineers, researchers and graduate students. But, the finite element code is an analysis tool not a design tool; the design process using the finite element code needs much trial and error, which requires considerable time and effort.

In the mechanical engineering, a large number of research papers and books for the optimal structure design are found and some commercial codes with the design sensitivity analysis are available. By contrast, there are only few books on the optimal design of the electromagnetic system. This book may be the first one devoted to the sensitivity analysis for the electromagnetic system.

This book aims to cover the theory and application of the shape sensitivity analysis for the electromagnetic system in a unified manner. The focus is on the continuum sensitivity analysis, which has great advantages over the other sensitivity methods: the finite difference method and the discrete method. The continuum design sensitivity is obtained as an analytical form; thus, it makes it easy to calculate the sensitivity and provides accurate sensitivity. In addition, it can be easily implemented with existing numerical analysis codes such as finite element method and boundary element method since its sensitivity calculation does not depend on the analysis method.

The continuum shape sensitivity for the electromagnetic system is derived by taking the material derivative of the design performance and the variational state equation. In this differentiation, the Lagrange multiplier method is introduced to deal with the implicit equality constraint of the variational state equation. An adjoint variable technique is also employed to express explicitly the sensitivity in terms of the design variables. The variational identities are used to transform the sensitivity of a domain integral into a boundary integral on the design surface. This continuum shape sensitivity analysis, which is applied to four electromagnetic systems: the electrostatic system, the magnetostatic system, the eddy current system and the DC conductor system, provides the sensitivity formulas for each electromagnetic system. The sensitivity formulas so obtained are the general three-dimensional ones of an analytical form. These analytical sensitivity formulas provides not only physical insight but also great advantages in numerical implementation.

The book contains eight chapters and four appendices. In Chap. 1 a brief review of optimal design process and design steps for the electromagnetic system is presented and the geometric design variables are classified. The Maxwell's equations and the governing differential equations are introduced and the characteristics of the electromagnetic system are described for comparison with the structural system in the mechanical engineering. An overview of design sensitivity calculation method is also provided.

In Chap. 2, the four variational state equations for the electrostatic system, the magnetostatic system, the eddy current system and the DC current-carrying conductor are formulated by the variational method of the virtual work principle. The variational equations are derived from the differential equations with boundary conditions and they are used for deriving the continuum sensitivity formulas for the four electromagnetic systems in Chaps. 3–6.

In Chap. 3, the general three-dimensional continuum shape sensitivities for the electrostatic system are derived by using the material derivative and are applied to design problems. The shape sensitivity for the electrostatic system is classified into two types according the design variable. One is for the design problem of outer boundary and the other is for the design problem of interface. Each one has also two different types of objective functions: domain integral and system energy. The sensitivity for the system energy is examined in the electric-circuit point of view to show its sign dependency on the source condition and to derive the capacitance sensitivity. The general sensitivity formulas are applied to analytical and numerical design examples to be validated.

In Chap. 4, the general three-dimensional continuum shape sensitivities for the magnetostatic system are derived and are applied to design problems. Unlike in the electrostatic system, the shape sensitivity for the magnetostatic system has only one type for the design problem of interface. The interface design problem has also two different types of objective functions: domain integral and system energy. The magnetostatic system may have four different material regions: ferromagnetic material, permanent magnet, source current, air; thus, the general sensitivity is expressed as the sensitivity formulas for nine interfaces. The system energy sensitivity is derived in the electric-circuit point of view, and it is used to the inductance sensitivity. The general sensitivity formulas are applied to analytical and numerical design examples to be validated.

In Chap. 5, the three-dimensional continuum shape sensitivities for the eddy current system are derived and are applied to design problems. Like in the magnetostatic system, the shape sensitivity for the eddy current system has only one type for the design problem of interface. The interface design problem has also two different types of objective functions: domain integral and system power. The eddy current system may have four different material regions: ferromagnetic material, conductive material, source current, air; thus, the general sensitivity is expressed as the sensitivity formulas for nine interfaces. The system power sensitivity is derived in the electric-circuit point of view, and then the inductance sensitivity and the resistance sensitivity are derived. The two sensitivity formulas are applied to numerical examples to be validated.

In Chap. 6, the general three-dimensional continuum shape sensitivity for the DC conductor system is derived and are applied to design problems. The design problem of the DC conductor system is similar to that of the electrostatic system, but it has only the design variable of outer boundary. The design problem of outer boundary has also two different types of objective functions: domain integral and system loss power. The derived sensitivity formula is expressed as a boundary integral of Dirichlet boundary and Neumann boundary. The loss power sensitivity is used to derive the resistance sensitivity. The general sensitivity formulas are applied to analytical and numerical design examples to be validated.

The shape optimal design using the sensitivity requires the optimization algorithms and the successive geometry modeling for evolving shapes. For this purpose, Chap. 7 introduces the level set method. The level set method expresses the shape variation with the velocity field; thus, it matches well with the continuum shape sensitivity, whose sensitivity formulas are expressed with the velocity. The level set method and the continuum sensitivity are coupled to transform the usual iterative optimization into the solving process of the level set equation, which is the transient analysis in the time domain. The adaptive level set method and the artificial diffusion technique are also presented for solving the coupled level set equation with existing finite element codes.

In Chap. 8, the hole and the dot sensitivity analyses are presented for the topology optimization of the electrostatic and the magnetostatic systems. The hole sensitivity formulas in the dielectric and the magnetic material regions are derived by using a hole sensitivity concept and the continuum sensitivity in the electrostatic and the magnetostatic system. The dot sensitivity formulas in the dielectric and the magnetic material regions are also derived by using a dot sensitivity concept and the continuum sensitivity. The derived hole and the dot sensitivity formulas are applied to numerical examples to show its usefulness.

The four Appendices A-D provide more analytical and numerical examples for the four electromagnetic systems, most of which are ones for other coordinates and interfaces not included in the examples of the Chaps. 3–6.

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Contents

Chapter 1
Introduction

1.1 Optimal Design Process

Optimal design of electromagnetic system consists of procedures to improve the performance by evolving design variables. There are many kinds of performance measures such as electric/magnetic field distribution, system energy, system power, force/torque, energy loss, equivalent circuit parameters, induced voltage, material volume, etc. Moreover, the electromagnetic system has various constraints and design variables since it is composed of many different materials such as dielectrics, conductor, insulator, charge, magnetic material, current, permanent magnet and electrolet.

The structure of the electromagnetic system is usually so complex and sophisticated that its design process has been dependent on the engineer's experience and intuition. A systematic design process will enable the designer to develop an improved device with less time and cost. For this purpose, simulation-based design is efficient for development and production of the better electromagnetic devices [1].

The simulation-based design consists of modeling, system analysis, sensitivity analysis, and optimization. The optimal design process is shown in Fig. 1.1, where the system analysis and the sensitivity analysis are important procedures [2].

1.2 Design Steps of Electromagnetic System

Choosing the design variables in system modeling is an important step to a successful design. It is often difficult to identify the design variables that have substantial influence on the performance. It is mainly due to system structure's complexity. Wrong choice of the design variables, which limits the size of design space for searching the design variables, results in a wrong design.

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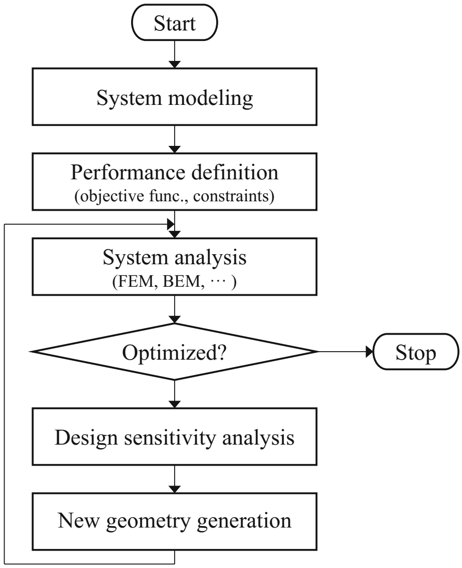

Fig. 1.1 Design process of electromagnetic system

The result of system analysis is used to evaluate the system performance. Nowadays, most of the system analyses for electromagnetic devices are carried out using numerical methods such as the finite element method, the boundary element method, etc. The finite element method, which is widely applicable to various electromagnetic systems including nonlinear system, provides reliable and accurate result; it is most frequently employed by the researchers and engineers. This book also employs the finite element method to analyze the electromagnetic system.

The objective function (performance measure, cost function), which is a criterion to ascertain whether the design is satisfactory or not, is evaluated with the results of the system analysis. Definition of the objective function, which has great influence on sensitivity evaluation and convergence, is also important to obtain a successful design. For example, the force/torque of the electromechanical system can be easily controlled with the objective function of the system energy in comparison with the objective function of force/torque, which often leads to difficulty and complexity of sensitivity evaluation. During the design optimization process, the objective function is minimized or maximized by the optimization algorithm of the mathematical programming.

The objective function for the electromagnetic system is usually nonlinear to the design variables. It is common to use the gradient-based method for the optimization algorithm. The gradient, which is called the design sensitivity of the objective function, is obtained by differentiating the objective function with respect to the design variables. The gradient information is used as the searching direction in the design space. The sensitivity analysis is the main concern of this book. The sensitivity, which means the effect of the design variables on the objective function, provides the information on how the design variables have influence on the objective function. The sensitivity information can be also used for identifying the key design variables.

The optimization problems are usually subject to constraints. If a design satisfies given constraints, it is called feasible design. If not, it is infeasible design. Whereas some design problems have simple constraints such as the upper/lower limits of the design variables and the constant volume of used material, others have complex constraints that are indirectly affected by the design variable. For example, when an electric field at a point of an electrostatic system is given as a constraint, it is usually impossible to explicitly express that with the design variables. This kind of constraint is called implicit constraint. The constraint of an explicit function of the design variables is simple, whereas the constraint of an implicit function of the design variables is complex to deal with. In this book, the electromagnetic state equation of the variational form is taken as a constraint and incorporated into the objective function by the Lagrange multiplier method.

1.3 Design Variables

The optimal design is the searching process for the better design variables providing the desired performance. Unless the design variables are well defined, the design space is limited so that a good design is not obtained no matter how accurate solution is used. For example, while a coarse geometrical modeling does not matter for an insensitive design region, a fine geometrical modeling is needed for a sensitive design region.

The material of the electromagnetic systems can be classified into two categories: active ones and passive ones. The active materials generate the source field, whereas the passive materials only react to the external field. The ferromagnetic material, the dielectrics, the electric conductor, the electric insulator, and the air belong to the passive materials. The electric charge, the electric current, the permanent magnet, and the electrolet are the active materials. For the optimal design of the electromagnetic system, the property of these materials is not taken as the design variable in this book, since it is neither controllable nor continuous in the available materials.

In this sensitivity analysis, only the geometric parameters of the material structure are taken as the design variables. The geometric design variables are classified into three categories: size, shape and topology as shown in Fig. 1.2. The size design variables such as width, height, depth, radius, angle, etc. are used for simple structures. The shape design variables, which cannot be defined with the size design variables, are used for more complex geometry. During the shape design process, its initial topology is maintained. The topology design variables are related to system layout. When a new material domain is generated outside a given material domain or an air hole is generated inside the material domain, the system topology changes. Recently, some topology design methods have been introduced to the electromagnetic system.

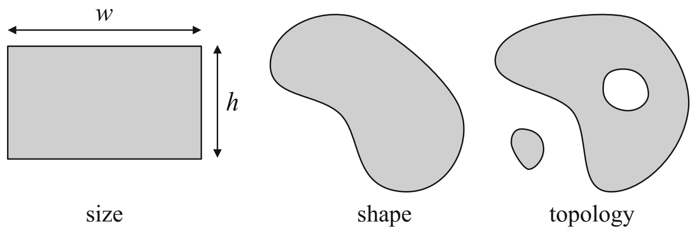

Fig. 1.2 Geometric design variables

Mathematically, the size design variables are a subset of the shape design variables, and the shape design variables are also a subset of the topology design variables. This book deals with the shape and topology design of the electromagnetic system. The optimal shape design is carried out by using the shape design sensitivity, which is derived as the analytical integral forms in the subsequent chapters. The sensitivity with respect to the size design variables are easily calculated by the design variable parametrization, which relates the size design variable to the shape design sensitivity. The topology sensitivity can be also derived with the concept of topology sensitivity and the shape design sensitivity.

1.4 Equations and Characteristics of Electromagnetic Systems

The electromagnetic systems, which are represented with the Maxwell's equations, are usually modeled by the partial differential equations for the electric and magnetic potentials. The electromagnetic systems are classified into four systems: electrostatic system, magnetostatic system, eddy current system, and wave system. These four systems are also represented with the governing partial differential equations: elliptic, parabolic and hyperbolic equations [3]. The governing equations for the electrostatic and magnetostatic systems are elliptic, and the ones for the eddy current and the wave systems are parabolic and hyperbolic, respectively. The understanding of the characteristics of the electromagnetic system is important to the development of the sensitivity analysis. In particular, recognition of differences between the electromagnetic system and the mechanical structure is helpful.

1.4.1 Maxwell's Equations and Governing Equations

The electromagnetic systems are generally represented with the Maxwell's equations and the constitutive relations [4–7]. The Maxwell equations in the differential form are:

$$\nabla \cdot \mathbf{D} = \rho \tag{1.4.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.4.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.4.3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{1.4.4}$$

where $\mathbf{D}$ is the electric flux density, ρ the volume charge density, $\mathbf{E}$ the electric field intensity, $\mathbf{H}$ the magnetic field intensity, $\mathbf{J}$ the volume current density and $\mathbf{B}$ the magnetic flux density. The constitutive relations are given as

$$\mathbf{D} = \varepsilon \mathbf{E} + \mathbf{P}_o \tag{1.4.5}$$

$$\mathbf{H} = v\mathbf{B} - \mathbf{M}_o \tag{1.4.6}$$

$$\mathbf{J} = \sigma \mathbf{E} \tag{1.4.7}$$

where ε is the electric permittivity, $\mathbf{P}_o$ the permanent polarization, v the magnetic reluctivity, $\mathbf{M}_o$ the permanent magnetization, and σ the electric conductivity. This book deals with only the low frequency system, where the displacement current in (1.4.4) is ignored. The wave system is out of the scope of this book.

The electrostatic system is represented by two equations from Maxwell's equations and one constitutive relation;

$$\nabla \cdot \mathbf{D} = \rho \tag{1.4.8}$$

$$\nabla \times \mathbf{E} = \mathbf{0} \tag{1.4.9}$$

$$\mathbf{D} = \varepsilon \mathbf{E} + \mathbf{P}_o \tag{1.4.10}$$

With the electric scalar potential ϕ introduced, the governing partial differential equation for the electrostatic system is obtained as Poisson equation;

$$-\nabla \cdot \varepsilon \nabla \phi = \rho - \nabla \cdot \mathbf{P}_o \tag{1.4.11}$$

The magnetostatic system is represented by two equations from Maxwell's equations and one constitutive relation;

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{1.4.12}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.4.13}$$

$$\mathbf{H} = \nu\mathbf{B} - \mathbf{M}_o \tag{1.4.14}$$

With the magnetic vector potential **A** introduced, the governing partial differential equation for the magnetostatic system is obtained as

$$\nabla \times \nu(\nabla \times \mathbf{A}) = \mathbf{J} + \nabla \times \mathbf{M}_o \tag{1.4.15}$$

The eddy current system is represented by three equations from Maxwell's equations and two constitutive relations;

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_\mathrm{e} \tag{1.4.16}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.4.17}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.4.18}$$

$$\mathbf{H} = \nu\mathbf{B} \tag{1.4.19}$$

$$\mathbf{J}_e = \sigma\mathbf{E} \tag{1.4.20}$$

where **J** is the source current density and $\mathbf{J}_e$ is the eddy current density. By introducing the magnetic vector potential **A** and the electric scalar potential ϕ, the governing partial differential equation for the eddy current system is obtained as

$$\nabla \times \nu\nabla \times \mathbf{A} = \mathbf{J} - \sigma\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\phi\right) \tag{1.4.21}$$

In the linear eddy current system without the term$\nabla\phi$, when the harmonic source is considered, the governing equation for the steady state is expressed using the complex variables as

$$\nabla \times \nu\nabla \times \mathbf{A} + j\omega\sigma\mathbf{A} = \mathbf{J} \tag{1.4.22}$$

In this book, the DC current-carrying conductor is separately described. The DC current-carrying conductor, although it has the same form of governing equation as the electrostatic system, is quite different in physics and related to the resistance of the equivalent circuit, the Joule loss, the current distribution, etc. The DC current-carrying conductor is represented by two equations from Maxwell's equations and one constitutive relation;

$$\nabla \cdot \mathbf{J} = 0 \tag{1.4.23}$$

$$\nabla \times \mathbf{E} = \mathbf{0} \tag{1.4.24}$$

$$\mathbf{J} = \sigma \mathbf{E} \tag{1.4.25}$$

where (1.4.23) is the continuity equation, which is implicit in (1.4.4) of Maxwell's equations. With the electric scalar potential ϕ introduced, the governing partial differential equation for the DC current-carrying conductor is obtained as Laplace equation;

$$-\nabla \cdot \sigma \nabla \phi = 0 \tag{1.4.26}$$

1.4.2 Characteristics of Electromagnetic Systems

The shape design sensitivity analysis has been well developed for optimal design of mechanical structures, for which a large number of research results are found in books and papers. Such a wealth of research results is very helpful for the sensitivity analysis of the electromagnetic system. There are, however, some differences between the electromagnetic system and the mechanical structure. Recognition of them helps to develop the sensitivity analysis for the electromagnetic systems.

The electromagnetic field exists even in the vacuum, whereas the mechanical fields such as stress, strain, fluidic velocity exist only where the media exist [8, 9]. In the electromagnetic system, the electric/magnetic field exists not only inside the materials but also in the air near the materials. In electromagnetics, the vacuum and the air have the material properties of dielectric constant ε_0 and magnetic permeability μ_0. Thus, the design variable of the electromagnetic system is basically the interface where two different materials meet. For example, the design problem of a magnet, of which the design objective is to produce a uniform magnetic field, is to optimize the interface shape between the ferromagnetic material, the air, and the current coil [10].

The sources of the electromagnetic field can be charge, current, permanent magnet, or electrolet, whereas the source of the mechanical field is only the force. In addition, the sources of the electromagnetic system are usually supplied by the voltage source or the current source through the circuit terminal. The permanent magnet and the electrolet are, however, treated as materials with the source. The electromagnetic system, which is connected to the external circuit, is driven or controlled by the external circuit. Thus, it is important to extract its equivalent circuit parameter. If the equivalent circuit parameter representing the electromagnetic system is incorporated into the external circuit system, the operating characteristics of the electromagnetic system can be easily obtained by analyzing the circuit system.

There are two kinds of nonlinearity in the structural system: the material nonlinearity and the geometrical nonlinearity. The geometrical nonlinearity comes from deformation of the structure geometry. But there is only the material nonlinearity in the electromagnetic system, which appears mainly in the magnetic saturation of the ferromagnetic material [11].

1.5 Design Sensitivity Analysis

The sensitivity calculation is the mathematical procedure of obtaining the derivatives of the objective function with respect to the design variables. The sensitivity calculation of state variables with respect to the design variables often costs the major computational time for optimization process. It is, therefore, crucial to have an efficient algorithm for calculating the sensitivity.

There are two approaches to obtain the design sensitivity. One is finite difference method and the other is analytic differentiation method. The analytic differentiation method is also divided into two methods: discrete one and continuum one [1, 12].

1.5.1 Finite Difference Method

The finite difference method is the simplest technique to obtain the sensitivity. When the objective function is given as a function $F(p)$ of a design variable p, its sensitivity can be approximated by comparing $F(p)$ with $F(p + \Delta p)$ perturbed by Δp in the design variable;

$$\frac{\mathrm{d}F}{\mathrm{d}p} \simeq \frac{F(p + \Delta p) - F(p)}{\Delta p} \tag{1.5.1}$$

This approximation method is so easy to implement that it is popular among engineers. This approximate sensitivity is frequently compared with the sensitivity obtained by the other methods for evaluating their efficiency and accuracy. When design variables are numerous, the finite difference method is computationally expensive. When the number of design variable is n, it requires $n + 1$ times analyses of the system matrix equation. In addition, it has a serious problem of accuracy since its accuracy is strongly dependent on the perturbation size Δp. Too-small perturbation causes numerical truncation errors, and too-large perturbation leads to inaccurate results. Thus, this method is unsuitable for the shape design problem with many design variables. The number of design variables for the shape design is the number of all nodes on the design surface.

1.5.2 Discrete Method

The discrete method of the analytical approach is based on the discretized system equation, which is obtained by numerical analysis methods such as finite element method, boundary element method. [1, 13–19]. The state equation of discretized model is expressed as an algebraic matrix equation;

$$[K(p)][\phi] = [f(p)] \tag{1.5.2}$$

where $[K(p)]$ is the $n \times n$ system matrix, $[\phi]$ the $n \times 1$ state variable vector at nodes, $[f(p)]$ the $n \times 1$ source vector, and n the number of nodes for unknown state variables. The system matrix $[K(p)]$ is determined by the system geometry and the passive material property. The source vector $[f(p)]$ is determined by the system geometry and the active material property of the source. The change in the system geometry causes the changes of $[K(p)]$ and $[f(p)]$, which result in the change of the state variable $[\phi]$. Since the state variable $[\phi]$ depends on the design variable, it can be written as $[\phi(p)]$, which is implicitly affected by the design variable in the system Eq. (1.5.2).

The objective function is usually a function of the design variables and the state variable;

$$F = F\{[p], [\phi(p)]\} \tag{1.5.3}$$

where $[p]$ is the $m \times 1$ design variable vector, $[\phi(p)]$ the $n \times 1$ state variable vector, and m the number of design variables.

The derivative of the objective function is obtained by taking the derivative of (1.5.3) with respect to the design variable vector;

$$\frac{\mathrm{d}F}{\mathrm{d}[p]} = \frac{\partial F}{\partial [p]} + \frac{\partial F}{\partial [\phi]}\frac{\mathrm{d}[\phi]}{\mathrm{d}[p]} \tag{1.5.4}$$

In this sensitivity expression, the two partial derivatives of F are easily obtained since F is an explicit function of $[p]$ and $[\phi]$. But the derivative of the state variable in the second term needs some calculations since the state variable is implicitly related to the design variable in (1.5.2). By taking the derivative of (1.5.2) with respect to the design variable vector, the derivative of the state variable is obtained as

$$\frac{\mathrm{d}[\phi]}{\mathrm{d}[p]} = [K]^{-1}\frac{\partial}{\partial [p]}\left[[f] - [K][\tilde{\phi}]\right] \tag{1.5.5}$$

where $[\tilde{\phi}]$ is the solution of (1.5.2). By inserting (1.5.5) into (1.5.4), the sensitivity is expressed as

$$\frac{\mathrm{d}F}{\mathrm{d}[p]} = \frac{\partial F}{\partial [p]} + \frac{\partial F}{\partial [\phi]}[K]^{-1}\frac{\partial}{\partial [p]}\left[[f] - [K][\tilde{\phi}]\right] \tag{1.5.6}$$

After the derivative of the state variable is calculated in (1.5.5), its values can be inserted into (1.5.5). But it requires m times analyses of the system Eq. (1.5.2). This problem is solved by introduction of an adjoint variable technique, which requires only one analysis. An adjoint variable equation is introduced;

$$[K]^{\mathrm{T}}[\lambda] = \frac{\partial F}{\partial[\phi]^{\mathrm{T}}} \tag{1.5.7}$$

where $[\lambda]$ is the $n \times 1$ adjoint variable vector, which is the nodal values like the state variable [20, 21]. By using the adjoint variable Eq. (1.5.7), the sensitivity is obtained as

$$\frac{\mathrm{d}F}{\mathrm{d}[p]} = \frac{\partial F}{\partial[p]} + [\lambda]^{\mathrm{T}} \frac{\partial}{\partial[p]} \left[[f] - [K][\tilde{\phi}] \right] \tag{1.5.8}$$

The adjoint variable vector, which is calculated in (1.5.7), is inserted into (1.5.8) to provide the sensitivity.

On the other hand, this sensitivity can be also derived using the Lagrange multiplier method. The system matrix (1.5.2), which is a kind of equality constraint, is taken a constraint subject to the objective function (1.5.3). The augmented objective function G with the Lagrange multiplier is written as

$$G = F\{[p], [\phi(p)]\} + ([f(p)] - [K(p)][\phi])[\lambda]^{\mathrm{T}} \tag{1.5.9}$$

where $[\lambda]$ is the $n \times 1$ Lagrange multiplier vector. The derivative of objective function is obtained by taking the derivative of (1.5.9) with respect to the design variable vector;

$$\frac{\mathrm{d}G}{\mathrm{d}[p]} = \frac{\partial F}{\partial[p]} + \frac{\partial F}{\partial[\phi]}\frac{\mathrm{d}[\phi]}{\mathrm{d}[p]} + \left(\frac{\partial}{\partial[p]} \left[[f] - [K][\tilde{\phi}] \right] - [K]\frac{\mathrm{d}[\phi]}{\mathrm{d}[p]} \right) [\lambda]^{\mathrm{T}} + ([f] - [K][\phi]) \frac{\mathrm{d}[\lambda]^{\mathrm{T}}}{\mathrm{d}[p]} \tag{1.5.10}$$

The last term of this equation vanishes by the system state equation (1.5.2);

$$\frac{\mathrm{d}G}{\mathrm{d}[p]} = \frac{\partial F}{\partial[p]} + \frac{\partial F}{\partial[\phi]}\frac{\mathrm{d}[\phi]}{\mathrm{d}[p]} + \left(\frac{\partial}{\partial[p]} \left[[f] - [K][\tilde{\phi}] \right] - [K]\frac{\mathrm{d}[\phi]}{\mathrm{d}[p]} \right) [\lambda]^{\mathrm{T}} \tag{1.5.11}$$

In order to avoid the calculation of $\frac{\mathrm{d}[\phi]}{\mathrm{d}[p]}$ and explicitly express this equation with the design variable, an adjoint equation is introduced:

$$[K]^{\mathrm{T}}[\lambda] = \frac{\partial F}{\partial[\phi]^{\mathrm{T}}} \tag{1.5.12}$$

where $[\lambda]$ is the adjoint variable vector, which is the Lagrange multiplier in (1.5.9). Inserting the relation (1.5.12) into (1.5.11) provides the sensitivity:

$$\frac{\mathrm{d}G}{\mathrm{d}[p]} = \frac{\partial F}{\partial [p]} + [\lambda]^{\mathrm{T}} \frac{\partial}{\partial [p]} \left[[f] - [K][\tilde{\phi}] \right] \tag{1.5.13}$$

This sensitivity is the same as the (1.5.8).

The Lagrange multiplier method is also used for the continuum method in the subsequent chapters. The discrete method is relatively simple to understand since the implicit relation between the state variable and the design variable is clearly shown. The analogy between the discrete method and the continuum method is helpful in developing the continuum sensitivity for the electromagnetic system.

The above sensitivity calculation by the discrete method is summarized as

(a) solve the state variable Eq. (1.5.2) for $[\phi]$.
(b) solve the adjoint variable Eq. (1.5.7) for $[\lambda]$.
(c) calculate the sensitivity (1.5.8) using the obtained $[\phi]$ and $[\lambda]$.

This sensitivity calculation requires only two analyses for the state and adjoint variables. In the adjoint equation, its source term in the right-hand side is easily obtained since the F is an explicit function of $[\phi]$. But the computation of $\frac{\partial [f]}{\partial [p]}$ and $\frac{\partial [K]}{\partial [p]}$ is dependent on discretization since $[K]$ and $[f]$, which are assembled with the element matrices, depend on the element such as the shape function and the mesh data. Thus, their computation requires access to the source code of the analysis program, which makes it difficult to implement the numerical program. It is unfortunate that most of the commercial programs do not provide access to the source code. It is desired to develop a sensitivity evaluation method that does not depend on discretization nor requires access to the inside of the source code.

1.5.3 Continuum Method

In the continuum method, the shape sensitivity is derived using the material derivative concept and the variational formulation for the governing equation of electromagnetic system. The continuum method is the core subject of this book. The material derivative concept of continuum mechanics is employed to relate the shape variation of electromagnetic system to the objective function [22–27]. For general application, the objective function is defined as arbitrary function of the state variables. The electromagnetic system is represented with the variational equation of the continuous model. This variational state equation for the electromagnetic system, which holds regardless of the shape variation, is taken as an equality constraint. For a systematic derivation of the continuum sensitivity, the Lagrange multiplier method is used for the equality constraint. The constraint of the variational state equation is added to the objective function to provide an augmented objective function. By taking the material derivative of this augmented objective function and using the variational identities, the continuum sensitivity formula is obtained. This shape sensitivity formula is expressed in the simple analytical form of surface integral on the design boundary. The integrand of the surface integral is written in terms of the shape variation and physical quantities such as the material properties, the state variable, and the adjoint variable.

If the exact solution for the state variable is given, the sensitivity, which is derived as an analytical form, will be exact. But the exact solution for complex electromagnetic system is not given; the sensitivity formulas are evaluated with the approximate solution by the numerical methods such as finite element method, boundary element method.

The major advantage of the continuum sensitivity is that since the variational system equation is differentiated before discretized, it does not only depend on discretization method but also provide more accurate sensitivity information than the discrete method.

In Chaps. 3–6, for deriving the shape sensitivity formulas, this continuum method is applied to the four electromagnetic systems: electrostatic system, magnetostatic system, eddy current system, and DC current-carrying conductor.

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Chapter 2
Variational Formulation of Electromagnetic Systems

In order to derive the continuum sensitivity for the electromagnetic system, the variational state equation is differentiated with respect to the design variables by using the material derivative concept in the subsequent Chaps. 3–6. In this chapter, the variational state equations for electrostatic system, magnetostatic system, eddy current system, and DC current-carrying conductor are formulated by the variational method of virtual work principle. Each variational equation is derived from its corresponding differential equation with boundary conditions. Electromagnetic systems are usually represented by a differential (point) form of Maxwell's equations that holds at all points of the field domain. Introducing the potentials such as the electric scalar potential and the magnetic vector potential, the governing differential equations are obtained as the second-order partial differential equations. Thus, the equations require continuous second-order derivatives of the potentials. The variational state equations reduce the required order of the derivatives by one so that the variational (weak) formulation provides a general solution that cannot be obtained by the differential equations. It is also the mathematical basis for the finite element method, which is widely applicable to the electromagnetic systems. Furthermore, since the variational state equation is expressed in integral form that contains the geometry information, it is more suitable to the shape design sensitivity analysis than the differential equation [1–7].

2.1 Variational Formulation of Electrostatic System

In this section, the differential state equation for electrostatic system is derived from Maxwell's equations by using the electric scalar potential ϕ, and then, its variational state equation is obtained by applying the variational formulation of the virtual work principle [8].

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2.1.1 Differential State Equation

The differential state equations for electrostatic field system are derived from Maxwell's equations with the electric scalar potential. The electrostatic system is represented by two equations from Maxwell's equations;

$$\nabla \cdot \mathbf{D} = \rho \tag{2.1.1}$$

$$\nabla \times \mathbf{E} = \mathbf{0} \tag{2.1.2}$$

where $\mathbf{D}$ is the electric flux density, ρ is the volume charge density, and $\mathbf{E}$ is the electric field intensity. The electric flux density is written with the electric field intensity and the permanent polarization by the constitutive relation;

$$\mathbf{D} = \varepsilon \mathbf{E} + \mathbf{P}_0 \tag{2.1.3}$$

where ε is the electric permittivity and $\mathbf{P}_0$ is the permanent polarization, The electric permittivity ε is $\varepsilon_r \varepsilon_0$ with the relative electric permittivity ε_r and the vacuum permittivity ε_0. ε_r is assumed to be constant. The permanent polarization for electrolet materials is included for general description of the electrostatic system. With the electric scalar potential ϕ introduced from (2.1.2), the electric field intensity is written as

$$\mathbf{E} = -\nabla \phi \tag{2.1.4}$$

Inserting (2.1.3) and (2.1.4) into (2.1.1), we obtain the Poisson equation of the electrostatic system, which is the governing equation for the state variable of the electric scalar potential ϕ;

$$-\nabla \cdot \varepsilon \nabla \phi = \rho - \nabla \cdot \mathbf{P}_0 \tag{2.1.5}$$

where $-\nabla \cdot \mathbf{P}_0 = \rho_P$ and ρ_P is the bound charge density of permanent polarization. The governing differential equation of the electrostatic system (2.1.5) has a unique solution with boundary conditions. We employ most common boundary conditions;

$$\phi = C(\mathbf{x}) \quad \text{on } \Gamma^0 \text{ (Dirichlet boundary condition)} \tag{2.1.6}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma^1 \text{ (homogeneous Neumann boundary condition)} \tag{2.1.7}$$

2.1.2 Variational State Equation

The differential state equation (2.1.5) for the electrostatic system can be reduced to a variational state equation by multiplying both sides by an arbitrary virtual potential $\bar{\phi}$ as

$$\int_{\Omega} -\nabla \cdot (\varepsilon \nabla \phi - \mathbf{P}_o)\bar{\phi} \mathrm{d}\Omega = \int_{\Omega} \rho \bar{\phi} \mathrm{d}\Omega \quad \forall\, \bar{\phi} \in \Phi \tag{2.1.8}$$

The arbitrary virtual potential $\bar{\phi}$ belongs to the space of admissible potential, defined as

$$\Phi = \left\{ \bar{\phi} \in H^1(\Omega) \middle| \bar{\phi} = 0 \text{ on } \quad \mathbf{x} \in \Gamma^0 \right\} \tag{2.1.9}$$

where Γ^0 is the Dirichlet essential boundary and $H^1(\Omega)$ is the Sobolev space of order one [3, 9]. $H^n(\Omega)$ is the Sobolev space of the order n, whose functions are continuously differentiable up to $n-1$, and nth partial derivatives belong to $L_2(\Omega)$, which is the space of square integrable functions such that

$$L_2(\Omega) = \left\{ f \middle| \int_{\Omega} |f(\mathbf{x})|^2 \mathrm{d}\Omega < \infty \right\} \tag{2.1.10}$$

By the vector identity $\nabla \cdot \left(\nabla \psi \bar{\psi} \right) = (\nabla \cdot \nabla \psi)\bar{\psi} + \nabla \psi \cdot \nabla \bar{\psi}$, (2.1.8) is written as

$$\int_{\Omega} \left[(\varepsilon \nabla \phi - \mathbf{P}_0) \cdot \nabla \bar{\phi} - \nabla \cdot \left((\varepsilon \nabla \phi - \mathbf{P}_0)\bar{\phi} \right) \right] \mathrm{d}\Omega = \int_{\Omega} \rho \bar{\phi} \mathrm{d}\Omega \quad \forall \bar{\phi} \in \Phi \tag{2.1.11}$$

By the divergence theorem $\int_{\Omega} \nabla \cdot \left(\nabla \psi \bar{\psi} \right) \mathrm{d}\Omega = \int_{\Gamma} (\nabla \psi \cdot \mathbf{n}) \bar{\psi} \mathrm{d}\Gamma$, (2.1.11) is rewritten as

$$\int_{\Omega} \left(\varepsilon \nabla \phi \cdot \nabla \bar{\phi} - \rho \bar{\phi} - \mathbf{P}_0 \cdot \nabla \bar{\phi} \right) \mathrm{d}\Omega = \int_{\Gamma} (\varepsilon \nabla \phi - \mathbf{P}_0) \cdot \mathbf{n} \bar{\phi} \mathrm{d}\Gamma \quad \forall\, \bar{\phi} \in \Phi \tag{2.1.12}$$

Inserting the relation (2.1.3) into the right side of (2.1.12) provides the variational identity for the state equation of electrostatic system;

$$\int_{\Omega} \left(\varepsilon \nabla \phi \cdot \nabla \bar{\phi} - \rho \bar{\phi} - \mathbf{P}_0 \cdot \nabla \bar{\phi} \right) \mathrm{d}\Omega = - \iint_{\Gamma} D_n(\phi) \bar{\phi} \mathrm{d}\Gamma \quad \forall\, \bar{\phi} \in \Phi \tag{2.1.13}$$