Springer Proceedings in Mathematics & Statistics

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Cohomology of Arithmetic Groups

Springer

Springer Proceedings in Mathematics & Statistics

Volume 245

Springer Proceedings in Mathematics & Statistics

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Cohomology of Arithmetic Groups

On the Occasion of Joachim Schwermer's 66th Birthday, Bonn, Germany, June 2016

Springer

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ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-3-319-95548-3 ISBN 978-3-319-95549-0 (eBook)
https://doi.org/10.1007/978-3-319-95549-0

Library of Congress Control Number: 2018947491

Mathematics Subject Classification (2010): 11F75, 11F70, 11G18, 22E40, 57Q45, 12F10, 11R23

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

In June of 2016, a conference was held at the Max Planck Institute for Mathematics, Bonn, to celebrate the 66th birthday of Joachim Schwermer. All such conferences offer us the opportunity to look back at the career of the honoree, both mathematical and otherwise. In the case of Joachim Schwermer, his career was very rich.

Schwermer received his Doctor rerum naturalium from the Rheinische–Friedrich–Wilhelms Universität, Bonn, in 1977, under the supervision of G. Harder and F. Hirzebruch, and his Habilitation there in 1982. After leaving Bonn in 1986, Schwermer held Professorships in Eichstätt (1986–1998) and Düsseldorf (1998–2000) before taking up residence as Professor at the University of Vienna in 2000. Simultaneously with his move to Vienna, he became Deputy Director of the Erwin Schrödinger Institute (ESI). He became Director in 2004 and in 2011 oversaw the transition of the ESI from an independent research institute supported by the Austrian Government to a research institute under the auspices of the University of Vienna. He retired from the ESI in 2016 and the University of Vienna in 2017.

There have been three separate but related focuses of Schwermer's career. The first is his mathematical career. His research interest has always been the cohomology of arithmetic groups and its relation with the theory of automorphic forms. This volume represents this side of his career, with many of the articles responses to the interests of Schwermer. The article of Grbac, in particular, describes his collaboration with Schwermer. But there are two more sides that have produced value in their scholarship and in the fostering of scholarship of others. The first of these is represented by Schwermer's interest in the history of mathematics, particularly that of the nineteenth and early twentieth centuries. This interest can be seen in his various articles and in particular in his books, one with Goldstein and Schappacher on Gauss' *Disquisitiones* and then with Dumbaugh on Emil Artin and Class Field Theory. Equally important is Schwermer's work in what might be termed the administration of mathematics. We would like to mention two aspects of this. One is the series of Oberwolfach meetings organized with S. Kudla, usually on cohomology of arithmetic groups, automorphic forms, representation theory, etc. These were very influential and well attended, and we owe them both a debt of gratitude for organizing these. More importantly is Schwermer's stint as the Director of the

ESI. Here too he hosted many workshops on similar topics, and many of us have enjoyed his hospitality and owe a debt of gratitude for the mathematics we have produced there. The ESI has been an important research center in Europe, and many of us recall Schwermer's heroic efforts to keep it open and move it under the umbrella of the University of Vienna. For a period it was unclear that the ESI would survive, and to a large part, its survival in its current form (which seems almost indistinguishable from its former form) is due to his efforts.

When we came together in June of 2016 to celebrate the mathematics of Joachim Schwermer, we did not forget his efforts in both history and administration. Knowing of them made his contributions to mathematics that much richer, both for the depth his historical interest provided and the appreciation of his work in light of his other concomitant administrative achievements. Not all of the speakers at the conference are represented in these proceedings, but we thank those that did contribute. We would also like to thank all of our colleagues that contributed to the volume by serving as anonymous referees for the contributions. We also thank Springer for seeing this volume through to fruition in spite our missing of various deadlines. And mostly, we thank Joachim himself for providing us with the opportunity to thank him for all he has done.

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Contents

Globally Analytic p-adic Representations of the Pro–p Iwahori Subgroup of $GL(2)$ and Base Change, II: A Steinberg Tensor Product Theorem

Laurent Clozel

Abstract In this paper, which is a sequel to Clozel (Globally analytic p-adic representations of the pro-p Iwahori subgroup of GL(2) and base change, I: Iwasawa algebras and a base change map, to appear in Bull. Iran Math Soc, [4]), we exploit the base change map for globally analytic distributions constructed there, relating distributions on the pro-p Iwahori subgroup of $GL(2)$ over $\mathbb{Q}_p$ and those on the pro-p Iwahori subgroup of $GL(2, L)$ where L is an unramified extension of $\mathbb{Q}_p$. This is used to obtain a functor, the 'Steinberg tensor product', relating globally analytic p-adic representations of these two groups. We are led to extend the theory, sketched by Emerton (Locally analytic vectors in representations of locally p-adic analytic groups, [6]), of these globally analytic representations. In the last section we show that this functor exhibits, for principal series, Langlands' base change (at least for the restrictions of these representations to the pro-p Iwahori subgroups.)

Keywords 11R23 · 11F70 · 14G22

Introduction

This is part II of a paper, the first part of which is [4]. In that article we considered the Iwasawa algebra of the pro–p Iwahori subgroup of $GL(2, L)$ for an unramified extension L of degree r of $\mathbb{Q}_p$ and gave a presentation of it by generators and relations, imitating [3]. A natural base change map then appears that, however, is well–defined

Dedicated to Joachim schwermer on his 66th birthday

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J. W. Cogdell et al. (eds.), *Cohomology of Arithmetic Groups*,
Springer Proceedings in Mathematics & Statistics 245,
https://doi.org/10.1007/978-3-319-95549-0_1

only for the *globally analytic distributions* on the groups, seen as rigid–analytic spaces.

In Sect. 1.1 of [4], we stated that this should be related to a construction of base change for representations of these groups, similar to Steinberg's tensor product theorem [13] for algebraic groups over finite fields.

In this paper we give such a construction, and we show that it is compatible with the (p–adic) Langlands correspondence in the case of the principal series for $GL(2)$.

By the previous remark, we have to limit ourselves to globally analytic representations. These representations have been considered by Emerton in his exhaustive introduction (unfortunately unpublished) to p–adic representation theory [6]. See in particular Sects. 3.3, 5.1 in his paper; the restriction of scalars, central to our constructions, is considered in his Sect. 2.3.

The first section of this paper contains preliminaries about rigid–analytic groups. The group associated to the pro–p Iwahori is (by Lazard's description) very simple, a product of copies of the rigid–analytic closed unit ball. In particular the algebras of functions we consider are all Tate algebras. We must, however, systematically consider restriction of scalars. Even for such simple spaces, this functor does not behave trivially, as was pointed out to me by Gaëtan Chenevier. See [1, 14]. However, this is the case for unramified extensions (Sect. 1.1.) It is then an easy matter to describe the natural functorial maps between Tate algebras (Defintion 1.4) and, dually, between (global) distribution algebras (Sect. 1.2). Nevertheless, the distribution algebra for a product is not a tensor product (even a completed tensor product.) This causes problems in the representation theory, which will be mentioned below; these "pathologies" are reviewed in the Appendix.

In Sect. 2 of this paper, we review the properties of these representations, adding some complements to Emerton's results. In particular, we study tensor products of representations (Theorem 2.3).

In contrast with the category of locally analytic representations, we can work here with (p–adic) Banach spaces rather than with Fréchet spaces, or spaces of compact type [6, 12]. Indeed, the spaces $\mathcal{A}$ and $\mathcal{D}$ of globally analytic functions (resp. distributions) are Banach spaces. The unfortunate consequence is that they are not reflexive. In particular we cannot systematically use duality as in the admissible Banach theory [11] or the locally analytic theory [12]. A related problem is that the spaces $\mathcal{D}$ of distributions are not Nœtherian. See the remarks in Sect. 2.3, as well as the Appendix.

In Sect. 3, we take up the construction of the base change functor, i.e., the Steinberg tensor product. Once the requisite property of the tensor product has been established in Sect. 2, this is totally natural. The main point is that a globally analytic representation will automatically extend, from the L–points of a rigid–analytic group G over L (we consider only very special groups, cf. Sect. 2) to the F–points for any finite extension F of L. Although this is not explicit in [6], it follows from his definitions. The construction is given in Sect. 3.2.

Of course this is meaningful only if it is compatible with the expected Langlands correspondence. The end of Sect. 3 is devoted to the proof of this fact for the principal series. We start with the pro–p Iwahori G of $GL(2, \mathbb{Q}_p)$. We must of course consider only the representations which have globally analytic vectors. This condition is specified in (3.4). In Proposition 3.6, we show that (under the same assumption as in [12]) the globally analytic representation of the pro–p Iwahori subgroup of $GL(2, \mathbb{Q}_p)$ is topologically irreducible.

In Sect. 3.4, we extend these results to the pro–p Iwahori subgroup of $GL(2, L)$ where $L/\mathbb{Q}_p$ is unramified. Here the similar irreducibility result is suggested by the work of of Orlik and Strauch [8]. We show that the formation of the Steinberg tensor product is compatible with Langlands functoriality (cf. Definition 3.2); the final result is Theorem 3.11 which exhibits base change in this context.

These results concern only the pro–p Iwahori subgroups, not the full groups $GL(2, \mathbb{Q}_p)$, $GL(2, L)$. In Sect. 3.5 we make some tentative remarks about the extension of base change to the full groups. Finally, the Appendix reviews some questions concerning the tensor products of distributions and the non-Noetherian character of these algebras.

While writing this paper I had the benefit of discussions or correspondence with Berthelot, Breuil, Chenevier, Raynaud and Schneider. Ariane Mézard corrected some mistakes in an early version. I am very grateful to them, and especially to Peter Schneider who explained to me the facts reviewed in the Appendix. I also thank the referee for useful comments.

1 Restriction of Scalars and Base Change Maps for Analytic Functions and Distributions on Rigid–Analytic Unit Ball Groups

1.1 .

We consider an unramified extension L/L_0, of degree r, of p–adic fields (finite extensions of $\mathbb{Q}_p$). Let $X = B^1/L$ be the closed unit ball over L, a rigid–analytic space whose affinoid algebra is

$$\mathcal{T}_L^1 = L < x > ,$$

the algebra of power series in x with coefficients tending to zero.

There is a functor of restriction of scalars, which to $X = X/L$ associates a rigid–analytic space $Y = \mathrm{Res}_{L/L_0} X/L_0$.

Lemma 1.1 (L/L$_0$ unramified)- *Y is isomorphic to the r–th power of B^1/L_0.*

This is a special case of the more general results of Bertapelle [1]. Let (e_i) be a basis of $\mathcal{O}_L$ over $\mathcal{O}_{L_0}$, and let B be an affinoid L_0–algebra. Consider $f \in Hom_L(L < x >, B \bigotimes_{L_0} L)$, thus

$$f(x) = \sum b_i e_i \qquad (b_i \in B) .$$

We want to define canonically

$$g \in Hom_{L_0}(L_0 < x_1, \dots x_r >, B),\ \text{with}\ g(x_i) = b_i\,.$$

(Thus $Y \cong B^r/L_0$ is canonical, given the choice of the basis (e_i).) This is possible if, and only if, $\|b_i\|_{\mathrm{Sup}} \leq 1$ assuming $\|b\|_{\mathrm{Sup}} \leq 1$ where the sup norms are relative to the affinoid algebras B and $B \bigotimes_{L_0} L$.

Assume first that B is a finite field extension of L_0. Then $B \bigotimes_{L_0} L$ is a product of finite, unramified extensions B_α of L, and the integers $\mathcal{O}(B \bigotimes_{L_0} L) = \prod_\alpha \mathcal{O}(B_\alpha)$ satisfy, the extensions being unramified, $\mathcal{O}(B \bigotimes_{L_0} L) = \mathcal{O}_B \bigotimes_{\mathcal{O}_{L_0}} \mathcal{O}_L = \bigoplus \mathcal{O}_B e_i$. To say that $\|b\|_{\mathrm{Sup}} \leq 1$ for $b \in B \otimes L$ is to say that $b_\alpha \in \mathcal{O}(B_\alpha)$, or $b \in \mathcal{O}(B \bigotimes_{L_0} L)$.

This implies that $\|b_i\| \leq 1$.

Now let B be a general affinoid algebra over L_0, and $B' = B \bigotimes_{L_0} L$. If $b = \sum b_i e_i \in B'$ $(b_i \in B)$, the computation in [1, p. 444] shows that

$$\|b\|_{\mathrm{Sup}} = \operatorname*{Sup}_{y \in \mathrm{Max} B} \operatorname*{Max}_{\substack{x \in \mathrm{Max} B' \\ x|y}} \left\| \left(\sum b_i e_i \right)(x) \right\|_{\mathrm{Sup}}.$$

However, y corresponds to a finite extension K_0 of L_0, x to a finite extension K of L contained in $L \bigotimes_{K_0} L_0$ so unramified over L. The previous result implies that $\| \sum b_i e_i(x) \|_{\mathrm{Sup}} = \mathrm{Sup} \|b_i(x)\|$. Thus $\|b_i\|_{\mathrm{Sup}} \leq 1$ if $\|b\|_{\leq 1}$. We note that we have in fact:

Lemma 1.2 *The isomorphism* $Y \overset{\simeq}{\to} (B^1/L_0)^r$ *is canonically defined by the choice of the basis* (e_i).

In fact, the function g (for instance if $B = K_0$ is a field extension of L_0) is defined by

$$g(x_1, \dots x_r) = f(\Sigma e_i x_i)\,. \tag{1.1}$$

$(|x_i| \leq 1)$. The e_i being integral, it is easy to check that for $f \in \mathcal{T}_L^1$, the infinite series in the right is convergent.

Since restriction of scalars is compatible with direct products [1, Proposition 1.8] we have likewise

$$\mathrm{Res}_{L/L_0}(B^1/L)^d = (B^1/L_0)^{dr}$$

the isomorphism being canonical once we have fixed the basis (e_i).

1.2 .

We now consider a rigid–analytic group G_L over L, isomorphic as a rigid–analytic space to $(B^1/L)^d$. (In particular $G_L(L)$ is dense in G_L for the Zariski topology.) Let $\mathcal{A}(G_L) \cong \mathcal{T}_d(L) = L < x_1, ..., x_d >$ [2, 5.1] be the space of analytic functions on G_L. The multiplication in G_L is associated to a (comultiplication) morphism

$$m^* : \mathcal{A}(G_L) \longrightarrow \mathcal{A}(G_L)\widehat{\otimes}\mathcal{A}(G_L)$$

(completed tensor product). In this case, the product is given, in co–ordinates, by integral functions, [2, Corollary 5.1.3.5] so

$$m^* : \mathcal{A}^0(G_L) \longrightarrow \mathcal{A}^0(G_L)\widehat{\otimes}\mathcal{A}^0(G_L)\,.$$

Then m^* defines naturally a map

$$\text{Res}\, m^* : \mathcal{A}^0(\text{Res}\ G_L) \longrightarrow \mathcal{A}^0(\text{Res}\ G_L)\widehat{\otimes}\mathcal{A}^0(\text{Res}\ G_L)\,,$$

Res being the restriction of scalars of G_L, a group over L_0.

Assume now that the group G_L is actually defined over L_0, i.e., is obtained by *extension* of scalars from L_0. Then $\mathcal{A}(G_L) = \mathcal{A}(G_{L_0}) \otimes L$. The map m^* is obtained by extension of scalars from

$$m_0^* : \mathcal{A}(G_{L_0}) \longrightarrow \mathcal{A}(G_{L_0})\widehat{\otimes}\mathcal{A}(G_{L_0})\,.$$

The integrality property for G_L and the property for G_{L_0} are equivalent.

Now the previous construction associates to $f \in \mathcal{A}(G_L)$ (with L–coefficients, i.e. in $\mathcal{T}_d(L)$) a function g in $\mathcal{A}(\text{Res}\ G_L) \otimes L$ (the function g defined by (1.1) will have coefficients in L). In particular we get a map $\mathcal{A}(G_{L_0}) \to \mathcal{A}(\text{Res}\ G_L) \otimes L$ by composition with the previous "tautological" map $\mathcal{A}(G_{L_0}) \to \mathcal{A}(G_L)$.

Definition 1.3 This map $b_1\colon \mathcal{A}(G_{L_0}) \to \mathcal{A}(\text{Res}\ G_L) \otimes L$ is the holomorphic base change map.

This map commutes with the comultiplications m_0^* and Res m^*: it is obvious if we consider m_0^* and m^*, and for m^* and Res m^* it follows from the formal properties of restriction of scalars. Furthermore b_1 sends $\mathcal{A}^0(G_{L_0})$ to $\mathcal{A}^0(\text{Res}\ G_L \otimes L)$.

The unramified extension L/L_0 is Galois. Thus the Galois group $\Sigma = \text{Gal}(L/L_0)$ acts naturally on G_L (by σ–linear automorphisms of the Tate algebra) and acts on Res G_L by L_0–automorphisms.

Definition 1.4 The map $b : \mathcal{A}(G_{L_o}) \to \mathcal{A}(\text{Res}\ G_L) \otimes L$ is defined by

$$b(f) = \prod_{\sigma\in\Sigma} b_1(f)^{\sigma}\,.$$

Since b_1 commutes with the comultiplication, the same is true for the product $\Pi\, b_1^\sigma$. We also note the following: Assume we *extend* scalars from L_0 to L for the L_0–groups. Then

$$\text{Res } G_L \otimes_{L_0} L$$

is naturally isomorphic to $\prod_\sigma G_L$. Indeed, if B is an L–algebra (in particular an affinoid algebra), $B \otimes_{L_0} L \cong \bigoplus_\sigma B_\sigma$ where $B_\sigma = \{\beta \in B \otimes L : \lambda_1\beta = \lambda_2^\sigma \beta$ where $\lambda \in L$ and λ_1 is the action of λ on $B \otimes L$ by the first component, λ_2 by the second component. Now, $\mathcal{A}$ denoting a Tate algebra:

$$\begin{aligned}\text{Hom}_L\ &(\mathcal{A}(\text{Res}G_L) \otimes_{L_0} L, B) \qquad (B/L)\\ &= \text{Hom}_{L_0}(\mathcal{A}(\text{Res } G_L), B_0)\\ &\ \ (B_0 \text{ being equal to } B/L_0)\\ &= \text{Hom}_L(\mathcal{A}(G_L), B_0 \otimes L)\\ &= \bigoplus_\sigma \text{Hom}_L(\mathcal{A}(G_L), B_\sigma)\,.\end{aligned}$$

In particular, after extension of scalars to L, $\mathcal{A}(\text{Res } G_L) \otimes L \cong \widehat{\bigotimes_\sigma} \mathcal{A}(G_L)$. The map b is then a tensor product: b_1 sends $\mathcal{A}(G_{L_0})$ to the functions on G_L that are L–holomorphic (given by power series $\Sigma a_m \underline{x}^m$, $\underline{x} = (x_1, \ldots, x_d)$ being the variable) while the component associated to σ sends a power series in $\mathcal{A}(G_0)$ to $\Sigma a_m \sigma(\underline{x})^m$.

We now agree to consider all Tate algebras as having coefficients in L, and we denote them by $\mathcal{A}_L$.

Summarizing, we now have the following result:

Proposition 1.5 *(i) There exists a natural map* b_1: $\mathcal{A}_L(G_{L_0}) \to \mathcal{A}_L(\text{Res } G_L)$. *It commutes with the comultiplications.*

(ii) There exists a natural map $b = \prod_{\sigma \in \Sigma} b_1^\sigma : \mathcal{A}_L(G_{L_0}) \to \mathcal{A}_L(\text{Res } G_L)$. *It commutes with the comultiplications.*

(iii) In the isomorphism $\mathcal{A}_L(\text{Res } G_L) \cong \widehat{\bigotimes_\sigma} \mathcal{A}(G_L)$ $(\mathcal{A}(G_L) = \mathcal{A}_L(G_L))$, $b = \bigotimes_\sigma b_1^\sigma$.

(iv) The maps b_1 *and* b *send the unit balls* $\mathcal{A}_L^0(G_{L_0})$ *to* $\mathcal{A}_L^0(\text{Res } G_L)$. *(The norm being the sup norm of coefficients).*

We now consider the spaces of (L–valued) global distributions on G_{L_0} and Res G_L. We denote them by $\mathcal{D}_L(G_{L_0})$, $\mathcal{D}_L(\text{Res } G_L)$. These are the Banach spaces dual to the Banach spaces of analytic functions (for the sup norms). We obtain, dually, a map

$$b_1^* : \mathcal{D}_L(\text{Res } G_L) \to \mathcal{D}_L(G_{L_0})$$

and also

$$b^* : \mathcal{D}_L(\text{Res } G_L) \to \mathcal{D}_L(G_{L_0}) .$$

These are homomorphisms, for the convolution of distributions. Using (iii) in the Proposition, we can write

$$\bigotimes_\sigma \mathcal{D}_L(G_L) \subset \mathcal{D}_L(\text{Res } G_L)$$

and b^* is then, on this subspace, given by

$$\bigotimes_\sigma T_\sigma \mapsto \mathop{*}_\sigma T_\sigma$$

(where $T_\sigma \in \mathcal{D}_L(G_{L_0})$ is σ–holomorphic). However, $\widehat{\bigotimes_\sigma} \mathcal{D}_L(G_L)$ is **not** equal to $\mathcal{D}_L(\text{Res } G_L)$. Since, after extension of scalars, our groups become products, this can be seen as follows.

We may forget for a moment the restriction of scalars, and consider two groups G, H isomorphic (as rigid–analytic spaces) to $(B^1)^d$, $(B^1)^{d'}$ over L. The spaces of analytic functions are $\mathcal{T}_d(L)$, $\mathcal{T}_{d'}(L)$, with the sup norm. The dual $\mathcal{D}_L(G)$ of the space of functions

$$f(x) = \sum_n a_n \, \underline{x}^n , \qquad a_n \to 0$$

($n \in \mathbb{N}^d$, $\underline{x} = (x_1, \ldots x_d)$, $|x_i| \leq 1$) is the space of distributions

$$T = \sum_n c_n \, \delta_n \qquad (|c_n| \leq C)$$

where $\delta_n(f) = a_n = n\,! \, \frac{\partial^n f}{\partial \underline{x}^n}(0)$. It is a Banach space, the norm being $\sup |c_n|$. The same description applies to a distribution S on H, and a distribution on $G \times H$. However, these Banach spaces are ℓ^∞ spaces in the indexes, and for three (countable) sets X, Y, $X \times Y$, it is not true that

$$\ell^\infty(X) \mathbin{\widehat{\otimes}} \ell^\infty(Y) = \ell^\infty(X \times Y) .$$

In order to form tensor products, we must consider the unit balls in $\mathcal{D}_L(G)$, $\mathcal{D}_L(H)$ (with their weak topology) and apply a result of Lazard. This was explained to me by Peter Schneider; we will return to it at the end.

2 Globally Analytic Representations

2.1 .

In this section we review some basic properties of globally analytic representations of a rigid–analytic group on a Banach space, mostly following Emerton [6]. We assume given L and G/L as in Sect. 1.2. We denote by $\mathcal{A} \cong \mathcal{T}_d(L)$ the space of globally analytic functions on G. We will often write G for $G(L)$ if this does not lead to confusion ; $G(L)$ is dense in G for the Zariski topology.

2.2 .

Let V be a Banach space over a field K containing L. We assume again K finite over $\mathbb{Q}_p$. If $g \mapsto \pi(g)$ is a continuous representation of G on V, we say that π (or V) is a globally analytic representation if the map

$$g \mapsto g \cdot v = \pi(g)v$$

is (globally) analytic on G for all $v \in V$. Thus, in coordinates $(x_1, \dots x_n)$:

$$g \cdot v = \sum_m \underline{x}^m v_m$$

where $v_m \in V$ and $\|v_m\| \to 0$.

Here $m = (m_1, \dots, m_d)$ and $\underline{x}^m = x_1^{m_1} \cdots x_d^{m_d}$, $m_i \in \mathbb{N}$. Such a representation is automatically differentiable. We will simply use the term "analytic" for "globally analytic". Note that it is relative to the L–structure on V.

In this situation V is endowed with two natural norms, the given norm and

$$\|v\|_\omega = \operatorname{Sup}_m \|v_m\| \, .$$

The second norm is the norm of the map $g \mapsto gv$ in the Banach space $\mathcal{A}(G, V) = \mathcal{A}(G)\hat{\otimes} V$ (for this isomorphism cf. e.g. [6, Sect. 2.1]). The map $(V, \| \|_\omega) \to (V, \| \|)$ is bijective and obviously continuous. Since V, with the norm $\| \|_\omega$, is complete [6, 3.3.1, 3.3.3] it is bicontinuous by Banach's isomorphism theorem [9, Corollary 8.7].

We recall the proof of the completeness of $(V, \| \|_\omega)$, as we will require similar arguments. Thus let $(v^\alpha)_\alpha$ be a Cauchy sequence in V for $\| \|_\omega$. For each α, $(v^\alpha_m)_{m \in M}$ is an element of $C^0(M, V)$ where $M = \mathbb{N}^d$ is the set of exponents. Since this space is complete, $(v^\alpha_m)_m \mapsto (v_m)$ in $C^0(M, V)$ for an element $(v_m) \in C^0(M, V)$. In particular $v^\alpha = v^\alpha_0 \to v := v_0 \in V$. Now $gv = \lim_\alpha g v^\alpha_0$ $(g \in G)$, so $gv = \lim_\alpha (\sum_m \underline{x}^m v^\alpha_m)$.

Since

$$\Big\| \sum_m \underline{x}^m (v_m^\alpha - v_m) \Big\| \leq \operatorname{Sup}_m \Big\| v_m^\alpha - v_m \Big\| \longrightarrow 0 \qquad (\alpha \to \infty)$$

we see that $gv = \sum \underline{x}^m v_m$, which implies that $\|v - v^\alpha\|_\omega \to 0$.

Corollary 2.1 *There exists a constant C_V (depending on V) such that $\|v\|_\omega \leq C_V \|v\|$ $(v \in V)$.*

In particular $\|gv\| \leq C_V \|v\|$ $(g \in G)$.

In fact the original norm can be replaced by an equivalent norm such that $\|gv\| = \|v\|$: see Emerton [6, Sect. 6.5].

Lemma 2.2 *Let $(V, \| \ \|)$ be a continuous Banach representation of G, and let $W \subset V$ be a subspace comprised of analytic vectors. Assume that $\|w\|_\omega \leq C\|w\|$ $(C > 0)$ for $w \in W$. Then any vector of $\bar{W} \subset V$ (the closure for the topology of V) is analytic.*

Proof — Consider a sequence $(w^\alpha)_\alpha$ of vectors in W, such that $\|w^\alpha - v\| \to 0$ $(v \in V)$. Then w^α is a Cauchy sequence for $\| \ \|$, so also for $\| \ \|_\omega$. If

$$g \cdot w^\alpha = \sum_m \underline{x}^m w_m^\alpha ,$$

the sequence $(w_m^\alpha)_{m \in M}$ has a limit (v_m) in $C^0(M, V)$. In particular $v_0 = v$. Again

$$g\, w^\alpha = \sum_m \underline{x}^m w_m^\alpha \longrightarrow gv \qquad (\alpha \to \infty)$$

and $\| \sum_m \underline{x}^m (w_m^\alpha - v_m)\| \leq \operatorname{Sup}_m \|w_m^\alpha - v_m\| \to 0$ $(\alpha \to \infty)$ which implies that $gv = \sum_m \underline{x}^m v_m$.

Consider now two rigid analytic groups G, H verifying our assumptions. Let V, W be analytic representation of G, H on Banach spaces. We assume the norms invariant, using Emerton's result. Then $G \times H$ acts on the algebraic tensor product $V \otimes W$. By [9, Proposition 2.1.7.5] this action extends to $V \hat{\otimes} W$, with $\|(g, h)u\| = \|u\|$ $(u \in V \hat{\otimes} W)$.

Now $V \otimes W$ is dense in $V \hat{\otimes} W$, and is comprised of analytic vectors : if $v \in V$, $w \in W$ and

$$gv = \sum_m \underline{x}^m v_m , \quad hw = \sum_p \underline{y}^p w_p$$

$(g \in G, h \in H)$ then

$$(g, h)(v \otimes w) = \sum_{m,p} \underline{x}^m \underline{y}^p v_m \otimes w_p .$$

Since $\|v_m \otimes w_p\| = \|v_m\| \, \|w_p\|$ (Schneider [9, Proposition 17.4]), this yields an analytic expansion.

Now endow $V \otimes W$ with its analytic norm $\| \ \|_\omega$, for the action of $G \times H$. We have

$$\begin{aligned}\|v \otimes w\|_\omega &= \operatorname*{Max}_{m,p} \|v_m \otimes w_p\| \\ &= \operatorname{Max}\|v_m\| \operatorname{Max}\|w_p\| \\ &= \|v\|_\omega \, \|w\|_\omega.\end{aligned}$$

Now consider any vector $u \in V \otimes W$. The tensor product norm is defined by

$$\|u\| = \inf \operatorname*{Max}_i \|v_i\| \, \|w_i\|$$

over the decompositions $u = \sum v_i \otimes w_i$. Choose $\varepsilon > 0$, and a decomposition such that

$$\begin{aligned}\|u\| \quad &\geq \operatorname{Max}\|v_i\| \, \|w_i\| - \varepsilon. \\ \text{Then } \|u\|_\omega &\leq \operatorname*{Max}_i \|v_i \otimes w_i\|_\omega \\ &\leq C_V C_W \operatorname*{Max}_i \|v_i \otimes w_i\| \leq C_V C_W (\|u\| + \varepsilon).\end{aligned}$$

Thus $\|u\|_\omega \leq C_V C_W \|u\|$, and $V \otimes W \subset V \hat{\otimes} W$ verifies the assumption of the Lemma. This implies:

Theorem 2.3 *If V, W are (globally) analytic representations of G, H, $V \hat{\otimes} W$ is a globally analytic representation of $G \times H$.*

(For a similar result, but for locally analytic representations, see Emerton [6, 3.6.18]).

We also note the following property. Let $\mathfrak{g}$ be the Lie algebra of G (over $\mathbb{Q}_p$).

Proposition 2.4 *If V is a globally analytic representation of G and $W \subset V$ is a closed subspace, W is G–invariant if and only if W is invariant by the enveloping algebra $U(\mathfrak{g})$.*

(Recall from [12] that the Lie algebra, or $U(\mathfrak{g})$, acts on a space of analytic vectors). If W is G–invariant, it contains the derivatives $Xw = \lim_{t \to 0} \left(\frac{e^{tX} - 1}{t} \right) w$ of its vectors by elements $X \in \mathfrak{g}$. Conversely, if

$$gw = \sum_m \underline{x}^m v_m$$

then $v_m = \frac{1}{m!} \frac{d^m}{d\underline{x}^m} \Big|_0 (gv)$, the derivative being computed with respect to the variables $\underline{x}$. However the enveloping algebra (acting via $uf = (u * f)(0)$ for $u \in U(\mathfrak{g})$, f an analytic function on G) also spans the space of derivatives at 0. If W is invariant by $U(\mathfrak{g})$, the coefficient v_m belong to W and therefore $gw \in W$.

By contrast with the case of complex unitary representations, we do not know if $V\hat{\otimes}W$ is (topologically) irreducible if V, W are topologically irreducible. The only, obvious, property is that $V\hat{\otimes}W$ is topologically cyclic (i.e., the closed subspace generated by a suitable vector is equal to $V\hat{\otimes}W$) if V and W are - in particular if they are irreducible. Indeed, if v spans V and w spans W, $v \otimes w$ spans $V\hat{\otimes}W$.

2.3 .

Finally, we also recall from Emerton's paper that there is a duality theory for globally analytic representations, similar to the duality for locally analytic (or Banach admissible) representations. If V is a globally analytic representation, the distribution algebra $\mathcal{D}_K(G)$ acts on the dual V'. There is a duality between closed submodules of $(\mathcal{A}(G) \otimes K)^n$ and quotients of $\mathcal{D}_K(G)^n$. See [6, Theorem 5.1.15]. We will not be able to use this, however. There are two obstacles: the algebra $\mathcal{D}_K(G)$ is not Nœtherian; furthermore, as noticed at the end of Sect. 1, it does not behave well with respect to the product of groups.

Let us define an *admissible globally analytic* representation as a globally analytic Banach representation that is a closed submodule of $(\mathcal{A}(G) \otimes K)^n$. Recall also from [6, 10] that there is a category of admissible (continuous) Banach representations and of admissible locally analytic Banach representations on spaces of compact type [11]. In general, an admissible globally analytic representation is not an admissible locally analytic representation (an infinite–dimensional Banach space is not of compact type) and is not an admissible Banach representation. Indeed, if E is such a representation and E^0 is its unit ball (for a given G–invariant norm), and if ϖ is a uniformising parameter of K, it is known that $E^0/\varpi E^0 = \bar{E}$ is a smooth admissible representation of G over the finite residue field k of K [10], [6, 6.5.7]. However, $\mathcal{A}(G)$ does not have that property.

For instance, if G is the additive unit ball, so $V = \mathcal{A}(G) \otimes K = \mathcal{T}_1(K)$, its unit ball is translation–invariant and the subgroup $\varpi_L \mathcal{O}_L$ of $G(L) = \mathcal{O}_L$ acts trivially on $\bar{V} = k[x]$, so this representation is not admissible.

Assume however that E is an admissible Banach representation. Then E is a closed subspace of $\mathcal{C}(G, K)^n$ for some n [10],[6, Sect. 6]. Let $V = E^{an}$ be the space of globally analytic vectors. Emerton's results (see the proof recalled before Corollary 2.1) show that V is complete for the norm $\| \ \|_\omega$. It is an analytic Banach representation [6, Corollary 3.3.6].

Assume $V = E^{an}$ is dense in E. Since $\mathcal{C}(G, K)^{an}$ is equal to $\mathcal{A}(G) \otimes K$, V is sent to $(\mathcal{A}(G) \otimes K)^n$. Let $j = (j_i)_{i=1,\dots n}$ be the closed embedding $E \to \mathcal{C}(G, K)^n$. By Banach's theorem $\|v\| \geq C \operatorname{Sup}_i \|j_i(v)\|$ for $v \in E$, C being a > 0 constant. This implies, the embedding being equivariant, that $\|v\|_\omega \geq C \operatorname{Sup}_i \|j_i(v)\|_\omega$ for $v \in V$. The canonical norm $\| \ \|_\omega$ on $\mathcal{A}(G)$ is the usual norm – the sup norm on coefficients. (See Proposition 2.7 below.) Thus V is a closed subspace of $(\mathcal{A}(G) \otimes K)^n$. Conversely, if V is such a subspace, we can consider its closure $E \subset \mathcal{C}(G, K)^n$. It is an admissible Banach representation in which V is dense. Clearly $V \subset E^{an}$, but it does not seem to follow that V is equal to E^{an}. To summarise:

Proposition 2.5 *Any admissible globally analytic representation is a dense subspace of an admissible Banach representation. If E is an admissible Banach representation, E^{an} is an admissible globally analytic representation.*

The admissible analytic representations have further interesting properties. Recall that in general, if V is an analytic representation, there is an action of $\mathcal{D}(G) \otimes K$ on the continuous dual V' [6, 5.1.8]. If V is admissible, we can say more.

Assume $T \in \mathcal{D}(G)$ (we forget the extension of scalars for simplicity of notation.) If $f \in \mathcal{A}(G)$, we can define a function $T * f$ by

$$T * f(x) = \int T(z) f(z^{-1}x) dz \tag{2.1}$$

in functional notation, i.e. T applied to the function of z, $z \mapsto z^{-1}x$. Since $f(z^{-1}x)$ is in the Tate algebra of $G \times G$, this is well–defined and, moreover, defines a function in $\mathcal{A}(G)$. Thus $\mathcal{D}(G)$ acts by convolution on $\mathcal{A}(G)$, and this is compatible with the convolution product.

Assume now that $V \subset \mathcal{A}(G)$ is a closed invariant subspace. Then V is invariant by the differential operators $\frac{1}{m!}\frac{d^m}{dx^m}$. If $f \in \mathcal{A}(G)$ and

$$T = \sum_m c_m \frac{1}{m!} \frac{d^m}{dx^m}\Big|_0$$

$\in \mathcal{D}(G)$ (with c_m bounded), $T * f$ is the limit in $\mathcal{A}(G)$ of $T_X f$,

$$T_X f = \sum_{|m| \le X} c_m \frac{1}{m!} \frac{d^m}{dx^m} f$$

as can be seen by expanding the function $f(z^{-1}x)$ in (2.1) in the Tate algebra of $G \times G$. Therefore V is invariant by $\mathcal{D}(G)$. The same extends to an embedding $V \to \mathcal{A}(G)^n$. Thus:

Proposition 2.6 *If V is an admissible globally analytic representation, the distribution algebra $\mathcal{D}(G)$ acts naturally on V. The action is continuous if $\mathcal{D}(G)$ is equipped with its weak dual topology.*

The continuity follows from the previous argument. It implies in particular that the action is intrinsic.

We recall that for locally analytic representations this construction is due to Schneider and Teitelbaum [11, Sect. 3]. However their proof relies on an isomorphism

$$\mathcal{L}(\mathcal{D}_{loc}(G), V) \cong \mathcal{A}_{loc}(G, V)$$

([11, Theorem 2.2]; here $\mathcal{A}_{loc}(G)$ is the space of locally analytic functions and $\mathcal{D}_{loc}(G)$ its dual space, and V is a suitable topological space. The analogue is not true in our context. Indeed

$$\mathcal{A}(G, V) = \mathcal{A}(G)\hat{\otimes}V \cong C_0(M, V)$$

where M is our set of exponents, while $\mathcal{D}(G) \cong \ell^\infty(M, L)$. Since $\ell^\infty(\mathbb{N})'$ is distinct from $C_0(\mathbb{N})$, we see a fortiori that these spaces are not isomorphic.

Because the comultiplication is given by integral series, we also have:

Proposition 2.7 *Consider the admissible representation $V = \mathcal{A}(G) \otimes K$ of G, with its usual norm (sup of the coefficients.) Then*

(i) V is a unitary representation.

(ii) On V, $\| \|_\omega = \| \|$.

(iii) For $T \in \mathcal{D}^0 = (V')^0$ and $f \in \mathcal{A}^0$, the function $g \mapsto < T, gf >$ is in $\mathcal{A}^0(G, K)$.

These facts easily follow from the property of the coproduct. Since an admissible analytic representation embeds as a closed subspace of $(\mathcal{A}(G) \otimes K)^n$, it follows that:

Corollary 2.8 *Properties (i)–(iii) of Proposition 2.7 are true for an admissible analytic representation.*

3 Unramified Base Change : The Pro–p Iwahori for $GL(2)$

3.1 .

The content of this section is twofold: we first describe a functor producing, for an unramified extension L/L_0 and a globally analytic representation of $G(L_0)$ (the assumptions are those of Sect. 1), a representation of $G(L)$ of the same kind. In fact, as in Sect. 1 for distribution algebras, there are two such functors. The first produces a "holomorphic" extension to $G(L)$. The second ("full base change") is the one that should be related to Langlands functoriality. It is the "Steinberg tensor product" described at the end of Sect. 1.1 of [4].

We then show that for $GL(2)$ and principal series representations of the pro–p Iwahori subgroup, this is compatible with base change for the principal series described by Schneider–Teitelbaum and Orlik–Strauch [8, 11]. In particular, we show that certain globally analytic representations are irreducible.

3.2 .

Let L/L_0 denote an unramified extension of p–adic fields and G a rigid analytic group over L_0 verifying the conditions of Sect. 1. We fix a p–adic field K (finite over $\mathbb{Q}_p$) and an injection $\iota : L \subset K$. If $\sigma \in \mathrm{Gal}(L/L_0)$, we then have the injection $\iota \circ \sigma : L \to K$.

Let V denote a (globally) analytic representation of $G(L_0)$ on a K–Banach space.

Proposition 3.1 *(i) V extends naturally to an analytic representation of $G(L)$.*
(ii) If V is admissible, the corresponding representation of $G(L)$ is admissible.

The group $G(L_0)$ acts on V by

$$g \cdot v = \sum_m \underline{x}^m v_m \tag{3.1}$$

with the notations of Sect. 2, and $v_m \to 0$. If $g \in G(L)$, the same expansion (with $\underline{x} = (x_1, \ldots, x_d) \in \mathcal{O}_L^d$) is convergent, and we define $g \cdot v$ by (3.1).We must check that this defines a group representation of $G(L)$. The map

$$\begin{array}{l} (g, h) \mapsto gh.v = F(g, h) \\ G(L) \times G(L) \longrightarrow V \end{array}$$

is the composition of the map $(g, h) \mapsto gh$, analytic in the two variables, and of an analytic map $G(L) \to V$. It is analytic in the two variables.

On the other hand we have for $g, h \in G(L_0)$:

$$g(hv) = g\, F(1, h)\,. \tag{3.2}$$

Write $\underline{y}$ for the coordinates of g and $\underline{x}$ for the co-ordinates of h. Then

$$F(1, h) = hv = \sum_m \underline{x}^m v_m\,.$$

On the other hand, for any v_m,

$$g v_m = \sum_p \underline{y}^p v_{m,p}$$

with $v_{m,p} \to 0$ $(|p| \to \infty)$.

Since $\|v_m\| \le C_V \|v\|$ for any m and $v \in V$,
$gF(1, h) = \sum_{m,p} \underline{x}^m \underline{y}^p v_{mp}$,
the double sum being convergent: if $|m| + |p| \to \infty$, either $m \to \infty$ and $\|v_{m,p}\| \le C_V \|v_m\| \to 0$ or m is bounded and, again, $\|v_{m,p}\| \to 0$. Thus the function $gF(1, h) : G(L_0) \times G(L_0) \to V$ is a Tate series (with coefficients in V) in the two variables, and

extends to an analytic function $G(L) \times G(L) \to V$. Since $F(g, h) = gF(1, h)$ for $g \in G(L_0)$, these two analytic functions coincide: indeed $G(L_0)$ is Zariski–dense in G, and the result follows (for instance, evaluate the two functions against a continuous linear form $\lambda \in V'$).

This proves (i). Assume now V is a closed subspace of $\mathcal{A}(G_{L_0}, K)$. Note that the same argument applies to $\mathcal{A}(G_{L_0}, K)$, an analytic representation of $G(L_0)$. But $\mathcal{A}(G_{L_0}, K) = \mathcal{A}(G_L, K)$ and now V, as a representation of $G(L)$, is a closed subspace of $\mathcal{A}(G_L, K)$.

We will call the extension of Proposition 3.1 the holomorphic base change of V. Its coefficients are L–analytic (for the given embedding $L \to K$): it is L–analytic in the sense of Emerton [6].

If $\sigma \in \mathrm{Gal}(L/L_0)$ we write V^σ for the representation of $G(L)$ associated to $\iota \circ \sigma$. It is L–analytic for $\iota \circ \sigma$.

Definition 3.2 The full base change of V is the globally analytic representation of $\mathrm{Res}_{L/L_0} G(L)$ on $W = \widehat{\bigotimes_\sigma} V^\sigma$.

It is analytic for $\mathrm{Res}_{L/L_0} G(L)$ by the results of Sect. 1. (Note that L/L_0 being unramified, $\mathrm{Res}_{L/L_0} G(L)$ is again a group of the same type.) The fact that the completed tensor product is globally analytic follows from Sect. 2.

When V is the restriction to $G(L)$ — the L–points of a rigid–analytic group deduced from a suitable integral structure on a reductive group $\mathcal{G}/L$ — of a representation (still denoted by V) of $\mathcal{G}(L)$, we conjecture that this will be compatible, in some sense, with Langlands base change (still conjectural) for p–adic Banach representations of $\mathcal{G}(L)$. Of course the relation between admissible Banach representations and globally analytic Banach representations (for $G(L)$) is not one–to–one, cf. Proposition 2.5. It would be interesting to determine which Banach spaces E give rise to a given V, for instance if V is irreducible. Furthermore, even in the case of irreducible principal series V for $\mathcal{G}(L)$, the restriction to $G(L)$ is not irreducible. The full base change of Definition 3.2 then describes only certain of its submodules. This will be clear for the principal series.

3.3 .

We now consider the case of the principal series for $GL(2)$. For simplicity we assume $L_0 = \mathbb{Q}_p$. We assume $p > 2$. Let G be the rigid-analytic group over $\mathbb{Q}_p$ defined by the pro-p Iwahori subgroup of $GL(2, \mathbb{Q}_p)$. It is checked in [4] that G is naturally a rigid-analytic group, with comultiplication given by integral series.[1] As a space G is a product of 1–dimensional balls. Thus $G(\mathbb{Q}_p) = \left\{ g \in GL(2, \mathbb{Z}_p) : g \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} [p] \right\}$.

[1] In that paper p is assumed greater than 5, in order to apply Lazard's theory. However this particular computation only requires $p > 2$.

(Recall that we write G for $G(\mathbb{Q}_p)$ if the meaning is clear.) The principal series is then described by Schneider and Teitelbaum [11]. (They define the Iwahori subgroup by matrices that are lower triangular mod p while in [4] we consider upper–triangular matrices. We have followed their choice.) We fix K (and an embedding $L \hookrightarrow K$) as in Sect. 3.2.

Let $B = \left\{ g \in GL(2, \mathbb{Z}_p) : g \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} [p] \right\}$, so our group $G = G(\mathbb{Q}_p)$ is a subgroup of B. Let $P_0 \supset T_0$ be the set of upper triangular (resp. diagonal) matrices in B. Let $\chi : T_0 \to K^\times$ be a locally analytic character, such that

$$\chi \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} = \exp(c(\chi) \log(t))$$

for $t \in T_0 = (\mathbb{Z}_p^\times)^2$ when t is sufficiently close to 1. Thus $c(\chi) \in K$.

We consider first, as they do, the locally analytic induced representation of B

$$J_{loc} = \mathrm{ind}_{P_0}^{B}(\chi) = \{ f \in \mathcal{A}_{loc}(B, K) : f(gb) = \chi(b^{-1}) f(g) \}$$

($b \in P_0$), where χ is naturally extended to P_0. We have

$$B = U P_0 , \quad U = \left\{ \begin{pmatrix} 1 & \\ z & 1 \end{pmatrix} , \quad z \in \mathbb{Z}_p \right\} . \tag{3.3}$$

Note that since χ is fixed, the restriction of the functions of J_{loc} to $G \subset B$ is injective. With

$$Q_0 = P_0 \cap G = \left\{ \begin{pmatrix} s & x \\ & t \end{pmatrix} : s, t \equiv 1 , \; x \equiv 0 \, [p] \right\}$$

we see that the space of J_{loc} is

$$I_{loc} = \{ f \in \mathcal{A}_{loc}(G, K) : f(gb) = \chi(b^{-1}) f(g) \}$$

($b \in Q_0$). With (3.3) replaced by $G = U Q_0$, we see that $I_{loc} \cong \mathcal{A}_{loc}(\mathbb{Z}_p, K)$ where $\mathbb{Z}_p$ is seen as the rigid analytic (additive) group $B^1(\mathbb{Z}_p)$. The group G acts by left translations, thus by $f(g) \mapsto f(h^{-1} g)$. We now have [11, Lemma 5.2]:

Lemma 3.3 *For $y \in \mathbb{Z}_p$, $x \in p\mathbb{Z}_p$, $s, t \equiv 1 \, [p]$:*

$$(i) \begin{pmatrix} 1 & \\ y & 1 \end{pmatrix} f(z) = f(z - y)$$

$$(ii) \begin{pmatrix} s & \\ & t \end{pmatrix} f(z) = f(st^{-1} z) \chi(s, t)$$

$$(iii) \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} f(z) = f\left(\frac{z}{1 - xz} \right) \chi((1 - xz)^{-1}, \; 1 - xz) .$$

We now seek conditions such that $\mathcal{A}(B^1, K)$, where B^1 is the unit ball in the z–variable, is a globally analytic representation. We simply denote this space by $\mathcal{A}$; we will similarly drop the subscript K in this section.

Lemma 3.4 *It suffices to check analyticity separately for the* 1*–parameter (rigid–analytic) subgroups of which G is the product.*

Changing notation, denote by x, y, u, w the variables in $\mathbb{Z}_p$ deduced from the natural variables. (So x is $p^{-1}x'$ where x' is the coordinate in (iii)). Assume for instance $yf = \sum_o^\infty y^m f_m$, $\|f_m\| \to 0$ for any f, where $\| \ \|$ is the natural norm on $\mathcal{A}$. Then, with obvious notation:

$$\begin{aligned} xyf &= x \sum_o^\infty y^m f_m \\ &= \sum_{m=0}^\infty y^m \sum_{p=0}^\infty x^p f_{m,p} \end{aligned}$$

where, for each m, $f_{m,p} \to 0$ with p.

However, the norm on $\mathcal{A}$ is equivalent to the norm $\| \ \|_{\omega,x}$ deduced from the action of the (rigid–analytic) x–group. Thus we can assume that $\|f_{m,p}\| \le C\|f_m\|$. Then $\|f_{m,p}\| \to 0$ when $|m| + |p| \to \infty$. The same argument applies to any number of variables.

For $f \in \mathcal{A}$ and $z \in \mathbb{Z}_p$, $f \mapsto f(z)$ is a continuous linear form. For $s = t$, (ii) yields:

$$\begin{pmatrix} s & \\ & s \end{pmatrix} f(z) = f(z)\chi(s,s)\,.$$

If the action is analytic, we see that $\chi(s,s)$ must be an analytic function of s for $s \equiv 1\ [p]$. Now $\chi(s,t) = \chi(t,t)\chi(st^{-1},1)$. We may then consider

$$\begin{pmatrix} s & \\ & 1 \end{pmatrix} f(z) = f(sz)\chi(s,1)\,.$$

Taking $f = 1$, we see that $\chi(s,1)$ must be analytic. Moreover, if

$$f(z) = \sum_{m \ge 0} a_m z^m$$

and $s = 1 + pu$, then

$$f(sz) = \sum_n (pu)^n \sum_{m \ge n} a_m \binom{m}{n} z^m = \sum_n u^n f_n(z)$$

yields an analytic expansion, in $\mathcal{A}$, of $f(sz)$.

The condition on the analyticity of $\chi(s,t)$ is as follows. Write $\chi=(\alpha,\beta)$ with

$$\alpha(1+pu)=e^{a\log(1+pu)},\ \beta(1+pu)=e^{b\log(1+pu)}$$

$(a,b\in K)$ for $u\in\mathbb{Z}_p$ close to 0. The exponential is analytic (in K) in the domain $v_p(z)>\frac{e}{p-1}$ where $e=e(K)$; v_p is always the normalized valuation, $v_p(p)=1$. Now

$$v_p(a\ \log(1+pu))=v_p(a)+1+v_p(u)$$

since $p>2$, so we must have $v_p(a)+1>\frac{e}{p-1}$, i.e.:

$$\begin{aligned} v_p(a),v_p(b) &> \frac{e}{p-1}-1 \\ &= \frac{-p}{p-1} \qquad \text{if } K \text{ is unramified.} \end{aligned} \tag{3.4}$$

Henceforth we assume that α,β verify these conditions (" α,β are analytic" for short.) Now the action of $\begin{pmatrix} s & \\ & t\end{pmatrix}$ is a twist of the action associated to $\chi=1$ by an analytic character. Thus (i), (ii) yield analytic actions.

Now $\alpha(1+v)$ belongs to the Tate algebra on the ball $|v|\le p^{-1}$, so $\alpha(1-xz)$ belongs to the Tate algebra of two variables on $B^1=B(1)\times B(p^{-1})$. In particular it has a convergent expression

$$\sum_{m\ge 0}x^m\ \alpha_m(z)\,,\qquad \alpha_m\in\mathcal{A}$$

on this domain, convergent (for $|x|\le p^{-1}$) as a series in $\mathcal{A}$. Now for $|v|<1$

$$(1-v)^{-m}=\sum_{q=0}^{\infty}\binom{m+q-1}{q}v^q$$

so, for $f=\sum_0^\infty a_m z^m$,

$$\begin{aligned} f\Big(\frac{z}{1-xz}\Big) &= \sum_0^\infty a_m z^m\sum_{q=0}^\infty\binom{m+q-1}{q}x^q z^q \\ &= \sum_{q=0}^\infty x^q\ \sum_{m=0}^\infty\binom{m+q-1}{q}a_m z^{m+q}\,. \end{aligned}$$

We have to remember that $x\in p\mathbb{Z}_p$, so the analyticity of the action (iii) must be seen in the variable $\xi=\frac{x}{p}\in B^1$. The expression now becomes

$$\sum_{q=0}^{\infty} \xi^q p^q \sum_{m=0}^{\infty} \binom{m+q-1}{q} a_m z^{m+q}$$
$$= \sum_{q=0}^{\infty} \xi^q f_q(z)$$

with obviously $\|f_q\|_{\mathcal{A}} \leq p^{-q}\|f\|_{\mathcal{A}}$. Thus the action (iii) is analytic.

Let $\mathcal{A}_{\text{loc}} \supset \mathcal{A}$ be the space of locally analytic functions. The representation of G on $\mathcal{A}_{\text{loc}}$ is studied by Schneider and Teitelbaum in [11]. Let $\mathcal{D}_{\text{loc}}$ be the space of distributions on $U = \left\{ \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix} \right\}$ in their sense, i.e. the topological dual of $\mathcal{A}_{\text{loc}}$. We recall that $\mathcal{A}_{\text{loc}} = \varinjlim_n \mathcal{A}(n)$ where $\mathcal{A}(n)$ is the space of functions globally analytic on each ball of radius p^{-n}. Thus $\mathcal{A} = \mathcal{A}(0)$. The transition maps are injective and compact, with dense image. Dually we have $\mathcal{D}_{\text{loc}} = \varprojlim \mathcal{D}(n)$. This is a projective limit of Banach spaces, the projection maps being compact with dense image; $\mathcal{D} = \mathcal{D}(0)$.

Similarly for the rigid–analytic group G, we have $\mathcal{A}_{\text{loc}}(G)$, $\mathcal{D}_{\text{loc}}(G)$ with similar properties. Consider the maps

$$r : \mathcal{D}_{\text{loc}} \longrightarrow \mathcal{D} = \mathcal{A}' \qquad \text{(continuous dual of } \mathcal{A})$$
$$R : \mathcal{D}_{\text{loc}}(G) \longrightarrow \mathcal{D}(G)\,.$$

We have natural actions of $\mathcal{D}_{\text{loc}}(G)$ on $\mathcal{D}_{\text{loc}}$ and of $\mathcal{D}(G)$ on $\mathcal{D}$ (see 2.3), which we denote by the convolution sign.

Lemma 3.5 *For $T \in \mathcal{D}_{\text{loc}}(G)$, $F \in \mathcal{D}_{\text{loc}}$, $r(T * F) = R(T) * r(F)$.*

The maps r and R are continuous. The map $(t, f) \mapsto t * f$ $(t \in \mathcal{D}(G), f \in \mathcal{D})$ is continuous in t; similarly $(T, F) \mapsto T * F$ $(T \in \mathcal{D}_{\text{loc}}(G), F \in \mathcal{D}_{\text{loc}})$ is continuous [11]. Furthermore the finite group algebra $K[G]$ is dense in $\mathcal{D}_{\text{loc}}(G)$. It suffices then to check the formula for a single Dirac measure $T = \delta_g$, where it is obvious. The results of Schneider–Teitelbaum now easily imply:

Proposition 3.6 *If $b - a \notin \mathbb{N} = \{0, 1, ...\}$, the globally analytic representation of G on $\mathcal{A}$ is topologically cocyclic and admissible.*

Here ‘topologically cocyclic’ means that its dual is topologically cyclic.

Consider the G–map $r : \mathcal{D}_{\text{loc}} \to \mathcal{D}$, and let $X \subset \mathcal{D}$ be a closed submodule (for the action of G). Then $r^{-1}X \subset \mathcal{D}_{\text{loc}}$ is a closed submodule, invariant by $\mathcal{D}_{\text{loc}}(G)$. In [11], Schneider and Teitelbaum consider in fact the action of $\mathcal{D}_{\text{loc}}(B)$. By [11, Theorem 5.4], $\mathcal{D}_{\text{loc}}$ is (algebraically) irreducible under $\mathcal{D}_{\text{loc}}(B)$. However a glance at their proof shows that it remains irreducible under $\mathcal{D}_{\text{loc}}(G)$: the proof involves only the action of the Lie algebra, except for the argument at the bottom of p. 460. Here it must be checked that a submodule V of $\mathcal{D}_{\text{loc}}$, under the action of B, is generated by distributions, the Amice transform of which has only zeroes in the set of elements of the form $\zeta - 1$ where ζ is a root of unity (in $\mathbb{C}_p$) of p^n-order. The argument relies on the action of T_0; however it is easily seen that the action of the group of elements congruent to 1 mod p (contained in G) is sufficient.

Thus $r^{-1}X$ is null or equal to $\mathcal{D}_{\text{loc}}$. Since X is closed and $r : \mathcal{D}_{\text{loc}} \to \mathcal{D}$ has dense image, we deduce that X is equal to $\mathcal{D}$ in the second case. If $r^{-1}X$ is null, $X \cap Im(\mathcal{D}_{loc}) = \{0\}$. Since $Im(\mathcal{D}_{loc})$ is dense, choosing a suitable vector in $\mathcal{D}$ implies that $\mathcal{D}$ is cyclic. However $\mathcal{D}$ is the Banach dual of $\mathcal{A}$. Thus $\mathcal{A}$ is cocyclic.

Finally, the representation on $\mathcal{A}$ is admissible: indeed, $\mathcal{A}$ is the subspace of $\mathcal{A}(G)$ defined by the conditions $f(gb) \equiv \chi(b^{-1})f(g)$ (f is then analytic on G since χ is so) and this is a closed subspace.

This suggests the stronger result:

Theorem 3.7 *If $b - a \notin \mathbb{N}$, the globally analytic representation of G on $\mathcal{A}$ is topologically irreducible (and admissible).*

Assume that $\mathcal{X} \subset \mathcal{A}$ is a closed, G-invariant subspace. Then it is stable by the action of the enveloping algebra of $\mathfrak{g}$. Let X, Y be the usual infinitesimal generators of the upper and lower unipotent subgroups, and let H be an infinitesimal generator of the diagonal subgroup T with entries $(s, 1)$. Thus T is identified with $\{s \in \mathbb{Z}_p^\times, s \equiv 1[p]\}$. We deduce from Lemma 3.3 the action of these elements on a function $f \in \mathcal{A}$. For

$$f(z) = \sum_0^\infty a_n z^n$$

the image of f by $s \in T$ is

$$\alpha(s)f(sz) = \sum_0^\infty \alpha(s)s^n a_n z^n.$$

Therefore, since $\frac{d}{dt}|_0(s^n) = nH$ for $s = \exp(tH)$, $H \in \mathbb{Z}_p$ (for instance $H = 1$),

$$\sum_0^\infty n a_n z^n \in \mathcal{X}. \tag{3.5}$$

Moreover,

$$\begin{aligned} -Yf &= f' = \sum_0^\infty n a_n z^{n-1} \\ Xf &= -(d/dx)|_0(f\left(\tfrac{z}{1+xz}\right)\alpha((1+xz)^{-1})\beta(1+xz)) \\ &= z^2 f'(z) + (a-b)zf(z). \end{aligned}$$

Let τ_N be the natural truncation $\mathcal{A} \to K[z]_N (N \geq 0)$ where $K[z]_N$ is the space of polynomials of degree $\leq N$. Then τ_N is equivariant for the action of T. On the finite-dimensional space $K[z]_N$, the operators given by $s \in T$ are simultaneously diagonalizable, in the basis $\{z^n\}$. The associated characters of T are linearly independent. Thus $\tau_N(\mathcal{X})$ is a direct sum of the monomials for the exponents $m \in M_N \subset [0, N]$. If $N \le N'$, the surjectivity of $K[z]_{N'} \to K[z]_N$ implies that $M_N \subset M_{N'}$ and in fact $M_N = M_{N'} \cap [0, N]$

Taking M equal to the union of the M_N for all N, we see that there exists $M \subset \mathbb{N}$ such that

$$\begin{aligned} &(a) \quad f \in \mathcal{X} \Rightarrow a_n = 0 \quad (n \notin M) \\ &(b) \quad n \in M \Rightarrow z^n \in \tau_N(\mathcal{X}) \ for\ any\ N \ge n \end{aligned}$$

In particular, if $n \in M$, there exists

$$f = z^n + \sum_{m>N} a_m z^m \in \mathcal{X} \tag{3.6}$$

for any $N \ge n$. Furthermore, using the action of Y, we see that $n-1 \in M$ if $n \in M$, so $M = [0, N](N \ge 0)$ or $M = \mathbb{N}$.

Lemma 3.8 *The constant function z^0 belongs to $\mathcal{X}$.*

The proof will rely on an analogue of the operator of ‘ordinary projection’ in Iwasawa theory. Start with

$$f = a_0 + \sum_{m>0} a_m z^m \in \mathcal{X} \quad (a_0 \ne 0).$$

Then the function deduced from $H^{p-1} f$,

$$\sum_{m>0} m^{p-1} a_m z^m \in \mathcal{X},$$

so

$$Af := a_0 + \sum_{m>0} (1 - m^{p-1}) a_m z^m \in \mathcal{X}.$$

If $p|m$, $(1 - m^{p-1})^n \to 1$ if $v_p(n) \to \infty$. If p does not divide m, this power tends to 0. Thus we see, with $E = \lim A^n$ (for such values of n) that

$$\begin{aligned} Ef &:= a_0 + \sum_{p|m} a_{pm} z^m \\ &= \sum_{m \ge 0} a_{pm} z^{pm} \in \mathcal{X}. \end{aligned}$$

Applying again the transformation given by formula (3.5), dividing by p, and iterating, we see that $\sum_{m>0} m^k a_{pm} z^{pm} \in \mathcal{X}$. Therefore

$$A_1 f := a_0 + \sum_{m>0} (1 - m^{p-1}) a_{pm} z^{pm} \in \mathcal{X}.$$

Defining E_1 as the limit of A_1^n for $v_p(n) \to \infty$, we see that $a_0 + \sum_{m>0} a_{p^2m} z^{p^2m} \in \mathcal{X}$. Iterating again, we see finally that the constant $a_0 z^0$ belongs to $\mathcal{X}$.

We can now finish the proof of Theorem 3.7. For $f = z^m$, the formula for the action of X yields $Xf = (m + a - b)z^{m+1}$. If $b - a \notin \mathbb{N}$, we see, starting with z^0, that all monomials are in $\mathcal{X}$. Since $\mathcal{X}$ is closed, it is equal to $\mathcal{A}$. On the other hand, if $b - a = N \in \mathbb{N}$, it is easy to see from Lemma 3.3, or from the derived representation, that the space of polynomials of degree $\leq N$ is stable by G: the corresponding representation is the irreducible representation of G of degree $N + 1$ and central character $\alpha\beta$.

3.4 .

Let now L be an unramified extension of $\mathbb{Q}_p$, of degree r. Denote by $I_{\mathbb{Q}_p}(\chi)$ the previous representation of G, on globally analytic functions. Let $I_L(\chi)$ be its extension to $G(L)$ by holomorphic base change (Proposition 3.1). It is an L-analytic representation , still given by the formulas of Lemma 3.3. Note that $L/\mathbb{Q}_p$ being unramified, the holomorphic extension of χ to $(\mathcal{O}_L[1])^2$, where $\mathcal{O}_L[1] = \{x \in \mathcal{O}_L^\times : x \equiv 1[p]\}$, verifies (3.4). The representation of $G(L)$ is realised on the L-analytic functions on B^1 , seen as an L-analytic space. More generally, if $\underline{\chi} = (\underline{\alpha}, \underline{\beta})$ is a pair of characters of $\mathcal{O}_L[1]$ verifying the condition extending (3.4) (see before Theorem 3.11), we can consider the globally $\mathbb{Q}_p$-analytic vectors in the induced representation of $G(L) = Res_{L/\mathbb{Q}_p} G(\mathbb{Q}_p)$ on $\mathcal{A}_{loc}(Res_{L/\mathbb{Q}_p} U(\mathbb{Q}_p), K)$. This representation will be denoted by $I(\underline{\chi})$.

Proposition 3.9 *If $b - a \notin \mathbb{N}$, the holomorphic base change $I_L(\chi)$ of $I_{\mathbb{Q}_p}(\chi)$ is topologically irreducible.*

(The irreducibility clearly follows from the irreducibility under $G(\mathbb{Q}_p)$.)

We now compare the base change functor we have constructed with the natural consequences of a (conjectural) Langlands functoriality for p–adic representations. We refer the reader to the Introduction to [3] for more motivation. The principal series representation of $G(\mathbb{Q}_p)$ is one of two summands (under $G(\mathbb{Q}_p)$) of an irreducible representation π of $\mathcal{G}(\mathbb{Q}_p) = GL(2, \mathbb{Q}_p)$ [11, Sect. 5], the principal series associated to the representation of the Galois group

$$\sigma \mapsto \alpha(\sigma) \oplus \beta(\sigma)$$

($\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p)$). Here we have assumed α, β extended to $\mathbb{Q}_p^\times$, thus giving characters of the Weil group $W_{\mathbb{Q}_p}$, and $\alpha(p)$, $\beta(p)$ units so the representation of $W_{\mathbb{Q}_p}$ actually extends to the Galois group.

In conformity with the general formalism, the base change π_L of π should be associated to the couple of characters $(\alpha \circ N_{L/\mathbb{Q}_p}, \beta \circ N_{L/\mathbb{Q}_p})$. Thus, instead of