Mathematics Education in the Digital Era

Nigel Calder · Kevin Larkin
Nathalie Sinclair *Editors*

Using Mobile Technologies in the Teaching and Learning of Mathematics

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Mobile Technologies: How Might Using Mobile Technologies Reshape the Learning and Teaching of Mathematics?

Nigel Calder, Kevin Larkin and Nathalie Sinclair

As our attention moves to the opportunities and constraints that mobile technologies (MT) might afford, app developers, teachers and researchers have become more adept at identifying and enacting opportunities for enhancing mathematical thinking. These opportunities emerge through the various environments, both hardware (i.e., tablets) and software (i.e., applications), and the mathematical activity that these facilitate. The features of MT, for instance the ability to use in-built video and audio tools, allows users to capture authentic data in their everyday world and use the data for modelling, or statistical inference. Processing this data in situ changes the nature of the learning experience. Likewise, the potential for visual, interactive engagement with some learning experiences, coupled with the haptic and oral/aural affordances of the technology, change the nature of the mathematical activity. By inference, this changes the nature of the mathematical thinking. Engaging with number sequences by creating sets of objects that represent numbers as you touch the screen, using an oral count, using concrete materials, or learning a sequence by rote, are all different representations of number that might evoke a variety of understandings and ways of thinking mathematically. Being able to elegantly connect these various representations, and move between them, appears to offer opportunities for deeper conceptual understanding of mathematics concepts.

With MT being a relatively recent addition to the scope of digital technologies that might facilitate learning in mathematics, there is a need for research on the use

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of MT, and for this research to be presented in a coherent, multi-faceted manner. Such research is of particular importance as schools are investing heavily in mobile devices, often without a concomitant investment in developing practices regarding how such devices may be used to develop conceptual rather than procedural or declarative knowledge (e.g., Calder, 2011), nor of how such devices connect with other resources in the classroom. The effectiveness of their engagement in shifting conceptual understanding is also contingent on associated professional learning for teachers (O'Malley et al., 2013). Theoretical frameworks for teaching and learning with MT might also be influential in our understanding of the ways that using MT in the learning of mathematics might be examined.

Some researchers have noted a lack of theoretical rigour regarding the use of MT (e.g., Larkin, 2015), identifying issues in relation to the lack of mathematical quality of many mathematics apps. He also reported the lack of time and expertise for teachers to accurately evaluate them or their use. Nevertheless, MT offer vast potential to enhance mathematical learning. This book builds on international research (e.g., Attard, 2015; Calder & Campbell, 2016; Moyer-Packenham et al., 2015; Sinclair & Heyd-Metzuyanim, 2014) into the use of MT in mathematics education. It includes an examination of the ways MT might influence student engagement, cognition, collaboration and attitudes, through reshaping the learning experiences across a diverse range of year levels and contexts.

Central to learning mathematics through using MT is the nature of the tools and the apps utilised, the learning intentions of the teacher, and the type of activity that the students are engaged with. While there is frequently a focus in schools, and in the media, on *consumable* apps that is, those where students follow a set task at a specified level; more recently there has been a focus on apps that: enable students to create screencasts of their mathematical thinking; can be used for coding, including the programming of small robotic devices; enable students to create visual, dynamic representations of mathematical situations.

In this relatively new field of engaging with mathematics learning through using MT, this book reflects the growing understanding of how the learning experience might be reshaped to harness the opportunities that MT afford. It also incorporates an examination of using MT for developing mathematical thinking, enhancing teacher pedagogy, and understanding the embodied cognition inherent when using mobile, touch-screen devices. In addition, the broader assemblages incorporating underlying discourses and political elements are hugely influential in using MT effectively. The book proposes emerging frameworks, or new uses for existing frameworks that encourage educators to better interrogate student engagement and learning aspects as well as to evaluate the vast range of apps that continuously appear. As the field is always in flux, with researchers and teachers often scrambling to keep up with recent innovative developments, we are reminded to look beyond the specific to the general (Mason, 2005). Thus we look to find common themes and trends that enable us to gain insights into and evaluate MT, and the associated learning/teaching practices, from vantage points not dependent on understanding specific examples or apps. Not withstanding the need for a global approach, the examination of individual apps and experiences are crucial as they mark new initiatives and point toward potential innovation. And what of the methodologies that we use to theorise the terrain? There is an advantage in looking through a range of lenses, as what one lens doesn't highlight, another may. However, with the continual developments in technology, we need to also consider how MT might open up innovative approaches to methodology and research design. As a community, we need to continue to explore the edges, while incorporating the generic ways these innovations inform practice, reshape the learning experience and might enhance students' mathematical thinking.

This book draws from a diverse range of international studies, where MT have influenced the ways that learning might occur across a range of educational contexts. The book is divided into four sections: *Looking across the terrain; Traversing the learning and teaching landscape; Navigating content: focussing on particular concepts*; and *Exploring new forms of communication to make mathematical learning visible*. While the purpose of these groupings is to draw the reader's attention to particular themes within the chapters, there is nevertheless considerable overlap between the sections, and most chapters could have easily been situated in more than one section.

The first section, *Looking across the terrain* considers some generic aspects that straddle the diversity in the field. Larkin and Milford use cluster analysis to group apps based on particular processes, features and concepts. Undertaking this process enables the apps to be grouped independently from developers' marketing and highlights particular aspects that mathematics educators consider as important. Through this educators are supported in their selection of apps for the user's intended purpose, and hence enhance app use in mathematics classrooms. Calder and Murphy analyse aspects of a 2-year study involving primary children learning mathematics through the use of mobile devices and apps. With the teachers as co-researchers, they examined teacher practice and the inter-connectivity between teacher pedagogy and the affordances of the apps. The teachers used a diverse range of apps, including ones for screencasting and coding, with the various learning experiences and opportunities for influencing the learning outlined and considered. An interesting theme to emerge across the use of various MT was socio-material assemblages. The final chapter in this section also investigates the interplay between a range of contexts and the types of apps used in these contexts. Attard considers the notion of student engagement when using apps for mathematical learning. Her framework for interrogating aspects of engagement incorporates cognitive elements. She draws together common threads about engagement, while synthesising insights into a collection of different studies related to engagement in various contexts.

The second section, *Traversing the teaching and learning landscape,* includes chapters that link teaching and learning related to various learning processes that utilise particular processes or affordances of MT. Kyriakides and Meletiou-Mavrotheris discuss a multifaceted programme designed to provide a group of in-service teachers with the knowledge, skills, confidence, and practical experience required to effectively use tablet devices for enhancing mathematics teaching and learning. The teachers integrated the app *A.L.E.X* into their lesson plans and thus reshaped the students' learning experience. Sollervall, de la Iglesia, and Zbick explore how mathematics classroom teachers can implement an innovative mobile learning activity. They report on an ongoing, 5-year study into using a GPS app for geometry, with the focus of the activity involving GPS and spatial orientation tasks that are executed in outdoor settings. Sedaghatjou and Rodney's chapter considers how a particular multitouch app called *TouchCounts*, along with children's collaborative engagements, can enhance mathematical learning of number. They utilise *StudioCode* software to better understand children's collaborative, gestural practices within the *TouchCounts* environment. In the final chapter of this section, Bokhove, Clark-Wilson, and Pittalis consider two cases of how MT provided opportunities for "mathematics outside the classroom". The examples describe how using mobile phones with augmented reality allowed students to bridge between formal and informal mathematics learning. Their examples, a dynamic Ferris wheel and a static cathedral are used to demonstrate how educators can use *geo-location* and *augmented reality* to enhance the learning of mathematics through MT.

In the third section of the book, *Navigating content: focussing on particular concepts*, the authors primarily attend to specific mathematical concepts or processes. Pelton, Milford, and Francis Pelton use an app to develop children's understanding of time. Their chapter details the integration of a researcher-designed iPad app into a series of collaboratively created lessons to facilitate the learning of clock-reading and time concepts. The authors used a lesson study approach to design and refine the intervention that included teacher-led activities and structured use of the iPad app. Lommatsch, Tucker, Moyer-Packenham, and Symanzik examine what patterns were revealed when heatmaps were used with hierarchical clustering to examine pre-schoolers' performance with two touchscreen mathematics apps in two different learning sequences: counting and seriation. Their analysis highlighted changes in children's performance, speed, and developmental progressions after using the two apps. The use of hierarchical clustering analysis facilitated the analysis of individual and whole group data leading to the identification of young children's developmental progressions. Rosen, Palatnik, and Abrahamson explore an embodied-design for engaging particular mathematical concepts with an action level where the virtual objects were either generic (e.g., a circle), or situated, (e.g., a hot-air balloon). They evaluate an instructional methodology whereby students first learn to physically move objects on the screen before eventually generalising these movements as formal mathematical rules. Chorney and Sinclair describe a research project with first-grade children using a multi-touch, dynamic geometry app called *WebSketchpad* to study how the concept of symmetry arises. They analyse the data through the lens of inclusive materialism, considering the intra-actions involved in the child-device-geometry assemblages, and how new mathematical ideas might emerge from these assemblages. Their particular focus is how the multi-touch environment can provide the basis for emerging geometrical ideas. Ferrara and Savioli discuss a classroom-based intervention with a group of first-grade children using the multi-touch app *TouchCounts* to develop children's number sense. They investigate how understanding might emerge out of the relational entanglement of numbers, iPads, and learners, engendering new kinds of mathematical experiences with number and providing the basis for emerging relational meanings of number. In the final chapter of this section, Soldano and Arzarello examine an approach to geometry in a secondary-school context. Their chapter illustrates a way of using MT to support the transition from an empirical to a theoretical approach to geometry. Drawing on Zbiek's et al., (2007) notions of pedagogical, mathematical and cognitive fidelities, they implement group game-activities whereby students investigate the geometric property upon which the game is designed.

In the final section of the book, *Exploring new forms of communication to make mathematical learning visible,* the notion of screencasting is the focus. The use of screencasting opens up opportunities for mathematical thinking of learners to become more transparent as students and teachers might create individual explanations of their thinking using a blend of both digital tools and their associated social elements. Galligan and Hobohm examine a case study of the use of mobile devices and screencasting in university mathematics education teaching. They incorporate this with an evaluative tool for teachers and students to evaluate their own and others' screencasts, with the intention of developing pre-service teachers' understanding of mathematics and ways to teach it. The chapter concludes with recommendations for using screencasting to assist with developing mathematical understanding and pedagogical content knowledge. Prescott and Maher explore the ways primary-school students worked collaboratively to solve a problem, explaining their mathematical thinking. The students used screencasting apps such as *Explain Everything* and *Educreations* to produce create-alouds, which helped them to collaboratively understand and explain mathematical concepts. The apps also assisted teachers in providing formative assessment and feedback to the students. In the final chapter of this section, Ingram, Pratt, and Williamson-Leadley discuss how a Show and Tell app can make the students' thinking more observable in problem solving. They consider how using Show and Tell apps for problem solving can lead to improvements in the level and quality of student engagement. Students were encouraged to socially negotiate their understandings, making student thinking more visible during this process. The apps can also scaffold students in reflecting upon the processes they used for problem solving.

The learning opportunities provided, and the evolution of the ways of promoting engaging mathematics learning and thinking through MT, exist in a fast moving, dynamic space; one where the comparative costs for mobile devices and connectivity are dropping markedly. There are also emergent MT that might quickly come to dominate the field: virtual reality is already developing rapidly, as is artificial intelligence and robotics. Some trusts and educational systems are distributing, to all schools in their community, 3-D printers that can print using materials as diverse as wood and titanium. Due to the reduction in the costs of MT, many of the previous equity and accessibility issues are alleviated. The potential to envisage space, location, shape, number, movement and rates of change has already been transformed. Likewise, ways of analysing data to model real-life situations in situ have changed. Yet, despite the rapid change we have witnessed in recent times, we do not really know where the technology, or its potential as a digital pedagogical medium, is headed. What might the landscape look like in 5 years, let alone in 20 years?

Underpinning each of the chapters in this book is the understanding that mathematical thinking must be given primacy—in the end it is our guiding premise and intention. We need to ask ourselves whether what we do with MT enhances mathematical thinking and understanding and to reflect on how the MT might be changing what counts as mathematical activity. The chapters in the book contribute to mathematics teaching and learning by providing readers with opportunities to reflect on their practice. It is not possible, nor wise, to ignore the role of MT in enhancing teaching and learning in mathematics. Therefore, all educators need to be alert to the potential that MT provide to enhance student learning.

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Part I
Looking Across the Terrain

Mathematics Apps—Stormy with the Weather Clearing: Using Cluster Analysis to Enhance App Use in Mathematics Classrooms

Kevin Larkin and Todd Milford

Abstract Mathematical apps are now used in many school settings. To support teachers in making appropriate pedagogical decisions regarding their increased use, empirical, quantitative analyses of apps are required. This chapter initially explores how cluster analysis can be used to identify elements within individual apps so that similar apps may be grouped together. This will assist teachers to make decisions regarding which apps might be most appropriate, either singularly or in groups, for various elements of their practice. Based upon selection criteria and ranking via four criterion-based scales, the cluster structure of 57 apps, primarily supporting number and algebraic thinking in elementary mathematics classrooms, is reported. The chapter then explores the homogeneity and heterogeneity of these clusters of apps and indicates when and how these apps may be used to enhance student mathematical learning. The chapter therefore makes both methodological and pedagogical contributions to the broader discussion of the use of apps in primary mathematics classrooms.

Introduction

This research is an extension of a broader research project that has been on-going by the lead author since 2013. In brief, the initial part of the project (Phase 1), investigated the usefulness of modified versions of three measures, already used in published research, in evaluating the pedagogical appropriateness of approximately 200 apps categorised as educational at the iTunes store. The three modified

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measures used in Phase 1 were the Haugland Scale (Haugland, 1999), Productive Pedagogies (Education Queensland, 2004), and Gee's Principles (2003). The use of these three measures is detailed in full later in this chapter. Based on the successful use of cluster analysis in evaluating 53 Geometry Apps (Larkin & Milford, 2018) we felt it necessary to revisit the earlier evaluation of Phase 1 apps (See Larkin, 2015) using cluster analysis to further understand why these apps were identified as being of high quality or otherwise.

Our research questions for this part of the research project are:

1. Whether and how the overall cumulative data on each app, gathered and discussed in Phase One, can be made more useful for teachers and researchers by additional analysis at a more granular level via cluster analysis; and
2. Whether and how specific number, algebra and a small number of non-number/algebra apps might be made more useful for specific teaching and learning activities if used in concert with other apps.

We initially argued in Larkin and Milford (2018) that cluster analysis provided a greater depth of valid and reliable information regarding the use of Geometry apps than had been provided in the earlier analysis by Larkin (2016). However, we seek here to increase the efficacy of cluster analysis for evaluating apps. The intent of this chapter is to, via a review of 57 apps using cluster analysis, assist teachers in choosing from these apps the most appropriate ones for their teaching whilst at the same time promote a methodology that more rigourously evaluates the quality of apps used individually and also in concert with other apps. Consequently, in this chapter, we offer an enhanced methodological approach to app research and uncover further pedagogical insights regarding app use.

Literature Review

It is encouraging that since 2013 there has been an increase in the number of researchers investigating the use of mathematics apps in elementary and primary school classrooms (See Calder, 2015; Sinclair & Pimm, 2014). This is an important addition to the body of mathematics knowledge on the why and how of app use as apps are often used in schools without a strong, conceptual, pedagogical, or methodological underpinning. The use or misuse of novel technologies in mathematics is, of course, not a new educational experience, as previous waves of technology—e.g. calculators, computers, and virtual applets—have each impacted upon schools. What is perhaps different about the use of tablets is that, due to their rapid uptake in use in non-school contexts, there is additional pressure on schools to incorporate their use in classrooms.

App Affordances

One strand of current research investigates the use of single or small groups of apps. There are currently very few such studies, and their focus is not specifically on supporting teachers choose which apps to use in their classrooms. For example, Sinclair, Chorney and Rodney (2016) investigated the affordances (academic, social and affective) of [*TouchCounts*], an app specifically designed and created for counting and doing arithmetic for early years students (3–8 years old). In this research, the authors identified rhythm as the primary unit of analysis and uncovered that the design of the app, which incorporated rhythm, worked as a motivational tool in terms of mathematics engagement as well as fostering the development of mathematical understanding in relation to early arithmetic. This work develops the notion of "finger gnosis" where direct and tactile engagement with [*TouchCounts*] fostered understanding of cardinality (Sinclair & Pimm, 2014).

Holgersson et al. (2016) used the app [*Fingu*], a multitouch virtual manipulative for understanding and mastering part-whole relationships for numbers 1–10 and established that the app provided valuable opportunities for early number development. In addition, they note that app design is a dynamic process and that [*Fingu*] continues to improve as an educational tool as newer versions are released. Lange and Meaney (2013), use Bishop's six mathematical activities and a Bernsteinian framework, to evaluate whether mathematical apps can support learning in preschool students. They suggest that mathematical apps provide opportunities for young learners to make their mathematical thinking more visible. Moyer-Packenham et al. (2016) report on the use of video to record 3–8 year olds interacting with 18 mathematics apps. Overall they found that although the apps aided student development, the level of development was varied, highlighting the difficulty in making broad educational claims regarding the use of mathematics apps. More recently, Lommatsch, Tucker, Moyer-Packenham, and Symanzik (2018 this volume) used cluster analysis to examine changes in the development progression of counting and seriation when supported by the use of pre-selected apps.

The work conducted by the range of authors above is promising and of significant value to teachers; however, as yet it is limited in scope to evaluating the affordances of pre-selected apps. As such, this research is limited in usefulness for teachers in determining the value of other apps.

Generic Reviews of Apps

A second strand of recent research has involved reviews of apps, mainly iPad apps, due to the prevalence of iPads as the tablet of choice in elementary or primary schools. For example, Highfield and Goodwin (2013) evaluated 360, iTunes store apps in relation to age appropriateness, curriculum content and an initial classification of constructive-manipulable and manipulable-instructive. Powell (2014), in providing advice for teachers to find apps for their students, suggests that they begin with “an app search using the standard iTunes categories: ‘Best New Apps,’ ‘Top Free Apps,’ or ‘Top Paid Apps’” (Powell, 2014, p. 21). Given the approximate 250 000 educational apps available (PocketGamer.Biz, Sept, 2016), the approach suggested by Powell is likely to be very time consuming and may not result in the discovery of useful mathematics apps.

Whilst the examples noted above often provide a useful starting point, reviews such as these are often generic and thus only provide teachers with a broad overview of the apps. In many cases, the broad overview may appear to meet some basic teacher requirements—e.g. drill and practice apps, apps that may keep early finishers engaged etc. As mathematics educators, what we suggest is also required is specific information regarding the types of mathematical knowledge inherent in the apps, the fidelity of the mathematics contained in the apps (Dick, 2008), or how the apps might be used productively in classrooms to support deeper mathematical learning. More recently, however, perhaps as a response to the growing demand for robust research into the quality of apps, a number of research articles have been published. For example, Namukasa, Gadanidis, Sarina, Scucuglia, and Aryee (2016) designed an instrument to evaluate apps according to their curriculum content, the range of affordances available in the app as a learning tool, user interactivity, and the quality of the overall design of the app. Their instrument offers a mechanism for teachers to evaluate apps use in upper primary and junior secondary mathematics, an area where there is a dearth of quality apps (Larkin, 2013, 2015) and is the type of broad based, robust, peer-reviewed research that is further required to enhance mathematics education using mathematics apps. While we support the work of Namukasa et al. (2016), our goal in this chapter is to add to extant research by evaluating a large number of apps using cluster analysis, and then examining of the types of mathematics promoted within each cluster of apps. We suggest that this approach provides more specific information regarding how teachers can coordinate the use of various elements of different apps to support mathematical learning beyond what can be achieved using individual apps.

Study Design—Preparing the Data and Cluster Analysis

To answer the research questions we initially used cluster analysis, a collection of multivariate techniques that group individuals or objects (in our case apps) into clusters so that objects in the same cluster are more similar to one another than they are to objects in other clusters (Hair, Black, Babin, Anderson, & Tatham, 2006). In this way, cluster analysis maximizes the homogeneity within clusters, while at the same time maximizing the heterogeneity between clusters. In essence the approach attempts to keep more ‘like things’ together while simultaneously keeping ‘unlike things’ separate. In terms of apps and the clusters they are grouped into, this is useful information for teachers in that apps located within a cluster likely provide similar teaching and learning opportunities. Subsequently, teachers can then make more informed choices regarding selections of different apps to support different types of mathematical knowledge and skills.

Target Population and Criteria for Inclusion

The initial population in this study were elementary mathematics apps labelled as educational for school children aged (5–11) at the iTunes App store. An initial search in 2013 on the term "mathematics education" returned 3740 apps thus the population was reduced with a targeted search using the following terms: elementary, primary, junior or infant mathematics. This still generated over 200 apps. A second level of quality control was then used (Larkin, 2013, 2015) by evaluating the apps using The Haugland Software Developmental Scale (Haugland, 1999). The Haugland Scale is a criterion based tool used to evaluate the appropriateness of web based applications and software for use by children. The Scale includes ten items including is the child in control of the learning, does the software cater for expanding complexity, is the software ethically sound and does the design of the app support independence and real world experiences. The scale was further modified for this research by clustering the ten items into three sub-dimensions (Child-Centred, Technical Design and Learning Design), and relating each dimension to an aspect of mathematics education. Each sub-dimension contributed to the overall score with child-centred scoring (0–4), technical design (0–3) and learning design (0–3). At the end of this evaluation procedure, 57 apps were determined as age appropriate and form the data set for this chapter. See http://tinyurl.com/ACARA-Apps for a full list of the reviewed apps. This link provides further details about each of the apps including price, Curriculum strand and sub-strand, Year Level appropriateness, type of knowledge developed (conceptual, procedural or declarative) as well as a lengthy review of the app in terms of its strengths and weaknesses.

Materials and Procedures

The 57 apps were then evaluated by the lead author using two further measures Productive Pedagogies (Education Queensland, 2004) and Gee's Principles (2003). It is acknowledged here that this initial evaluation was based on a subjective evaluation; however, the lead author has over 30 years experience as a primary educator and 5 years teaching primary mathematics education at university. He has also written extensively about the review process in both professional and academic publications. In addition, in this chapter, international colleagues supported the further evaluation of the apps. These two measures were further modified to be quantitative and better targeted to evaluate mathematics apps. A full account of how Productive Pedagogies and Gee's Principles were used to score the initial apps is

Table 1 Productive pedagogies (Education Queensland, 2004)

Dimensions	Sub dimensions
Intellectual quality *Total possible sub-dimension score = 30*	Higher order thinking/5 Deep knowledge/5 Deep understanding/5 Substantive conversation/5 Knowledge as problematic/5 Metalanguage/5
Supportive environment *Total possible sub-dimension score = 25*	Student direction/5 social support/5 Academic engagement/5 Performance criteria/5 Self regulation/5
Connectedness *Total possible sub-dimension score = 20*	Knowledge integration/5 Background knowledge/5 Connectedness to the world/5 Problem based/5
Total overall possible score = 75	

provided in Larkin (2015); what follows is a summary of the key aspects so that readers of this chapter understand how the apps were scored (see Tables 1 and 2).

The Productive Pedagogies are grouped under four dimensions: intellectual quality, supportive classroom environment, connectedness, and recognition of difference (Table 1). As very few apps attempted to cater for recognition of

Table 2 Modified learning principles with definitions. (Adapted from Gee, 2003)

Learning principle	Modified definitions
Active learning	All aspects of the app environment are set up to encourage active and critical, not passive, learning
Semiotic	Learning about and coming to appreciate interrelations with and across multiple sign systems as a complex system is core to tech learning experience
Achievement	For all learners there are intrinsic rewards from the beginning, customised to each learner's level and signalling the learner's ongoing achievements
Regime of competence	The learner operates within, but at the outer edge, of his/her level of competence so that there is both safety and challenge
Probing	Learning is a cycle of probing the world; reflecting in and on this action and, on this basis, forming a hypothesis for future testing
Multiple routes	There are many ways to complete the app, each of which caters for the strengths and interests of the learner
Situated learning	The meaning of signed are situated in embodied experiences and generated meanings are discovered bottom up
Practice	Learners get lots and lots of practice in a context where the practice is not boring and they therefore spend lots of time on the task
Discovery	Overt telling is kept to a minimum, allowing ample opportunity for the learner to experiment and make discoveries
Transfer	Learners are given ample opportunity to practice and transfer what they have learned to problems requiring adaptations and transformation

difference, this dimension was discarded leaving three dimensions and fifteen Productive Pedagogies. The second measure used was a modified version of Gee's (2003) Principles. Based on the experience of the earlier evaluations, it became clear that many of the original 36 principles were not applicable for evaluating apps and that the entire 36 criteria would be too cumbersome (Jorgensen & Lowrie, 2012). For these reasons the number of principles was reduced to 10 (Table 2).

Each of the 57 apps in this study was evaluated by the first author on each of the sub-dimensions of the Productive Pedagogies and modified learning principles of the Gee. Each sub-dimension or learning principle was scored on a scale from low (1) to high (5), the range and score was variable. For example, there were 6 sub-dimensions for Intellectual Quality (IQ) resulting in a maximum possible score of 30. Following this break-down, the maximum possible score for Supportive Environment (SE) was 25, for Connectedness (C) 20 and, for the modified Gee's Principles (GP), 50 as there were 10 modified learning principles. A low score (1) on, for example the criteria of Transfer, indicates that any activity in the app is only relevant within the app (feeding an avatar to earn points) whereas a high score (5) indicates that the app fosters learning more broadly applicable (visualising rotations and reflections). A truncated version of descriptive statistics for the apps used in this study, in rank order, and based upon scores on the productive pedagogies and Gee's Principles, is presented in Table 3.

The internal reliability across the Productive Pedagogies in this study was calculated at $\alpha = 0.897$ and the reliability of the GP was calculated at $\alpha = 0.861$. To add robustness to the evaluation of the app and to offer further evidence of the psychometric quality of the scales used here, inter-rater reliability was calculated to determine whether these scales were consistent across more than one rater. Fifteen apps were randomly selected from the 57 and sent, along with accompanying documentation on each of the scales, to three graduate students who work with the second author at the University of Victoria. Fifteen apps were selected for the graduate students to confirm internal reliabilities as 25% of the total number of apps exceeds the informal 20% that is suggested when reliability is estimated from a sample. Other graduate students were given the same apps to evaluate. Their responses were then compared to the responses completed by the first author for

Table 3 Top and bottom app scores for productive pedagogies and modified Gee Principles

Selected apps	Productive pedagogies			Gee learning principles
	IQ	SE	C	
Mathemagica—Kids math	28	23	20	36
Area of rectangles	28	22	16	37
Early numbers: maths wizard counting	24	22	14	26
…	–	–	–	–
Telling time free	11	10	7	18
Math party	11	11	6	16

inter-rater reliability. The alpha for each of IQ, SE, C and GP was 0.714, 0.766, 0.758, and 0.790 respectively (generally a value > 0.7 is considered acceptable for the inclusion of scales with non-critical consequences). It is not surprising that these values for alpha were lower than the authors, as they were not teachers nor experienced mathematics educators.

Building upon the internal consistency and inter-rater reliability calculated above, the rating of each of the 57 apps, as generated by the four scales (i.e., IQ, SE, C and GP), was used as data for a subsequent cluster analysis using SPSS v.22. We initially measured similarities as the squared Euclidian distances between each pair of apps on each of the four scale characteristics. In this way, smaller distances were viewed as indicating greater similarity. Once the similarity measures were calculated, a hierarchical procedure via the centroid cluster—which joins the apps in a weighted combination of the central points of the two individual clusters, where the weights are proportional to the sizes of the clusters—was applied to the clusters. Lastly, the number of clusters was determined, based upon the output, with the objective of generating the simplest structure possible while still representing homogeneous groupings. The number of clusters was determined by both the output and also a decision by the researchers to identify the simplest structure possible while still representing homogeneous groupings.

Findings

Initial descriptives for the scales used to run the cluster analysis are presented in Table 4. All variables were presented in their original scale here (i.e., 30, 25, 20 and 50 respectively). There is no specific sample size required for cluster analysis; however, the data was screened for outliers and none were uncovered.

A correlational analysis (Table 5) was subsequently run on the four scales to determine if their inclusion in the cluster analysis would be appropriate, or if any overlap (i.e., multicolinearity—where two or more of the scales are highly correlated) might account for double counting (Hair et al., 2006). For example, the scores for Connectedness and Intellectual Quality are correlated at 0.812 and share over 64% of their variance.

Based upon this table, it was determined that the scales were all moderately to highly correlated (i.e., between 0.443 and 0.812) and thus multicolinearity was an

Table 4 Scale descriptives

Variable	N	Mean	Median	SD
Intellectual quality (30)	57	17.2	17	4.23
Supportive environment (25)	57	15.6	15	4.10
Connectedness (20)	57	11.4	12	3.20
Gee's principles (50)	57	24.1	22	7.62

Table 5 Correlations of the 4 scales

	Intellectual quality	Supportive environment	Connectedness	Gee
Intellectual quality	1.00			
Supportive environment	0.789**	1.00		
Connectedness	0.812**	0.671**	1.00	
Gee	0.597**	0.433**	0.553**	1.00

**Correlation is significant at the 0.01 level (2-tailed)

issue with this data set. Hair et al. (2006) suggest that using Mahalanobis distance (D^2)—which bases clusters upon the central distance between clusters—can account for correlation among variables as it weights each variable equally. To account for this issue in SPSS, each scale score was standardized (i.e., mean deviated and divided by the standard deviation)—as distance scores are quite sensitive to differing magnitudes among the variables—and the centroid cluster option, which proportionally weights the apps, was applied to the clusters. The results of this second analysis are provided below.

Because we used an agglomerative method to determine clusters (i.e., each app started out as its own cluster), the dendrogram detailed in Fig. 1 should be read from left to right. Starting on the left with each of the 57 apps as its own cluster, using the centroid method of similarity, apps are combined one step at a time, based upon which two are the most similar, and are formed into a new cluster. The horizontal lines are indicative of homogeneity. The longer the horizontal line the more dissimilar the clusters are that are merged. For example, Fig. 1 indicates that [Case 38] is very homogenous to [Case 37]; in contrast the length of the connecting horizontal line indicates that [Case 1 and 2] are more heterogeneous to each other and also as a pair to [Case 6]. Based upon this distance measure, the vertical line was placed on the dendrogram to highlight the three-cluster solution.

From the dendrogram there are a number of apps (i.e., 1, 2, 6, 27, 36, 43, and 54) that are not captured in the three-cluster solution presented here. In order to convey any meaningful information, clusters need to contain at a minimum three objects and based upon this solution, none of these apps combined into a cluster. This is possibly due to the sample size as these apps may capture additional attributes not detailed in three clusters or these apps may be outliers. One additional display that helps to demonstrate why the three-cluster solution was selected is provided in Table 6. What is evident is that [*Mathemagica*] and [*Area of Rectangles*] always combine and are always separate from all other apps (except [*Math Galaxy Fractions*] until Cluster 6) regardless of where the cluster solution is placed. Likewise [*Hands on Maths*] remains separate to all other apps after Cluster 2. This indicates that whether a three, four, six or six cluster solution was tried, these seven apps consistently demonstrated heterogeneity from the other 50 apps evaluated via cluster analysis. This is significant as it means that (a) these seven apps are

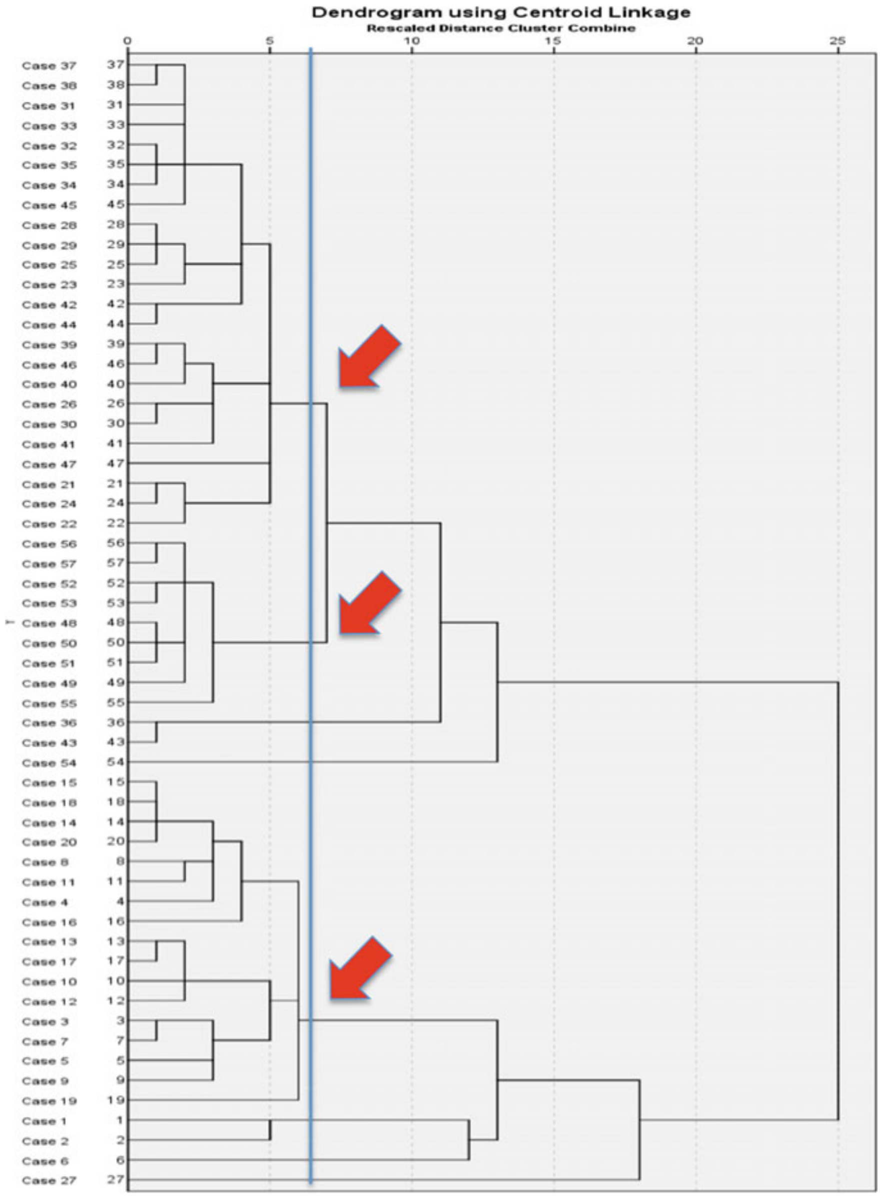

Fig. 1 Dendrogram—with vertical line and arrows indicating point of cluster formation

un-clustered and (b) that there must be distinguishing aspects within each app which can be uncovered to account for their heterogeneity.

A final criterion for the selection of number of clusters is based upon the location where the distance coefficient makes the biggest jump (i.e., a simple percentage

Table 6 Various possible cluster solutions of the 57 apps

App	8 cluster	7 cluster	6 cluster	5 cluster	4 cluster
Mathemagica—Kids math	1	1	1	1	1
Area of rectangles	1	1	1	1	1
Math galaxy fractions fun	3	3	3	1	1
Hands-on maths	5	5	5	4	4
Probability tools	6	6	4	3	3
Geometry 4 Kids	6	6	4	3	3
Tens frame	8	7	6	5	3

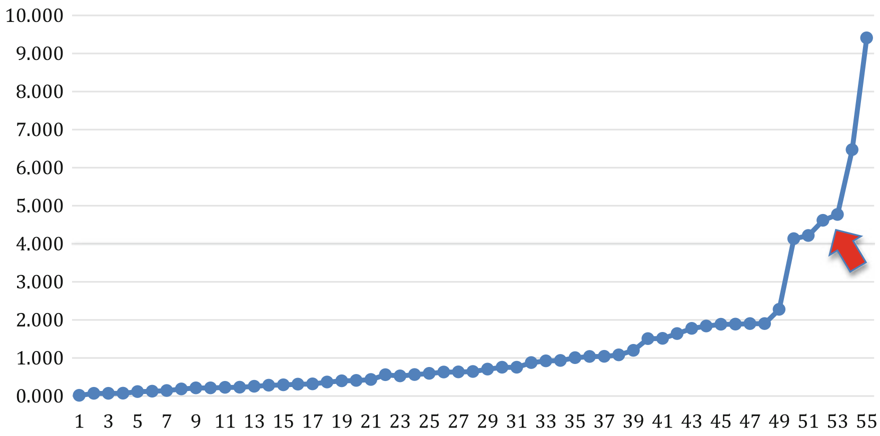

Fig. 2 The scree plot

change in heterogeneity of the clusters). This is provided in the Scree Plot detailed in Fig. 2. In this case, the largest jump in the distance coefficient was from stage 54 to stage 55 and the number of clusters generated is based on the following algorithm [*the number of cases* subtract the largest *distance coefficient* equals the *number of clusters*]. In our example, this equates to 57 – 54 = 3. Thus, based upon both a graphical depiction, and a percentage change in heterogeneity, a three-cluster solution was accepted as best representing the homogeneity and heterogeneity of the apps.

Discussion

As indicated earlier, a limitation of our previous research (Larkin, 2015, 2016; Larkin & Milford, 2018) was that only the lead author performed the analysis and synthesis of the clusters formed. Therefore, in order to enhance the validity of this research, two international mathematics educators, one from Canada and one from New Zealand, as well as the lead author, independently examined the formation of the clusters and independently identified themes that were apparent from the clusters that were formed.

Types of Mathematics Knowledge

In analysing the formation of the clusters, in terms of their homogeneity and heterogeneity, and reflecting upon the contributions of our international colleagues, one logical way to explain the formation of the clusters is in relation to the types of mathematical knowledge that they develop. Here we draw on the work of Miller and Hudson (2007) and others who proposed three types of knowledge—conceptual, procedural and declarative. A full description of how the apps were evaluated for high versus low conceptual and procedural knowledge can be found in Larkin (2015, 2016). Here we are evaluating whether cluster analysis adds further information as to how the apps, regardless of their high/low quality, might be used either individually, or in concert with other apps, for specific pedagogical purpose. According to Goldman and Hasselbring (1997) conceptual knowledge refers to a "connected web of information in which the linking relationships are as important as the pieces of discrete information that are linked" (p. 4). A student's conceptual knowledge is increased, for example, when they recognise relationship between multiplication and division or common and decimal fractions as opposed to when these concepts are only understood in isolation.

Procedural knowledge is the ability to follow a set of sequential steps to solve a mathematical task (Goldman & Hasselbring, 1997; Miller & Hudson, 2007) and is primarily used to solve computational tasks—e.g. finding areas or calculating change. Declarative knowledge is knowledge that students are able to efficiently recall from memory without hesitation—e.g. subitising small amounts or fluent processing of number facts. Of some concern to Miller and Hudson (2007), and also the Australian Curriculum, Assessment and Reporting Authority ACARA (2016), is the observation that mathematics educators have traditionally placed a heavy emphasis on the development of declarative and procedural knowledge and this emphasis is reflected in the large percentage of mathematics apps that develop these latter two forms of knowledge (Larkin, 2013, 2015). Our argument here is that mathematics apps require a balance between the three knowledge areas; either within one app [e.g. *Mathemagica; Area of Rectangles*] or in groups of apps on a specific topic, e.g. Fractions [*Hands On Number Sense*—Conceptual; *Fraction Time*—Procedural; and *Subtracting Like Fractions*—Declarative]. Unfortunately, in our view, this balance is not evident in the range of apps that are available to school mathematics educators and mirrors the findings of Namukasa et al. (2016) who reported that only four of the 80 apps they reviewed "focused on building understanding of concepts" (p. 290).

We now turn our attention to the constitution of each of the clusters. The labels of the clusters were generated after determining the types of mathematics content and pedagogy they contained. In discussing these apps, an important observation is that apps did not cluster according to content; therefore, there are apps spread across the three clusters developing content from a wide range of curriculum sub-strands e.g. Fractions, Place Value, Patterning, Chance or Statistics. This is valuable knowledge as teachers may suspect that quality apps are more likely to develop particular content and poor apps other content, when the reverse is true; there are both quality apps and poor apps developing the same content areas, e.g. [*Early Numbers: Maths Wizard Counting*] and [*Letz Learn Counting*] are respectively very high and very poor in the early counting domain.

Cluster 2—Conceptual and Procedural Knowledge

By and large the 17 apps within this cluster develop both conceptual and procedural knowledge across a range of content areas including early number, computations, algebra, statistics and place value. The exceptions to this general rule are the apps [*Marble Math Junior*—Procedural only] and [*Math Model*—Conceptual only]. In both cases, an examination of where these particular apps are positioned in the dendrogram indicates that they are only loosely connected to this Cluster; in other words, a different cluster formation, or a review of more apps, might see these apps clustered with other apps that are solely procedural or solely conceptual.

What was consistently the case for the remaining 15 apps in this cluster is that they all had multiple components such that the students could be developing conceptual knowledge using one component of the app and procedural knowledge when using a different component of the same app. Variance within the cluster occurred as some apps were stronger at one knowledge element than others (while still developing both). For example, [*Place Value Chart*] was very strong in developing conceptual knowledge, as students were free to explore the app and modify a range of settings. At the other end of the spectrum within this cluster [*Friends of Ten*] was highly scaffolded and thus developed procedural knowledge around place value but did not afford the level of self-direction likely needed for deep conceptual knowledge. Thus teachers can use both apps concurrently depending on the pedagogical intent of the learning.

From a design perspective, these apps all used representations or icons that can be manipulated/moved, but in simple ways. Variance within this cluster is largely accounted for in terms of the Gee Principles with some of the more supportive apps in terms of student environment in Productive Pedagogies attaining a high score in this sub-dimension at the expense of opportunities for Probing, Discovery and Multiple Routes in Gee; all considered important elements in conceptual knowledge development. Overall, this cluster is very useful for teachers in the early conceptual development stage of a range of content areas and then; using different elements of the same app, they can later develop procedural knowledge around the initial concept.

Cluster 4—Procedural and Declarative or Solely Procedural Knowledge

Cluster 4 was the largest cluster and contained 24 apps. Given its size, it is perhaps unsurprising that it was the most disparate in terms of the types of knowledge developed with apps either solely procedural; largely procedural with some declarative knowledge aspects; or in two cases conceptual [*Fun Count App*] and [*Patterns, Colors and Shapes*] but at a very low level which excluded these two apps from Cluster 2 where both conceptual and procedural knowledge were developed. In examining the dendrogram for the other 22 apps, it is evident that at the 6 cluster stage they are linked with the final Cluster 7 (see below). This might indicate a close relationship with Cluster 7 in terms of the skills and processes developed as they pertain more towards declarative knowledge rather than conceptual understanding of the mathematics as noted in Cluster 2. Many of the apps that scored more poorly in this cluster offered a behaviourist approach that uses rewards for correct answers and lack of progress for incorrect ones. Hence the focus, in these lower scoring apps, is on extrinsic motivators that work more effectively in declarative than procedural development modes. From a design perspective, there is a range in the quality of the visual representations in each app; in some the visual representations are used creatively and promote student thinking; however, in most, the representations are used solely to represent procedures or processes rather than supporting students to make conjectures or establish their own patterns of thought evident in the more conceptually oriented apps in Cluster 2. Most of the apps in Cluster 4 are, therefore, of limited use as they are largely procedural apps that use extrinsic motivators, and the visual representations do not necessarily enhance learning. However, they may have specific use in targeted scenarios such as developing fluency or reinforcing area formulas once conceptual understanding has been developed.

Cluster 7—Declarative Knowledge or Declarative and Minimal Procedural Knowledge

This cluster consisted of nine apps that generally scored poorly in both Productive Pedagogies and to a slightly lesser extent Gee Principles. Perhaps as a consequence of predominantly dealing with the content areas of time and mass, two areas where conceptual development is difficult due to the abstract nature of both concepts in terms of mass being independent of size and time being a non-visible attribute; these apps were all highly directive in nature focussing mainly on reading clock faces or conversions of mass units. The predominant design feature of the apps was accuracy and speed of feedback without any opportunity for any exploration and investigation (important for conceptual development) or process/skills work (necessary for procedural development). In terms of the Productive Pedagogies they