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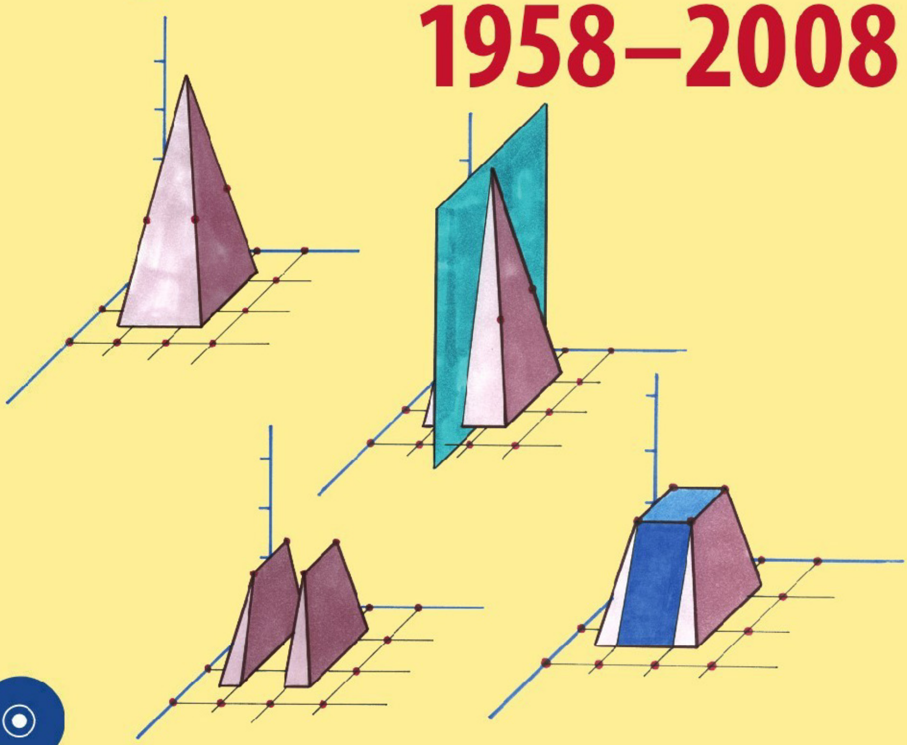
50 Years of Integer Programming 1958–2008



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50 Years of Integer Programming 1958–2008

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50 Years of Integer Programming 1958–2008

From the Early Years to the State-of-the-Art

Springer

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We dedicate this book to the pioneers of Integer Programming.

Preface

The name integer programming refers to the class of constrained optimization problems in which some or all of the variables are required to be integers. In the most widely studied and used integer programs, the objective function is linear and the constraints are linear inequalities. The field of integer programming has achieved great success in the academic and business worlds. Hundreds of papers are published every year in a variety of journals, several international conferences are held annually and software for solving integer programs, both commercial and open source, is widely available and used by thousands of organizations. The application areas include logistics and supply chains, telecommunications, finance, manufacturing and many others.

This book is dedicated to the theoretical, algorithmic and computational aspects of integer programming. While it is not a textbook, it can be read as an introduction to the field and provides a historical perspective. Graduate students, academics and practitioners, even those who have spent most of their careers in discrete optimization, will all find something useful to learn from the material in this book. Given the amount that has been accomplished, it is remarkable that the field of integer programming began only fifty years ago.

The 12th Combinatorial Optimization Workshop AUSSOIS 2008 took place in Aussois, France, 7–11 January 2008. The workshop, entitled *Fifty Years of Integer Programming*, and this book, which resulted from the workshop, were created to celebrate the 50th anniversary of integer programming. The workshop had a total of 136 participants from 14 countries ranging in experience from pioneers who founded the field to current graduate students. In addition to the formal program, the workshop provided many opportunities for informal discussions among participants as well as a chance to enjoy the spectacular Alpine setting provided by Aussois.

The book is organized into four parts. The first day of the workshop honored some of the pioneers of the field. Ralph Gomory's path-breaking paper, showing how the simplex algorithm could be generalized to provide a finite algorithm for integer programming and published in 1958, provided the justification of the anniversary celebration. The activities of the first day, led by George Nemhauser and Bill Pulleyblank, included a panel discussion with the pioneers who attended the workshop (Egon Balas, Michel Balinski, Jack Edmonds, Arthur Geoffrion, Ralph Gomory and Richard Karp) as well as three invited talks by Bill Cook, Gérard Cornuéjols and Laurence Wolsey on integer programming and combinatorial optimization from the beginnings to the state-of-the-art. The whole day is captured in two Video DVDs which come with the book (Part IV). Parts I, II, and III contain 20 papers of historical and current interest.

Part I of the book, entitled *The Early Years*, presents, in order of publication date, reprints of eleven fundamental papers published between 1954 and 1979. Ten of these papers were selected by one or more of the authors of the paper, who also wrote new introductions to the papers that explain their motivations for working on the problems addressed and their reason for selecting the paper for inclusion in this volume. The authors are Egon Balas, Michel Balinski, Alison Doig, Jack Edmonds, Arthur Geoffrion, Ralph Gomory, Alan Hoffman, Richard Karp, Joseph Kruskal, Harold Kuhn, and Ailsa Land. Each of these heavily cited papers has had a major influence on the development of the field and lasting value. The eleventh selection, which starts this section, is a groundbreaking paper by George Dantzig, Ray Fulkerson, and Selmer Johnson, with an introduction by Vašek Chvátal and William Cook. The introduction to Part I closes with a list, in chronological order, of our selection of some of the most influential papers appearing between 1954 and 1973 pertaining to the many facets of integer programming.

Part II contains papers based on the talks given by Cornuéjols, Cook, and Wolsey. The paper *Polyhedral Approaches to Mixed Integer Programming* by Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli presents tools from polyhedral theory that are used in integer programming. It applies them to the study of valid inequalities for mixed integer linear sets, such as Gomory's mixed integer cuts. The study of combinatorial optimization problems such as the traveling salesman problem has had a significant influence on integer programming. *Fifty-plus Years of Combinatorial Integer Programming* by Bill Cook discusses these connections. In solving integer programming problems by branch-and-bound methods, it is important to use relaxations that provide tight bounds. In the third paper entitled *Reformulation and Decomposition of Integer Programs*, François Vanderbeck and Laurence Wolsey survey ways to reformulate integer and mixed integer programs to obtain stronger linear programming relaxations. Together, these three papers give a remarkably broad and comprehensive survey of developments in the last fifty-plus years and their impacts on state-of-the-art theory and methodology.

Six survey talks on current hot topics in integer programming were given at the workshop by Fritz Eisenbrand, Andrea Lodi, François Margot, Franz Rendl, Jean-Philippe P. Richard, and Robert Weismantel. These talks covered topics that are actively being researched now and likely to have substantial influence in the coming decade and beyond.

Part III contains the six papers that are based on these talks. *Integer Programming and Algorithmic Geometry of Numbers* by Fritz Eisenbrand surveys some of the most important results from the interplay of integer programming and the geometry of numbers. *Nonlinear Integer Programming* by Raymond Hemmecke, Matthias Köppe, Jon Lee, and Robert Weismantel generalizes the usual integer programming model by studying integer programs with nonlinear objective functions. *Mixed Integer Programming Computation* by Andrea Lodi discusses the important ingredients involved in building a successful mixed integer solver as well as the problems that need to be solved in building the next generation of faster and more stable solvers. Symmetry is a huge obstacle encountered in solving mixed integer programs efficiently. In *Symmetry in Integer Programming*, François Margot presents several techniques that have been used successfully to overcome this difficulty. Semidefinite programming is a generalization of linear programming that provides a tighter relaxation to integer programs than linear programs. In *Semidefinite Relaxations for Integer Programming*, Franz Rendl surveys how semidefinite models and algorithms can be used effectively in solving certain combinatorial optimization problems. In the 1960s Ralph Gomory created a new tight relaxation for integer programs based on group theory. Recently the group theoretic model has been revived in the study of two-row integer programs. In *The Group-Theoretic Approach in Mixed Integer Programming*, Jean-Philippe P. Richard and Santanu S. Dey provide an overview of the mathematical foundations and recent theoretical and computational advances in the study of the group-theoretic approach.

We close with the hope that the next fifty years will be as rich as the last fifty have been in theoretical and practical accomplishments in integer programming.

November 2009

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About the Cover Illustration

The four figures on the cover illustrate adding Gomory mixed integer cuts to a polyhedron of dimension 3. The x-axis is horizontal, the y-axis is vertical and the z-axis is orthogonal to the cover. The starting polyhedron P shown in Fig. 1(a) is a cone with a square base and a peak having $y = 4.25$. P contains twelve integer lattice points. Suppose we solve the linear program: maximize y, subject to $y \in P$. The unique optimum will have $y = 4.25$. However, if we add the constraint that y be integral, then there are four optima, the lattice points illustrated on the edges of P having $y = 2$.

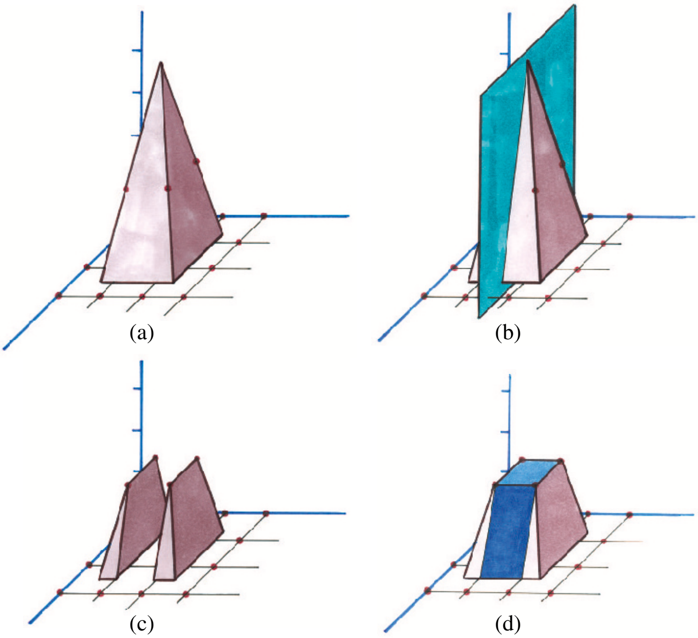
(a) (b) (c) (d)

Fig. 1 The Cover Illustration.

This example is a 3-D version of a 2-D example, first shown to us by Vašek Chvátal, which Bill Cook told us that Vašek attributes to Adrian Bondy. A "standard" Chvátal-Gomory cut (CG cut) is obtained by taking a hyperplane that supports a polyhedron and which contains no lattice points in space, then moving in a direction orthogonal to the hyperplane into the polyhedron until it hits a lattice point somewhere in space (not necessarily in the polyhedron). This gives a new valid inequality for all lattice points in the polyhedron, and which cuts off part of the original polyhedron. Gomory's fundamental result described a finite algorithm that, given any integer program, would automatically generate a finite sequence of CG cuts such that when they were added, the resulting linear program would have an integer optimum.

What cuts must be added to P to remove all points having $y > 2$? How do we generate the inequality $y \leq 2$ which must be added if the resulting linear program is going to have an integral optimum? The Bondy-Chvátal example showed that, even for dimension 2, the number of CG cuts that would have to added was unbounded, depending only on the height of the peak of the pyramid (provided that we adjust the base so that the lattice points in P having $y = 2$ continue to lie on the edges). In particular, the number of CG cuts that need to be added to solve an integer program is independent of the dimension of the polyhedron, and is not polynomial in the size of a linear system necessary to define the original polyhedron.

In 1960, Gomory described a method to generate so-called mixed integer cuts. These cuts have turned out to be very powerful in practice, both for integer and mixed integer programs. They work as follows: Take a hyperplane that intersects the polyhedron and passes through no lattice points in space. In Fig. 1(b), we chose the hyperplane $x = 1.5$. Note that it passes right through P. Consider the inequalities $x \leq 1$ and $x \geq 2$ which are obtained by shifting the hyperplane left and right respectively, until it hits a lattice point in space. We construct two new polyhedra P_1 and P_2 from P, one by adding the inequality $x \leq 1$ and one by adding $x \geq 2$. Then every lattice point in P will belong to one of P_1 and P_2.

These two polyhedra are the two wedges shown in Fig. 1(c). Note that every lattice point contained in P is in one of the two wedges.

The final step is to take the convex hull of the union of P_1 and P_2. This is the polyhedron shown in Fig. 1(d). Note that one hyperplane was used to create two subproblems. Then by maximizing y over these two subproblems, we get the solution we are seeking. Balas, Ceria and Cornuéjols describe a method called *lift-and-project* for generating a cut after a polyhedron has been split into two subpolyhedra. This is discussed in Balas' introduction to Chapter 10.

Also, everything we have done remains valid if x and z are allowed to be continuous variables and only y is required to be integral. For this reason, these types of cuts are usually called "mixed integer cuts".

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- the pioneers, without whom …

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Part I The Early Years

Part II From the Beginnings to the State-of-the-Art

Part III Current Topics

Part IV DVD-Video / DVD-ROM

Part I
The Early Years

In 1947, George Dantzig created the simplex algorithm, which was subsequently published in 1951. This landmark paper described a finite method for optimizing a linear objective function subject to a finite set of linear constraints. It was already recognized that this type of problem, called a linear programming problem, occurred in a great many situations. Moreover, Dantzig's simplex method was proving to be very effective in practice.

It was recognized that adding integrality constraints on some or all of the variables significantly increased the applicability of these models. A great many problems, including combinatorial optimization problems, could be modeled using linear functions and integer variables, but no method was known for modeling these problems using linear functions and continuous variables. (It is noteworthy that now, more than fifty years later, it is still not known whether integer programming is more powerful than linear programming.) Moreover, no general method was known for solving this type of problem, called mixed-integer linear programming problem.

In 1958, Ralph Gomory published a short paper which described how, with relatively straightforward modifications, Dantzig's simplex algorithm could be adapted to provide a finite algorithm for finding an optimal integral solution to a linear program. He showed how the simplex tableau could be used to generate new inequalities which were valid for all solutions satisfying the integrality constraints, but which were violated by the current linear program's optimum solution. The study of these inequalities, called cuts, quickly became a major area of activity both for theoretical reasons and because of the promise they showed as a computational tool. Recall that at the end of the decade of the 50s, digital computers were emerging as a force in the way that business was conducted with the potential to actually optimize business processes.

The year 2008 marked the 50th anniversary of the appearance of Gomory's groundbreaking paper. There is an annual meeting on combinatorial optimization and integer programming held each year at the French ski resort of Aussois. The editors of this volume proposed dedicating the January 2008 meeting to a celebration of the development of the field of integer programming together with an overview of the field today, including state-of-the-art surveys and recent results on selected hot topics. Our plan was to invite a number of pioneers of the field of integer programming who had been active in the fifties and sixties to provide a historical perspective and to participate in the scientific agenda. The first person we contacted was, of course, Ralph Gomory who enthusiastically accepted. (This may have been influenced by the fact that Ralph is an avid skier.) Each of these pioneers agreed to select one of their papers for inclusion in this volume, and to write a new introduction for it that would provide a historical and mathematical perspective.

We include two of Gomory's foundational papers on the cutting plane method for integer programming. The second dealt with the mixed integer problem and introduced a method of cut generation that has proved to be very effective in practice. Previously, this paper was only available as a Rand report.

The earliest paper we reprint here contains the solution to a 49 city traveling salesman problem using linear programming and cuts by George Dantzig, Ray Fulkerson, and Selmer Johnson. In addition to showing how a small set of cuts could be sufficient to prove optimality of a solution to an integer programming problem, this paper laid a foundation for much of the subsequent work on computational polyhedral combinatorics. Because the authors are deceased, Vašek Chvátal and William Cook, two of the coauthors of a recent book on the traveling salesman problem, volunteered to write the introduction.

The 1955 paper by Harold Kuhn describes a combinatorial algorithm for a specially structured integer program, the assignment problem. This work provided an early example of a specialized method for solving a structured problem and was one of the first uses of a primal-dual linear programming algorithm.

Alan Hoffman and Joseph Kruskal's 1956 paper showed the importance of the notion of total unimodularity to finding integer solutions to linear programs. They showed that this property characterized when this would happen automatically for all linear objective functions and choices of integral right-hand sides.

The 1960 paper by Ailsa Land and Alison Doig introduced the other method that has been so important in obtaining solutions to integer programming problems, branch-and-bound. In fact, most successful modern computer codes integrate cuts with branch-and-bound.

Michel Balinski's 1965 paper described the power of integer programming models to a range of real world problems. It provided the first comprehensive survey and introduced integer programming to a much broader audience.

Jack Edmonds' 1968 paper on matroid partition is one of a remarkable series of papers that he wrote showing a number of cases for which a combinatorially described set of cuts added to a linear program would yield the integer hull and would provide the basis for a polynomial run-time algorithm to solve the integer problem.

The importance of polynomial algorithms for combinatorial algorithms reached a broader audience in the early 1970s with the introduction of the classes P (polynomial) and NP (nondeterministic polynomial) in the theoretical computer science community. Steven Cook's fundamental result showed that there was a set of so-called NP-complete problems with the property that if any were solvable in polynomial time, then so too were all problems in the class NP. Richard Karp's 1972 paper highlighted the importance of these results to the mathematical programming community and showed that a long list of specially structured integer programs, for which no polynomially bounded algorithm was known, belonged to the class of NP-complete programs.

Art Geoffrion's 1974 paper showed how Lagrangean methods provided an alternative method for solving integer programming problems by incorporating certain constraints into the objective function and then alternating between solving primal and dual problems. He also established connections between the Lagrangean approach and Dantzig-Wolfe decomposition.

Egon Balas' 1979 paper showed that the class of integer programming problems could be extended to a much broader class defined by considering disjunctions of polyhedra, and that methods for this broader framework had specializations to integer programming that have turned out to have computational as well as theoretical importance.

We conclude with a list, in chronological order, of our selections of some of the most influential papers pertaining to the many facets of integer programming appearing between 1954 and 1973.

20 YEARS OF MIXED-INTEGER PROGRAMMING: MILESTONES (1954–1973)

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Chapter 1
Solution of a Large-Scale Traveling-Salesman Problem

George B. Dantzig, Delbert R. Fulkerson, and Selmer M. Johnson

Introduction by *Vašek Chvátal* and *William Cook*

The birth of the cutting-plane method

The RAND Corporation in the early 1950s contained “what may have been the most remarkable group of mathematicians working on optimization ever assembled” [6]: Arrow, Bellman, Dantzig, Flood, Ford, Fulkerson, Gale, Johnson, Nash, Orchard-Hays, Robinson, Shapley, Simon, Wagner, and other household names. Groups like this need their challenges. One of them appears to have been the traveling salesman problem (TSP) and particularly its instance of finding a shortest route through Washington, DC, and the 48 states [4, 7].

Dantzig’s work on the assignment problem [1] revealed a paradigm for minimizing a linear function $f : \mathbb{R}^n \to \mathbb{R}$ over a finite subset $\mathscr{S}$ of $\mathbb{R}^n$: first describe the convex hull of $\mathscr{S}$ by a system $Ax \leq b$ of linear constraints and then solve the linear programming problem

$$\text{minimize } f(x) \text{ subject to } Ax \leq b$$

by the simplex method. Attempts by Heller and by Kuhn to apply this paradigm to the TSP indicated that sets of linear constraints describing the convex hull of all tours are far too large to be handled directly. Undeterred, Dantzig, Fulkerson, and Johnson bashed on. The preliminary version of their paper [2] includes a discussion of the convex hull of all tours, nowadays called “the TSP polytope”. The version submitted for publication four months later (and eventually published and

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reproduced here) breaks free of the dogma: without letting the TSP polytope obscure their exposition, the authors just go ahead and solve the 49-city instance. (Regarding this change, Fulkerson writes in a September 2, 1954, letter to *Operations Research* editor George Shortly "In an effort to keep the version submitted for publication elementary, we avoid going into these matters in any detail.")

This case study ushered in the *cutting-plane method*. To solve a problem

$$\text{minimize } f(x) \text{ subject to } x \in \mathscr{S} \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a linear function and $\mathscr{S}$ is a finite subset of $\mathbb{R}^n$, choose a system $Ax \leq b$ of linear inequalities satisfied by all points of $\mathscr{S}$ and use the simplex method to find an optimal solution x^* of the linear programming problem

$$\text{minimize } f(x) \text{ subject to } Ax \leq b, \tag{1.2}$$

called the *linear programming relaxation* of (1.1). If x^* belongs to $\mathscr{S}$, then it is an optimal solution of (1.1); else there are linear inequalities satisfied by all points of $\mathscr{S}$ and violated by x^*, called *cutting planes*. Find one or more such inequalities, add them to $Ax \leq b$, and iterate. (The method actually used by Dantzig, Fulkerson, and Johnson—described also in [2, 3]—is a slight variation on this theme: rather than introducing cutting planes only when an optimal solution x^* of (1.2) lies outside $\mathscr{S}$, they introduce them after each simplex pivot leading to a basic feasible solution x^* of (1.2) that lies outside $\mathscr{S}$.)

The role played by the convex hull of $\mathscr{S}$ in this new paradigm is only implicit: we have to be able to find a cutting plane whenever one exists, which is the case if and only if x^* lies outside the convex hull of $\mathscr{S}$. In particular, the number of linear constraints in a description of the convex hull of $\mathscr{S}$ is irrelevant here. Another important difference between the two paradigms is that the cutting-plane method is an engineering rather than mathematical method: unlike the simplex method, it carries no guarantee that the sequence of its iterations will terminate. (But then again, a guarantee of termination after finitely many iterations is a far cry from a guarantee of termination before the end of our solar system.) Our three authors write "...what we shall do is outline a way of approaching the problem that sometimes, at least, enables one to find an optimal path and prove it so."

Until 1954, no one had an inkling of a way to solve large instances of the TSP. The lament about the number of tours through n cities being too large to allow their listing one by one marked the vanguard of scientific progress on this front. Then Dantzig, Fulkerson, and Johnson let the light in and inaugurated a new era. All successful TSP solvers echo their breakthrough. This was the Big Bang.

This Big Bang reverberates far beyond the narrow confines of the TSP. It provides a tempting template for coping with any NP-complete problem of minimizing a linear function over a finite set $\mathscr{S}$. For each problem of this kind, the challenge lies in finding cutting planes quickly. In the special case of integer linear programming, where $\mathscr{S}$ consists all integer solutions of a prescribed set of linear constraints, this challenge was met with remarkable elegance (and termination after finitely many iterations guaranteed) by Gomory in a series of papers beginning with [5].

Great new ideas may transform the discipline they came from so profoundly that they become hard to discern against the changed background. When terms such as “defense mechanism” and “libido” are in the common vocabulary, it is easy to forget that they came from Sigmund Freud. The cutting-plane method of George Dantzig, Ray Fulkerson, and Selmer Johnson had the same kind of impact on the discipline of mathematical programming.

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SOLUTION OF A LARGE-SCALE TRAVELING-SALESMAN PROBLEM*

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It is shown that a certain tour of 49 cities, one in each of the 48 states and Washington, D. C., has the shortest road distance.

THE TRAVELING-SALESMAN PROBLEM might be described as follows: Find the shortest route (tour) for a salesman starting from a given city, visiting each of a specified group of cities, and then returning to the original point of departure. More generally, given an n by n symmetric matrix $D=(d_{IJ})$, where d_{IJ} represents the 'distance' from I to J, arrange the points in a cyclic order in such a way that the sum of the d_{IJ} between consecutive points is minimal. Since there are only a finite number of possibilities (at most $\frac{1}{2}(n-1)!$) to consider, the problem is to devise a method of picking out the optimal arrangement which is reasonably efficient for fairly large values of n. Although algorithms have been devised for problems of similar nature, e.g., the optimal assignment problem,[3,7,8] little is known about the traveling-salesman problem. We do not claim that this note alters the situation very much; what we shall do is outline a way of approaching the problem that sometimes, at least, enables one to find an optimal path and prove it so. In particular, it will be shown that a certain arrangement of 49 cities, one in each of the 48 states and Washington, D. C., is best, the d_{IJ} used representing road distances as taken from an atlas.

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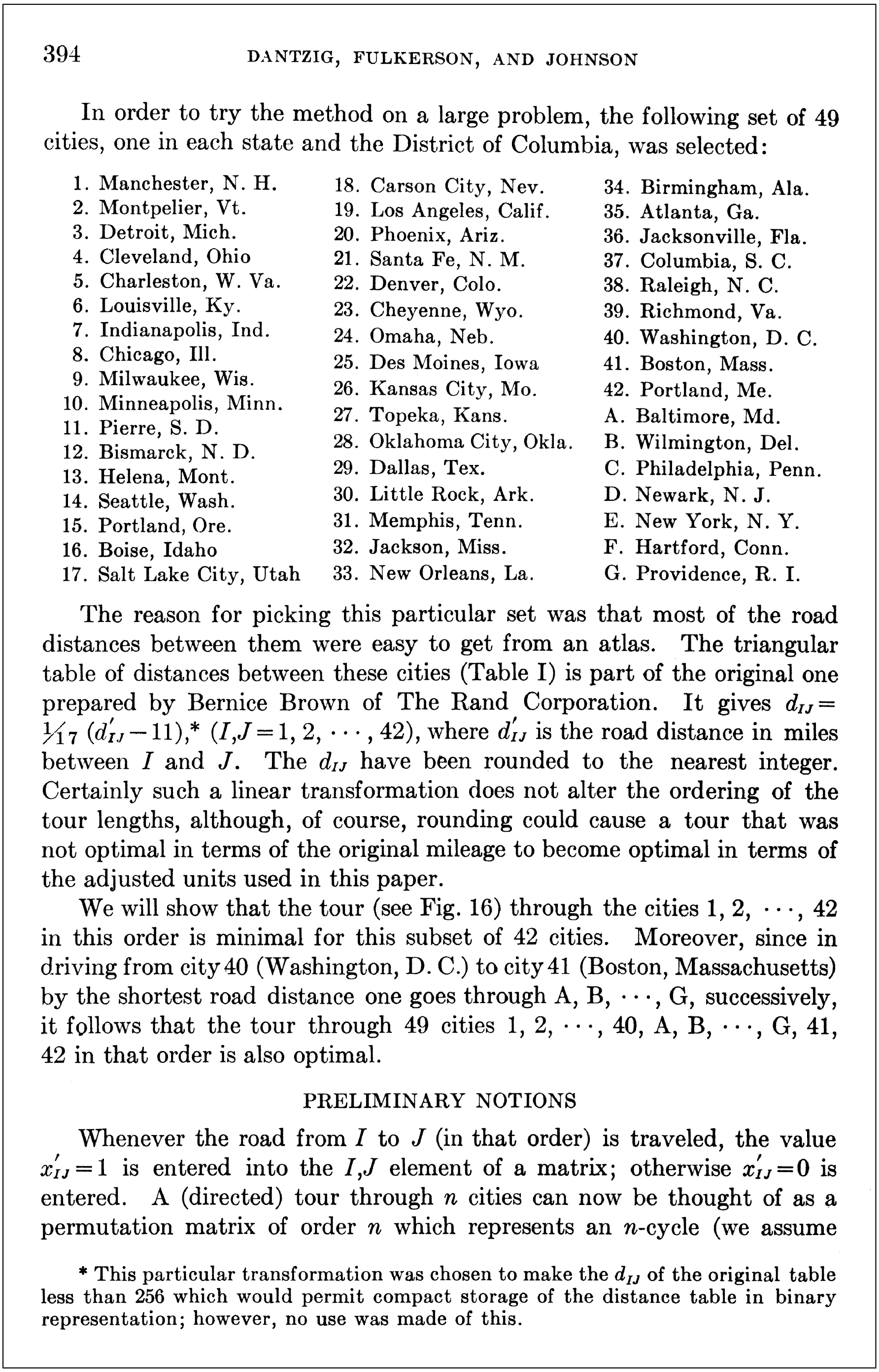
* HISTORICAL NOTE: The origin of this problem is somewhat obscure. It appears to have been discussed informally among mathematicians at mathematics meetings for many years. Surprisingly little in the way of results has appeared in the mathematical literature.[10] It may be that the minimal-distance tour problem was stimulated by the so-called Hamiltonian game[1] which is concerned with finding the number of different tours possible over a specified network. The latter problem is cited by some as the origin of group theory and has some connections with the famous Four-Color Conjecture.[9] Merrill Flood (Columbia University) should be credited with stimulating interest in the traveling-salesman problem in many quarters. As early as 1937, he tried to obtain near optimal solutions in reference to routing of school buses. Both Flood and A. W. Tucker (Princeton University) recall that they heard about the problem first in a seminar talk by Hassler Whitney at Princeton in 1934 (although Whitney, recently queried, does not seem to recall the problem). The relations between the traveling-salesman problem and the transportation problem of linear programming appear to have been first explored by M. Flood, J. Robinson, T. C. Koopmans, M. Beckmann, and later by I. Heller and H. Kuhn.[4,5,6]



In order to try the method on a large problem, the following set of 49 cities, one in each state and the District of Columbia, was selected:

1. Manchester, N. H.
2. Montpelier, Vt.
3. Detroit, Mich.
4. Cleveland, Ohio
5. Charleston, W. Va.
6. Louisville, Ky.
7. Indianapolis, Ind.
8. Chicago, Ill.
9. Milwaukee, Wis.
10. Minneapolis, Minn.
11. Pierre, S. D.
12. Bismarck, N. D.
13. Helena, Mont.
14. Seattle, Wash.
15. Portland, Ore.
16. Boise, Idaho
17. Salt Lake City, Utah
18. Carson City, Nev.
19. Los Angeles, Calif.
20. Phoenix, Ariz.
21. Santa Fe, N. M.
22. Denver, Colo.
23. Cheyenne, Wyo.
24. Omaha, Neb.
25. Des Moines, Iowa
26. Kansas City, Mo.
27. Topeka, Kans.
28. Oklahoma City, Okla.
29. Dallas, Tex.
30. Little Rock, Ark.
31. Memphis, Tenn.
32. Jackson, Miss.
33. New Orleans, La.
34. Birmingham, Ala.
35. Atlanta, Ga.
36. Jacksonville, Fla.
37. Columbia, S. C.
38. Raleigh, N. C.
39. Richmond, Va.
40. Washington, D. C.
41. Boston, Mass.
42. Portland, Me.

A. Baltimore, Md.
B. Wilmington, Del.
C. Philadelphia, Penn.
D. Newark, N. J.
E. New York, N. Y.
F. Hartford, Conn.
G. Providence, R. I.

The reason for picking this particular set was that most of the road distances between them were easy to get from an atlas. The triangular table of distances between these cities (Table I) is part of the original one prepared by Bernice Brown of The Rand Corporation. It gives $d_{IJ} = \frac{1}{17}\,(d'_{IJ} - 11)$,* $(I,J = 1, 2, \cdots, 42)$, where d'_{IJ} is the road distance in miles between I and J. The d_{IJ} have been rounded to the nearest integer. Certainly such a linear transformation does not alter the ordering of the tour lengths, although, of course, rounding could cause a tour that was not optimal in terms of the original mileage to become optimal in terms of the adjusted units used in this paper.

We will show that the tour (see Fig. 16) through the cities $1, 2, \cdots, 42$ in this order is minimal for this subset of 42 cities. Moreover, since in driving from city 40 (Washington, D. C.) to city 41 (Boston, Massachusetts) by the shortest road distance one goes through A, B, $\cdots$, G, successively, it follows that the tour through 49 cities $1, 2, \cdots, 40$, A, B, $\cdots$, G, 41, 42 in that order is also optimal.

PRELIMINARY NOTIONS

Whenever the road from I to J (in that order) is traveled, the value $x'_{IJ} = 1$ is entered into the I,J element of a matrix; otherwise $x'_{IJ} = 0$ is entered. A (directed) tour through n cities can now be thought of as a permutation matrix of order n which represents an n-cycle (we assume

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* This particular transformation was chosen to make the d_{IJ} of the original table less than 256 which would permit compact storage of the distance table in binary representation; however, no use was made of this.

TABLE I

ROAD DISTANCES BETWEEN CITIES IN ADJUSTED UNITS

The figures in the table are mileages between the two specified numbered cities, less 11, divided by 17, and rounded to the nearest integer.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
2	8																																								
3	39	45																																							
4	37	47	9																																						
5	50	49	21	15																																					
6	61	62	21	20	17																																				
7	58	60	16	17	18	6																																			
8	59	60	15	20	26	17	10																																		
9	62	66	20	25	31	22	15	5																																	
10	81	81	40	44	50	41	35	24	20																																
11	103	107	62	67	72	63	57	46	41	23																															
12	108	117	66	71	77	68	61	51	46	26	11																														
13	145	149	104	108	114	106	99	88	84	63	49	40																													
14	181	185	140	144	150	142	135	124	120	99	85	76	35																												
15	187	191	146	150	156	142	137	130	125	105	90	81	41	10																											
16	161	170	120	124	130	115	110	104	105	90	72	64	34	31	27																										
17	142	146	101	104	111	97	91	85	86	75	51	59	29	53	48	21																									
18	174	178	133	138	143	129	123	117	118	107	83	84	54	46	35	26	31																								
19	185	186	142	143	140	130	126	124	128	118	93	101	72	69	58	58	43	26																							
20	164	165	120	123	124	106	106	105	110	104	86	97	71	93	82	62	42	45	22																						
21	137	139	94	96	94	80	78	77	84	77	56	64	65	90	87	58	36	68	50	30																					
22	117	122	77	80	83	68	62	60	61	50	34	42	49	82	77	60	30	62	70	49	21																				
23	114	118	73	78	84	69	63	57	59	48	28	36	43	77	72	45	27	59	69	55	27	5																			
24	85	89	44	48	53	41	34	28	29	22	23	35	69	105	102	74	56	88	99	81	54	32	29																		
25	77	80	36	40	46	34	27	19	21	14	29	40	77	114	111	84	64	96	107	87	60	40	37	8																	
26	87	89	44	46	46	30	28	29	32	27	36	47	78	116	112	84	66	98	95	75	47	36	39	12	11																
27	91	93	48	50	48	34	32	33	36	30	34	45	77	115	110	83	63	97	91	72	44	32	36	9	15	3															
28	105	106	62	63	64	47	46	49	54	48	46	59	85	119	115	88	66	98	79	59	31	36	42	28	33	21	20														
29	111	113	69	71	66	51	53	56	61	57	59	71	96	130	126	98	75	98	85	62	38	47	53	39	42	29	30	12													
30	91	92	50	51	46	30	34	38	43	49	60	71	103	141	136	109	90	115	99	81	53	61	62	36	34	24	28	20	20												
31	83	85	42	43	38	22	26	32	36	51	63	75	106	142	140	112	93	126	108	88	60	64	66	39	36	27	31	28	28	8											
32	89	91	55	55	50	34	39	44	49	63	76	87	120	155	150	123	100	123	109	86	62	71	78	52	49	39	44	35	24	15	12										
33	95	97	64	63	56	42	49	56	60	75	86	97	126	160	155	128	104	128	113	90	67	76	82	62	59	49	53	40	29	25	23	11									
34	74	81	44	43	35	23	30	39	44	62	78	89	121	159	155	127	108	136	124	101	75	79	81	54	50	42	46	43	39	23	14	14	21								
35	67	69	42	41	31	25	32	41	46	64	83	90	130	164	160	133	114	146	134	111	85	84	86	59	52	47	51	53	49	32	24	24	30	9							
36	74	76	61	60	42	44	51	60	66	83	102	110	147	185	179	155	133	159	146	122	98	105	107	79	71	66	70	70	60	48	40	36	33	25	18						
37	57	59	46	41	25	30	36	47	52	71	93	98	136	172	172	148	126	158	147	124	121	97	99	71	65	59	63	67	62	46	38	37	43	23	13	17					
38	45	46	41	34	20	34	38	48	53	73	96	99	137	176	178	151	131	163	159	135	108	102	103	73	67	64	69	75	72	54	46	49	54	34	24	29	12				
39	35	37	35	26	18	34	36	46	51	70	93	97	134	171	176	151	129	161	163	139	118	102	101	71	65	65	70	84	78	58	50	56	62	41	32	38	21	9			
40	29	33	30	21	18	35	33	40	45	65	87	91	117	166	171	144	125	157	156	139	113	95	97	67	60	62	67	79	82	62	53	59	66	45	38	45	27	15	6		
41	3	11	41	37	47	57	55	58	63	83	105	109	147	186	188	164	144	176	182	161	134	119	116	86	78	84	88	101	108	88	80	86	92	71	64	71	54	41	32	25	
42	5	12	55	41	53	64	61	61	66	84	111	113	150	186	192	166	147	180	188	167	140	124	119	90	87	90	94	107	114	77	86	92	98	80	74	77	60	48	38	32	6