# David Hilbert's Lectures on the Foundations of Physics 1915–1927

**David Hilbert's Lectures on the Foundations of Mathematics and Physics, 1891–1933**

*General Editors*
William Ewald, Michael Hallett, Ulrich Majer and Wilfried Sieg

Volume 1
David Hilbert's Lectures on the Foundations of Geometry,
1891–1902

Volume 2
David Hilbert's Lectures on the Foundations of Arithmetic and Logic,
1894–1917

Volume 3
David Hilbert's Lectures on the Foundations of Arithmetic and Logic,
1917–1933

Volume 4
David Hilbert's Lectures on the Foundations of Physics, 1898–1914
Classical, Relativistic and Statistical Mechanics

Volume 5
David Hilbert's Lectures on the Foundations of Physics, 1915–1927
Relativity, Quantum Theory and Epistemology

Volume 6
David Hilbert's Notebooks and General Foundational Lectures

Tilman Sauer
Ulrich Majer
*Editors*

# David Hilbert's Lectures on the Foundations of Physics 1915–1927

Relativity, Quantum Theory and Epistemology

in collaboration with
Arne Schirrmacher and Heinz-Jürgen Schmidt

Springer

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ISBN 978-3-540-20606-4 e-ISBN 978-3-540-68285-1
DOI 10.1007/978-3-540-68285-1
Springer Dordrecht Heidelberg London New York

Library of Congress Control Number: 2009930650

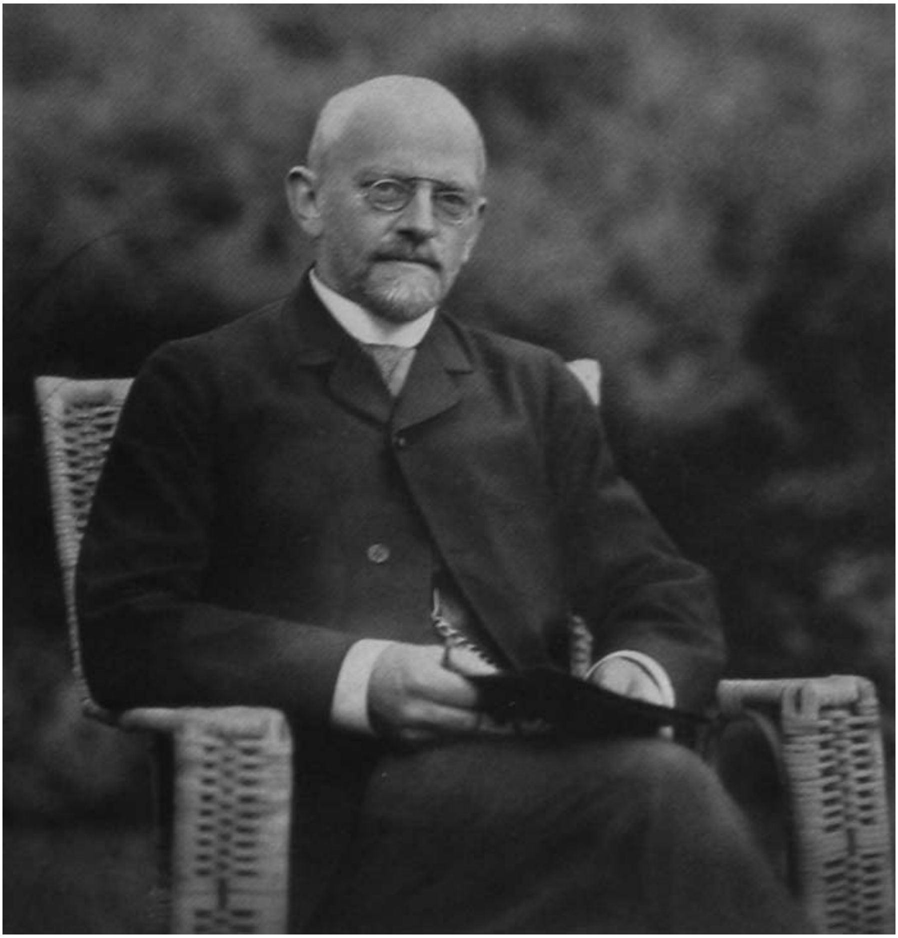


*Cover design:* WMXDesign, Heidelberg

Typeset by the editors.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)



David Hilbert, ca. 1921/22
(Voit Collection, Manuscript Division of the
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# *Preface*

The present Volume is the fifth in a series of six presenting a selection from Hilbert's previously unpublished lecture notes on the foundations of mathematics and physics during the period from 1890 to 1933. Hilbert's lecture courses represent an enormous fund of learning and invention, and embrace almost every subject common in the mathematical sciences of his day, including mathematical physics. The notes therefore provide a remarkable record, sometimes almost from day to day, of the development of his foundational ideas, and show, in addition, his engagement with the work of other scientific figures of the first rank. The present Volume treats Hilbert's lectures on relativity, quantum theory and epistemology from the fall of 1915 on. During this period, Hilbert reached the height of his research investigations into the foundations of the natural sciences.

The structure of this Edition, the nature, location, and condition of the Hilbert lecture notes, their provenance, and what we have been able to reconstruct of their history, are all described in the general 'Introduction to the Edition', which is to be found at the beginning of Volume 1. That Introduction also explains in detail the criteria for the selection of the texts, the way in which they were edited, and general matters of textual policy. Those matters are uniform for the entire Edition, and we have not repeated the full account here. We do, however, include a brief description of the textual policies in section 5 of the introduction to this Volume. This section is intended to provide all the basic information necessary to a reading of the texts, above all, information concerning the policies specific to this Volume.

That these lectures are finally being published is the result of the efforts, over nearly two decades, of many individuals and institutions. The series as a whole is under the supervision of four General Editors, William Ewald, Michael Hallett, Ulrich Majer, and Wilfried Sieg, who bear the collective responsibility for editorial policy. For each individual volume, Volume Editors were designated to produce the final selection of texts and to write the scholarly apparatus; this work was carried out in consultation with the General Editors. The designated Editors for this Volume were Tilman Sauer and Ulrich Majer. It should be noted that Arne Schirrmacher worked on Hilbert's lectures on radiation theory presented in Chapter 5, and Heinz-Jürgen Schmidt worked on Hilbert's lectures on quantum theory presented in Chapter 6.

All the Editors wish to express their thanks to the Deutsche Forschungsgemeinschaft (DFG) for its generous financial support from 1993 to 2003. To edit even the mere fragment of the voluminous Hilbert *Nachlaß* that appears in these six volumes required a considerable institutional apparatus located in proximity to the archives in Göttingen. Without the assistance of the DFG, which enabled us to establish a permanent staff in Göttingen, the present Edition could never have been realized. Ulrich Majer, the General Editor who was constantly 'vor Ort', supervised the permanent staff and thus had the task of dealing with all the technical problems that an edition of this sort must inevitably face. We again acknowledge the indispensable scholarly, editorial and technical contributions to the Edition as a whole of Ralf Haubrich, Albert Krayer, Tilman Sauer and Arne Schirrmacher, all at one time full-time members of the permanent staff.

The final work for this particular Volume was made possible through the generous gift of a donor who has asked to remain anonymous. These funds made it possible for Tilman Sauer to be freed from his project duties as an editor of the *Collected Papers of Albert Einstein* for several months during the summers of 2006 and 2007 and devote himself exclusively to work on the present Volume. We wish to thank Tom Ryckman for his interest in the project and his assistance in securing this additional funding. We are also extremely grateful to Diana Kormos Buchwald, general editor of the *Collected Papers of Albert Einstein*, for her encouragement, patience and unfailing support. Her generous sharing of the office resources of the Einstein Papers Project with its sister project of the Hilbert Edition enabled Tilman Sauer to work on the Hilbert project without being exiled from home.

We thank the Institut für Wissenschaftsgeschichte at the University of Göttingen (in particular Lorraine Daston, its former director) for giving the project its first physical home and for recognizing its significance. We are also grateful to the Philosophisches Seminar at the University of Göttingen for space and support.

Numerous other institutions and individuals provided significant support for the Edition. In Göttingen, from the first, formative stages of the project, we received encouragement and advice from the late Martin Kneser, Samuel Patterson, Günther Patzig and Helmut Rohlfing. The Mathematisches Institut and the Niedersächsische Staats- und Universitätsbibliothek in Göttingen (SUB), the holders of the original Hilbert documents, granted the necessary permission for publication, for which we are deeply grateful.

The Institute for Advanced Study in Princeton, through the offices of Harry Woolf and Phillip Griffiths, provided the Editors with a collective working environment in the summer of 1997.

Carnegie Mellon University, the Georg-August-Universität Göttingen, the Universität Bern, hosted a series of conferences on Hilbert's unpublished foundational writings. The Poincaré Project at the Université Nancy 2 hosted

a conference on editing scientific papers. These meetings and conferences, in addition to their intellectual focus, provided occasion of personal encounter without which a collaboration of this sort cannot thrive. We also thank Peter Aichelburg and the Arbeitsgruppe Gravitationsphysik at the University of Vienna for generously providing office space for Tilman Sauer during the summer of 2008.

Catriona Byrne of Springer Verlag has given the Edition abundant support and advice, and has been patient with the inevitable delays.

A large number of people have been of assistance in various technical and research capacities. For their help we thank: Volker Ahlers, Tobias Brendel, Willem Hagemann, Julia Hartmann, Nina Hehn, Arnim von Helmolt, Stefan Krämer, Pamela Klapproth, Michael Mai, Heiko Schilling, Rebecca Pates, Friedericke Schröder-Pander, Hans-Jakob Wilhelm, and many others. We thank Felicity Pors, Niels-Bohr-Archive Copenhagen, for help in locating documents, and Gudrun Staedel-Schneider, Munich, for her help in documenting Hilbert's Bucharest lecture. Special thanks go to Carol Chaplin and Rosy Meiron, Pasadena, for their gracious and meticulous help in proofreading in the final stages of preparing the Volume.

This series of volumes was originally set up under the supervision of Ralf Haubrich, who played an essential role in the design of the overall editorial apparatus, which was subsequently greatly advanced by Albert Krayer. The preparation, organization and presentation of the two volumes on the natural sciences were largely in the hands of Tilman Sauer.

Finally, the responsible Editors of this Volume wish to thank their wives and families for their tolerance, patience, and support during the arduous and protracted tasks of the preparation and finishing of this volume.

The General Editors

William Ewald, Michael Hallett, Ulrich Majer, Wilfried Sieg

# *Contents*

# *Introduction*

David Hilbert was concerned with the foundations of the mathematical sciences and physics throughout his career; this Volume is the second of two Volumes (4 and 5) concentrating on the foundations of physics. Most of the documents selected for these two Volumes were chosen from a large corpus of Hilbert's unpublished writings (primarily notes for lecture courses) on the natural sciences. These begin in 1898 with a four hour lecture course "Mechanics" (*Hilbert 1898/99**) and end in 1930 with the lecture course "Mathematical Methods of the New Physics."[1] The selection of documents and its distribution over the two Volumes was based on the fact that Hilbert's work prior to 1915 is concerned mainly with classical mechanics and electrodynamics, including the special theory of relativity, together with thermodynamics and statistical mechanics. Hilbert's work from 1915, on the other hand, deals with the new physics embodied in general relativity theory and quantum mechanics, and also the attendant epistemological implications. Volume 4 deals with the period before 1915, whereas the present Volume deals with material from 1915 and later. It presents a selection of Hilbert's writings and unpublished lectures from this period, beginning with his "First Communication" on the "Foundations of Physics" (*Hilbert 1915*) in late 1915 up to and including Hilbert's penultimate lecture course on physics, i.e., his course on the new quantum mechanics held in the winter semester 1926/27 (*Hilbert 1926/27**).

There are two exceptions to the temporal division between Volumes 4 and 5, both made on conceptual grounds. We have included in the present Volume parts of Hilbert's 1912 lecture course on radiation theory (*Hilbert 1912**) in order to document Hilbert's concerns with the old quantum theory based on Planck's law of blackbody radiation. By the same token, a lecture course on "Statistical mechanics" from 1922 (*Hilbert 1922**) is planned for inclusion in Volume 4 in order to complement the material presented there on statistical physics and thermodynamics.

Hilbert is known to have had a long-standing interest in the conceptual foundations and mathematical methods of physics. This is clear even from his published œuvre. Among other things, he made significant contributions to the debates about the proper foundations of kinetic theory, radiation theory,

[1] See "Hilbert's Lecture Courses, 1886–1934," this Volume, 709–726.

as well as to the foundations of general relativity and quantum mechanics. However, Hilbert's unpublished writings show his interest to be even deeper and more extended, a fact testified to by manuscripts for talks and individual lectures, and in notes and *Ausarbeitungen* for lecture courses. They show Hilbert not only as a skillful and ingenious mathematician, but also as a scientist with a profound understanding of the physics of his time, both in its technical details and in its conceptual foundations. We see a mathematician who contributes original ideas to the rapidly developing disciplines of theoretical physics, and we also see a *mathematical* logician pursuing an epistemologically motivated research program concerned with analyzing the most complex and deepest foundational problems of modern physics.[2] The unpublished papers also reveal him, even more than the published ones, as a mature philosopher of physical science, if one understands by philosopher someone who engages in the logical analysis of theories quite in the tradition of analytical philosophy. Hilbert reflects profoundly and in a deeply informed way on the central epistemological and ontological presuppositions and implications of the conceptual innovations of relativistic, statistical and quantum physics. In these lectures, the philosophical nature of Hilbert's thought emerges with striking clarity. The value of Hilbert's unpublished contributions to physics cannot be overestimated. In his appreciation of Hilbert's work published in 1922, Otto Blumenthal remarks that Hilbert had published relatively little on physics, at least when compared to his engagement and achievements. He then issues a general request:

> I would like to close this survey of Hilbert's scientific achievements with a request for him. He worked during his classical period, like Weierstraß did during his, mainly for himself and for the circle of his Göttingen students, but he brought very little to publication. The amount of the unknown that had to be worked through may have been the reason for this, just as much as his reluctance to give out anything imperfect or incomplete. But, on the other hand, would he consider the wealth of problems that have been opened up for us by his results, the solutions of which require the work of so many researchers and therefore would he make his insights accessible to us at least in the simple and little time-consuming way of copied lecture notes.[3]

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[2]Blumenthal reports Hilbert as saying that "physics is certainly far too difficult for the physicists." See *Blumenthal 1922*, 70.

[3]"Ich möchte diesen Überblick über Hilberts wissenschaftliche Leistungen mit einer Bitte an ihn schliessen. Er hat in seiner klassischen Periode, ebenso wie Weierstraß in der seinigen, in der Hauptsache für sich und seinen Göttinger Schülerkreis gearbeitet, wenig an die Öffentlichkeit gebracht. Die Menge des Neuen, das zu verarbeiten war, mag davon ebenso viel die Ursache sein wie die Scheu, etwas Unvollkommenes, Unvollständiges herauszugeben. Möge er demgegenüber bedenken, welche Fülle von Problemen uns seine bisherigen Ergebnisse erschließen, von Problemen, zu deren Bewältigung Massenarbeit gehört, und uns deshalb seine Erkenntnisse wenigstens in der anspruchslosen und wenig zeitraubenden Form vervielfältigter Vorlesungshefte zugänglich machen!" *Blumenthal 1922*, 71. Blumenthal repeats the

In these Volumes, we fulfill this request as far as the most important lecture notes on physics are concerned. Indeed, we present a selection of the unpublished work which attempts to bring out the significance and potential of Hilbert's contributions to the foundations of physics in all their aspects. For reasons made clear below, this selection of hitherto unpublished work is supplemented by two works on relativity theory that Hilbert published in 1915 and 1917 respectively.

In the following, we will comment only on the period from 1915 on, and focus on the most important contributions represented in the present Volume.

## 1 The Contents of the Volume

Three major themes pervade Hilbert's thinking about the foundations of physics in the period with which the present Volume is concerned: 1) the formulation of a generally covariant theory of gravitation and electromagnetism and his ideas about a unified theory of the gravitational and electromagnetic fields based on a generalization of Maxwell's equations; 2) the mathematical formulation of quantum theory and its physical interpretation; and 3) the reflection on epistemological questions suggested by the developments of general relativity and quantum theory.

The desire to document these three themes was of fundamental importance in the selection of the unpublished documents for this Volume. The selection is centered on three major documents, corresponding to the three themes mentioned above. The first theme is documented by Hilbert's own two major lecture courses entitled "Foundations of Physics" which deal with general relativity, the unification of gravitation and electromagnetism, and its mathematical foundations. To document the second theme, we include Hilbert's two major lecture courses on the mathematical methods of quantum theory. These two pairs of lecture courses speak more or less for themselves, i.e., they do not need much additional editorial explanation and conceptual clarification. Lastly, Hilbert's philosophical reflections are documented by manuscripts for lectures dealing with causality, irreversibility, and the completeness of science. In order to give a better understanding of these documents, we have supplemented our selection by a number of smaller documents that shed further light on the emergence and refinement of Hilbert's ideas. For completeness and purposes of comparison, we have also included Hilbert's two published papers on the general theory of relativity published in 1915 and 1917, both under the title "Foundations of Physics," distinguished by

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request, this time specifically concerning the lectures on physics, in *Blumenthal 1935*, 417. See also the plea in Sommerfeld's funeral oration held just before Hilbert's burial in *Sommerfeld 1943*, 213.

the appellation "First Communication" and "Second Communication" respectively, and both published in the *Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen.*

The Volume is divided into six chapters. The division is based on a mixture of thematic and chronological considerations, and aims to preserve the connection of documents as intended by Hilbert. The distribution is as follows:

Chapter 1: Hilbert's two "Communications" on the "Foundations of Physics" as published originally in 1915 and 1917 in the *Nachrichten* of the Göttingen Academy of Sciences.

Chapter 2: The *Ausarbeitungen* of a two-semester course entitled "Foundations of Physics" held at the University of Göttingen in the summer semester of 1916 and the winter semester of 1916/17.

Chapter 3: Proofs (a large section of which survived in the Hilbert archives) for Hilbert's "First Communication" in the *Nachrichten* of 1915 as well as three sets of notes for lectures on specific topics related to the subject matter presented in Chapters 1 and 2.

Chapter 4: Two sets of notes for lectures from 1921 and 1923 dealing with general epistemological questions raised by the new physics.

Chapter 5: Parts of an *Ausarbeitung* of a 1912 lecture course on classical and quantum radiation theory held at the University of Göttingen.

Chapter 6: An *Ausarbeitung* of Hilbert's two courses held at the University of Göttingen in the winter semesters 1922/23 and 1926/27 that were devoted to the old and new quantum theory.

Chapter 1 presents Hilbert's two "Communications" in their original published versions. The differences between these and a merged and revised version which appeared in 1924 in the *Mathematische Annalen* are pointed out in Editorial Notes. Also annotated are differences in the 1935 reprint of the 1924 version in Hilbert's *Gesammelte Abhandlungen.* Chapter 2 presents the two unpublished *Ausarbeitungen* of the course entitled "Foundations of Physics," which present the rich mathematical background of Hilbert's published "Communications" with the same title presented in Chapter 1. Chapter 3 presents the page proofs for the "First Communication." There are a number of small differences between the proofs and the published version that are already pointed out in the notes to the published paper in Chapter 1, but here we provide the full text of the proofs that survived. Chapter 3 also presents a set of notes for what appears to be a lecture on the subject matter of the "First Communication" given some time in late 1915 or 1916. Also included in this Chapter is an unpublished typescript with notes of a lecture on the "Causality Principle in Physics" given just a few weeks before the publication of Hilbert's "Second Communication" in 1917 and which covers somewhat similar material. Finally, Chapter 3 presents Hilbert's handwritten notes for a series of lectures on space and time delivered by Hilbert in Bucharest in the spring of

1918. In Chapter 4, we present Hilbert's handwritten notes for a lecture entitled "Nature and Mathematical Knowledge" given in Copenhagen in 1921 on the occasion of the award of an honorary doctorate. Chapter 4 also contains Hilbert's handwritten notes for a set of three lectures given in the summer of 1923 at the University of Hamburg. These lectures were announced under the title "Epistemological Questions of Modern Physics" and represent, perhaps, Hilbert's most explicit and most original discussion of his understanding of the epistemological and ontological implications of modern physics. Chapter 5 contains selections from the *Ausarbeitung* of the 1912 lecture course on radiation theory. This selection (we publish only Chapters 1 and 5–13) is presented here because it contains among other things a discussion of the early quantum theory of radiation. (Chapters 2–4 are omitted since they reproduce material on special relativity that is covered elsewhere.) In Chapter 6, we present in full the *Ausarbeitung* of Hilbert's first course on quantum theory, given in 1922/23 at the University of Göttingen, and the second part of the *Ausarbeitung* of Hilbert's second course on quantum theory, given in Göttingen in 1926/27. The first part of the 1926/27 course is a repetition of the 1922/23 course with only minor deviations, so it is omitted here. The 1922/23 *Ausarbeitung* has comments, corrections and additions in Hilbert's hand; the differences between the two *Ausarbeitungen* as well as Hilbert's handwritten notes are explicitly noted.

The general principles governing the selection of unpublished documents for the Edition as a whole are set out in detail in the "Introduction to the Edition" presented in Volume 1. A further guiding principle for this Volume was above all to select manuscripts which show clearly the originality of Hilbert's contributions to, and reflection on, modern physics, in particular quantum theory and the general theory of relativity. For this reason, we have included Hilbert's first exposition of general relativity and the new theory of gravitation rather than his later lectures on the same topic ("Advanced Mechanics and New Theory of Gravitation," *Hilbert 1920**). We have omitted later lecture courses by Hilbert that emphasize particular applications of physical theories ("Electron Theory," *Hilbert 1917/18**) or which discuss the unity of the mathematical sciences ("Nature and Mathematical Knowledge," *Hilbert 1919**, "On the Unity of Science," *Hilbert 1923/24a**). Likewise, we have not included later lectures on physics of a more popular character addressed to a broader audience ("Space and Time," *Hilbert 1918/19**, "Basic Ideas of Relativity Theory," *Hilbert 1921/22**). These lecture courses (or parts thereof) will be considered among others for publication in Volume 6 of this series. Finally, lecture courses whose focus is not primarily the natural sciences but in which Hilbert nevertheless comments on physics are also omitted from Volumes 4 and 5. Two cases in point are Hilbert's 1905 lectures "The Logical Principles of Mathematical Thought" (*Hilbert 1905**), and Hilbert's 1924/25 lecture course "On the Infinite" (*Hilbert 1924/25**). These will be included in Volumes 2 and 3 of this series respectively, since properly speaking their primary focus is the foundations of pure mathematics. One might say the same of Hilbert's published lectures on "Axiomatic Thought" (*Hilbert 1918*) and his 1930 Königsberg lecture on "Science and Logic" (*Hilbert 1930*). Although they contain specific considerations about physics, their main concerns are with foundational questions about mathematics and science in general.

It should also be kept in mind that the public perception of Hilbert's ideas on physics was mediated to a large extent by two later published works. One is the widely-read book *Methods of Mathematical Physics*, co-authored with Richard Courant (*Courant and Hilbert 1924*, *Courant and Hilbert 1937*). As Courant indicates in the Preface, the book was largely composed by Courant himself and it lists Hilbert as a co-author, mainly to give credit to the general influence of his ideas. The second is the joint paper by Hilbert, John von Neumann, and Lothar Nordheim "Foundations of Quantum Mechanics" (*Hilbert, von Neumann and Nordheim 1928*), which emerged from Hilbert's 1926/27 lectures on the subject and was inspired by Hilbert's program for the axiomatization of quantum physics. As indicated in the paper itself, the final version of the published article was composed by Nordheim and includes important mathematical contributions by von Neumann. We have omitted this publication from the present Volume, although we stress that the lectures given in Chapter 6 contain the relevant background material for this very important paper.

## 2 Hilbert's Work in Physics Represented in this Volume

The period covered by this Volume, i.e., from 1915 to 1926/27, was the most productive one in all of Hilbert's engagement with physics.[4] The year 1915 marks the point in Hilbert's investigations into the foundations of the natural sciences in which he no longer merely reflects on the conceptual problems and logical gaps in existing physical theories, but begins to advance his own original solutions to some of the more fundamental problems. This year, therefore, marks an important shift in Hilbert's understanding of physics. A new interest in current fundamental research into the natural sciences remained characteristic for his subsequent work, despite extended periods of concern with other problems across many areas. This interest pertains to the general theory of relativity and a unified field theory of gravitation and electromagnetism, including a fundamental theory of matter. But it is also characteristic for his later investigations in the mid-1920s into the conceptual and logical structure of the emerging and the maturing quantum theory.

[4]This assessment is somewhat at odds with Weyl's periodization, which singles out the years between 1910 and 1922 as Hilbert's most intense period in his work on physics; see *Weyl 1944*. For a more comprehensive discussion, see *Corry 2004* and *Majer 2006*.

This is not to say that there is no continuity between Hilbert's work after 1915 and his earlier work. Hilbert's main interest in foundational problems and the methodological means that he employs in their solution derive from his early axiomatic work in geometry. It is his interest in an analysis of the logical structure of physical theory and the axiomatic method as a general approach to such an analysis that provide the background for Hilbert's work both before and after the year 1915. But the application of the axiomatic method to such advanced and abstract physical theories like the theory of relativity and quantum theory poses a formidable challenge. In these fields many different aspects of scientific reasoning are interwoven, which means, for one thing, that the variation or addition or deletion of axioms is by no means as easy as it is in geometry.

An outstanding and lasting fruit of Hilbert's engagement with the natural sciences is represented by the two communications entitled "Foundations of Physics" from late 1915 and late 1916 (*Hilbert 1915*, *Hilbert 1917*) presented in Chapter 1 of this Volume. These two publications contain the essence of Hilbert's insights in a very condensed form. A broader and more comprehensive exposition of Hilbert's perspective and his ideas is set out in the two major lecture courses from the summer of 1916 and the winter of 1916/17, the *Ausarbeitungen* of which were given the same title: "Foundations of Physics." These lecture courses are presented in Chapter 2 and it is in these much more detailed expositions that we get an idea of how Hilbert envisaged a true axiomatic presentation of the new physics. In an impressive and illuminating exposition, Hilbert here goes back to the foundations of non-Euclidean, Riemannian differential geometry, and shows how these more refined geometrical concepts provide the background for treating the problem of the relationship between geometry and experience in the new relativistic physics. In the first part of these lectures, Hilbert also gives the first axiomatic analysis of the conceptual foundations of the special theory of relativity, an endeavor that was taken up some years later by Constantin Carathéodory (*Carathéodory 1924*) and Hans Reichenbach (*Reichenbach 1924*). A number of intriguing and complex aspects of Hilbert's work on the foundations of physics from this period are documented by the short manuscripts reproduced in Chapter 3. This Chapter also reproduces Hilbert's first presentation of the new concepts of space and time designed for a broader audience.

In the manuscripts for lectures at Copenhagen, Hamburg, and Zürich from 1921–1923, which are given in Chapter 4, Hilbert explicitly addresses the philosophical and metaphysical consequences of the theories of relativity and quantum theory. In particular, he addresses the metaphysical question of what part of the respective theories can be considered as objective and real, and what part only stems from our subjective and anthropomorphic point of view. Hilbert's special contribution is to identify a number of crucial epistemological problems on which the new physics casts an entirely new light.

These problems are the question of causality in a generally covariant theory, and the related problem of the time-reversal invariance of fundamental laws in view of the irreversibility observed in our practical world. More generally, he also reflects on the issue of the completeness of physical theories, and considers the question of the apriori and aposteriori with respect to geometrical concepts, and also the debate about geometrical conventionalism. In Chapter 5, we give an example of Hilbert's earlier work, demonstrating the difficulties of achieving satisfactory conceptual clarity in the emerging quantum theory. The contentions Hilbert presents here led to a significant controversy among physicists, and the lectures vividly document Hilbert's reflections on the difficulties arising from the conceptual integration of the idea of quanta into the statistical theory of radiation in the early quantum theory.

Many of the difficulties Hilbert raises here were clarified in the later quantum theory that he discusses in full detail in his lectures of 1922/23 and again in 1926/27; these lectures make up Chapter 6. The 1922/23 lecture course on quantum theory deals with the quantization conditions underlying Bohr's theory of atomic spectra, and, in particular, with the problem of generalizing those conditions to systems of more than one dimension and several degrees of freedom as advanced in works by Sommerfeld, Schwarzschild, Epstein, Einstein and others.[5] To this end, Hilbert begins his presentation with an extensive discussion of the variational calculus applicable to classical mechanics with several degrees of freedom and derives the Hamilton-Jacobi equations in some detail. This discussion provides the basis for the introduction of a concept that Hilbert calls the "quantrix" and which he regards as the proper mathematical entity subject to the quantization rules.

The first part of Hilbert's second exposition of quantum mechanics in 1926/27 repeats the 1922/23 exposition. It then presents the "new quantum mechanics," discussing the latest work by Heisenberg, Schrödinger, Jordan, Born, and other quantum physicists, with particular emphasis on the mathematical concepts employed in the new matrix and wave-mechanics.[6] Heisenberg's matrix mechanics is presented as a natural application of Hilbert's own theory of linear integral equations developed from 1909 on, which in part is a linear algebra for systems with infinitely many variables, and Hilbert then discusses the eigenvalue problems for the harmonic oscillator, the rotator and the hydrogen atom respectively. In the penultimate chapter of the lectures, Hilbert takes up the problem of applying quantum theory to statistical mechanics, discussing the distinction between Bose-Einstein and Fermi-Dirac

[5] See, e.g., *Sommerfeld 1915*, *Sommerfeld 1916*, *Schwarzschild 1916c*, *Epstein 1916a*, *Epstein 1916b*, *Einstein 1917b*, and, for a general historical discussion, *Mehra and Rechenberg 1982a*, ch. II.4.

[6] For a comprehensive historical account, see *Mehra and Rechenberg 1982b*, *Mehra and Rechenberg 1982c*, *Mehra and Rechenberg 1982d*, *Mehra and Rechenberg 1987a*, *Mehra and Rechenberg 1987b*, *Mehra and Rechenberg 2000*, *Mehra and Rechenberg 2001*.

statistics. Finally, Hilbert discusses Born's recently proposed probability interpretation of the wave function,[7] and sets this interpretation in the context of a new axiomatic analysis of the fundamental concepts of quantum mechanics, which follows the then recently published work of Jordan.[8]

So much for a general description of the contents of the Volume. In the next two sections, a more detailed discussion of the historical context of Hilbert's two communications on the "Foundations of Physics" deals with subjects and themes involving the contents of Chapters 1–3. It is presented here because it spans those Chapters collectively, and because the division of material we have adopted means that those Chapters do not represent a strictly chronological presentation of Hilbert's treatment of general relativity, thus making a general survey desirable.

## 3 Hilbert's "First Communication" on the "Foundations of Physics" and the Energy Concept in General Relativity

Recent historical scholarship has paid a great deal of attention to the *original* version of Hilbert's "First Communication" on the "Foundations of Physics" (*Hilbert 1915*), and to its prehistory.[9] This interest is motivated mainly by Hilbert's prominent role in the genesis of general relativity. Hilbert's note was published almost simultaneously with Einstein's celebrated paper (*Einstein 1915d*, submitted for publication to the Prussian Academy on 25 November 1915), in which he marked the completion of his search for a general theory of relativity by publishing his gravitational field equations.[10] Hilbert's paper presented generally covariant field equations using a variational formulation, but it also contained the gravitational field equations much as they were presented in Einstein's communication. Therefore, it has become customary to acknowledge Hilbert's contribution to the genesis of general relativity by

[7] *Born 1926a*, *Born 1926b*, *Born 1927*; for historical discussion, see *Pais 1982b* and *Mehra and Rechenberg 2000*, ch. I.3.

[8] *Jordan 1926a*.

[9] For further literature discussing Hilbert's two communications, see *Pauli 1921*, *Born 1922*, *Mehra 1974*, *Earman and Glymour 1978*, *Eisenstaedt 1982*, *Pais 1982a*, *Norton 1984*, *Stachel 1992*, *Vizgin 1994*, *Corry, Renn and Stachel 1997*, *Corry 1999a*, *Corry 1999b*, *Rowe 1999*, *Sauer 1999*, *Stachel 1999*, *Rowe 2001*, *Vizgin 2001 Sauer 2002*, *Corry 2004*, *Goenner 2004*, *Logunov, Mestvirishvili and Petrov 2004*, *Sauer 2005*, *Renn and Stachel 2007*, *Brading and Ryckman 2008*.

[10] For a comprehensive historical discussion of Einstein's path toward general relativity, see *Janssen et al. 2007a*, *Janssen et al. 2007b*.

referring to classical general relativity as the "Einstein-Hilbert Theory," particularly when the formulation of the theory by means of a variational principle is to be emphasized.

The interest in Hilbert's original paper was reawakened about ten years ago when it was pointed out that an extensive (but incomplete) set of proofs for this note exists among Hilbert's papers (*Corry, Renn and Stachel 1997*). These proofs bear the printer's stamp of 6 December 1915, and display a number of significant differences when compared to the published version. The debate with respect to these proofs initially focussed on the issue of priority, since it had been insinuated that Hilbert made substantial changes only *after* having seen Einstein's paper and without proper acknowledgement. As one of the Editors of this Volume has argued, however, the proofs do not in fact justify any such claim (*Sauer 1999*). We recapitulate here the main points.

Einstein visited Göttingen in the summer of 1915 to deliver a series of six Wolfskehl lectures to the Göttingen mathematicians and physicists. In these lectures Einstein presented his so-called *Entwurf* theory, i.e., the precursor to the general theory of relativity, a theory which is expounded in the major review that Einstein published in 1914 under the title "The Formal Foundation of the General Theory of Relativity" (*Einstein 1914*). A major part of the exposition in the "Formal Foundations" paper was concerned with a mathematical derivation of the field equations of the *Entwurf* theory from a variational principle. But since the equations themselves were not generally covariant, the variational integral was not invariant either.

Hilbert had studied this paper carefully, and had understood its argument in all technical details.[11] It is therefore quite possible that he found problems with Einstein's derivation of the field equations, and sought to revert to a derivation that remained fully covariant and restricted the covariance only at a clearly defined juncture. His heuristic idea was to try to connect Einstein's theory to a generalized version of Maxwellian electrodynamics which had recently been proposed by Gustav Mie. Mie's non-linear generalization of

[11]See Einstein to Arnold Sommerfeld, 15 July 1915: "In Göttingen I had the great pleasure of seeing that everything was understood down to the details. I am quite enchanted with Hilbert. That's an important man for you!" (*CPAE8-A 1998*, Doc. 96). Immediately after his breakthrough to general covariance Einstein referred to Hilbert in a letter to Heinrich Zangger, 26 November 1915: "The theory is beautiful beyond comparison. However, only *one* colleague has really understood it, [...]," (*CPAE8-A 1998*, Doc. 152). The latter comment was written in a mood of misgivings about Hilbert's parallel publication of his "First Communication," as is clear from a similar quote in Einstein's letter to Zangger, written at about the same time, before 4 December 1915: "Currently I am also having quite a curious experience with my dear colleagues. All but one of them is trying to poke holes in my discovery or to refute the matter, if only so very superficially; just one of them acknowledges it, insofar as he is seeking to 'partake' in it, with great fanfare, after I had initiated him, with much effort, into the gist of the theory." (*CPAE9 2004*, p. 36).

Maxwellian electromagnetism was also presented in a variational formulation, albeit strictly Lorentz covariant. Hilbert's central idea had been to combine both Mie's and Einstein's theories by means of an invariant variational integral governing the theory.

In the meantime, Einstein himself had found fault with his theory, and had returned to general covariance, as is clear from the series of four communications which Einstein presented to the Prussian Academy in November of 1915. When Hilbert discovered this, he saw himself forced in November of 1915 to write up his insights somewhat prematurely.

Through the proofs of Hilbert's "First Communication," we have learned more about his original ideas and his own particular path towards general relativity. There are two significant differences between the proofs of Hilbert's paper and the published version. The first difference is actually a rather minor point, even though it has received a great deal of attention in the debates surrounding the issue of priority. The proofs do not yet contain the gravitational field equations published by Einstein in his fourth and final note of 25 November in their explicit form using the Ricci tensor and its trace. These equations were added in the published version of Hilbert's paper, but not only did Hilbert properly cite Einstein's fourth note in this context, he also pointed out that the derivation of the explicit form of the Einstein equations from the variational formulation is, in fact, rather trivial.[12] The other substantive changes that Hilbert introduced at proof stage concern the discussion of the energy concept in his theory. These changes are more interesting and more important, both from a historical and a systematic point of view.

At first sight, what seems to be the major difference between the proofs and the published version of Hilbert's "Communication" is the formulation in the proofs of a third axiom that was dropped altogether in the version published.

In essence, the argument presented in the proofs was the following. In order to derive an expression for energy, Hilbert considered his variational integral (proofs, p. 2, this Volume, p. 318)

$$I = \int H\sqrt{g}dS \tag{1}$$

He then looked at what we would now call the Lie variation of this integral. Since the Lagrangian $H$ was assumed to depend on the components of the metric tensor $g^{\mu\nu}$ and its first and second derivatives as well as on the components of the electromagnetic four-potential $q_s$ and its first derivatives, although *not* on the coordinates themselves, the integral should be invariant under those changes of the metric induced by an arbitrary infinitesimal transformation of the coordinates. Technically, Hilbert assumed $p^\mu$ to be an

[12]See *Sauer 2005* for a more detailed discussion of this point and further references.

arbitrary infinitesimal vector field. The induced Lie variation of the metric would then be $\delta g^{\mu\nu} \propto p^{(\mu;\nu)}$ or, closer to Hilbert's notation (proofs, p. 5, this Volume, p. 321),

$$\delta g^{\mu\nu} \equiv p^{\mu\nu} = g^{\mu\nu}_{,s} p^s - g^{\mu s} p^{\nu}_{,s} - g^{\nu s} p^{\mu}_{,s}. \tag{2}$$

In classical general relativity, the way to proceed from here would be to look at the formal variation of $I$ with respect to $\delta g^{\mu\nu}$, and to remove derivatives on $p^{(\mu;\nu)}$ by partial integrations; then, in a second step, one would remove the derivative involved in $p^{(\mu;\nu)}$ itself by a further partial integration. What results is a bulk integral depending linearly on the arbitrary vector field $p^\mu$, as well as surface integrals produced by the partial integrations. The fact that $p^\mu$ is arbitrary then implies vanishing of the bulk integrand itself; this then produces the desired results such as the vanishing of the covariant divergence of the energy-momentum tensor. The procedure also produces special cases of Noether's theorems to be considered in a moment.

In contrast to what would occur in a more modern treatment, Hilbert does not carry out the calculations on the level of the action integral, but rather concentrates on the changes of the Lagrangian density itself. These still have to be integrated, while also dealing with intermediate terms that later cancel out when the integrations are actually carried out. In the proofs, however, Hilbert proceeded in an even more unusual manner. In performing the partial integrations, Hilbert's first step is not to replace *all* derivatives of $p^\mu$ terms involving $H$; rather he retains those terms proportional to the first derivative of $p^\mu$. Consequently, he arrives at an expression of the form (proofs, p. 6, this Volume, p. 322)

$$E = \sum \left( e_s p^s + \sum e^l_s \frac{\partial p^s}{\partial w^l} \right) \tag{3}$$

where $e_s$ and $e^l_s$ are to be interpreted by looking at the explicit integral obtained after performing the necessary partial integrations.

Hilbert now shows that, using the field equations, $E$ can be transformed into a pure divergence. He also argues that if you now proceed from equation (3) and also proceed to convert the derivative acting on $p^\mu$ to $e^l_s$, then the resulting expression multiplying the arbitrary vector $p^\mu$ must vanish. Hilbert now identifies the vanishing of the divergence of $e^l_s$ with the energy theorem of the old, Lorentz-covariant theory, and he concludes that this divergence can only vanish independently if the vector $e^s$ vanishes independently. This condition is postulated as a third axiom to the theory, and since it is not in itself a covariant equation, it represents a coordinate condition, restricting admissible coordinate systems to those in which $e^s \equiv 0$.

In the published version of Hilbert's note, a very different expression is derived to represent the energy vector. Moreover, the vanishing of the divergence of this vector is shown to hold for all coordinates. Hence the conservation of energy no longer serves to provide an effective restriction of the covariance of the theory, and, consequently, the third axiom of the proofs was dropped in the published version.

More than two years later, in March 1918, Hilbert's Göttingen colleagues, including Felix Klein and Carl Runge, were deeply engaged in disentangling the intricacies of the problem of energy-momentum conservation in general relativity. At one point during the debates that ensued, Klein and Runge had an idea that must have struck Hilbert as reminiscent of his early attempts to come to grips with the problem of energy conservation. In any case, he sent the original proofs of his "First Communication" to Klein, pointing out that he, too, at one time had similar ideas, and in particular had tried to recover the conservation laws of the classical theory by using special coordinates. But, he goes on,

> But I suppressed the whole thing, since the thing did not seem clear to me.[13]

The problem of energy conservation is not only fascinating from a historical persepctive, but also points to a philosophically interesting issue in the foundations of general relativity.[14]

One way to view Hilbert's "First Communication" on the "Foundations of Physics" is as one of the decisive steps in a line of development that originates in the restricted covariance of Einstein's original *Entwurf* theory and eventually culminates in Emmy Noether's celebrated theorems about the relation between symmetries and conservation laws. Briefly, the main steps of this line of development were the following:

- Einstein's "Formal Foundations" (*Einstein 1914*): Restricted covariance in the variational derivation of the gravitational field equations of the *Entwurf* theory.
- Hilbert's proofs for his "First Communication": Generally covariant field equations but restricted covariance introduced by a third axiom that postulates the vanishing of an energy expression.
- Hilbert's "First Communication" itself (*Hilbert 1915*): Generally covariant field equations without restrictions of the covariance group. No further third axiom, and a complete treatment of the invariance properties by means of the Lie variation.

[13] "Ich habe aber die ganze Sache später unterdrückt, weil die Sache mir nicht reif erschien." *Hilbert and Klein 1985*, p. 144.

[14] See, e.g., *Hoefer 2000*, *Brading 2005* for some further discussion.

- Klein's "First Note" (*Klein 1917*) consisting of extracts of the correspondence between Klein and Hilbert on this matter: A first comparison of Einstein's and Hilbert's treatment of the energy problem by looking at the Lie variation of the action integral rather than at the changes of the Lagrangian alone. Speculations about "improper" conservation laws by Hilbert. For the Lie variation, the vanishing of the boundary terms is considered from the beginning.

- Klein's "Second Note" (*Klein 1918*): A further investigation of the various energy concepts, in particular those advanced by Einstein and Hilbert. The Lie variation is considered, taking into full account all boundary terms. Einstein's and Hilbert's energy expressions are identified as the contributions from the bulk integral and the surface integrals respectively.

- Noether's Note (*Noether, E. 1918*): An extension of Klein's treatment endowing it with full generality, i.e., without restriction to four dimensions and for arbitrary local and global group transformations.

It is particularly intriguing to trace back to Hilbert's "First Communication" Emmy Noether's distinction between "proper" and "improper" conservation laws. In the last section of her famous paper on "invariant variational problems," using her second theorem Noether gives a definitive justification for what she calls an "assertion of Hilbert's [*eine Hilbertsche Behauptung*]" concerning the validity of "proper" energy theorems:

> From the preceding finally follows the proof of a theorem due to Hilbert about the connection between the failure of proper energy theorems with "general relativity" (first note by Klein, Göttinger Nach. 1917, response, 1. paragraph) in a generalized group-theoretic way.[15]

The distinction between "proper" and "improper" conservation laws is introduced by Noether in the following passage:

> The invariance with respect to the translation group, as is well-known, asserts that the $x$ in $I = \int \ldots \int f\left(x, u, \frac{\partial u}{\partial x}, \ldots\right) dx$ do not appear explicitly. The corresponding divergence relations
>
> $$\sum \Psi_i \frac{\partial u_i}{\partial x_\lambda} = \mathrm{Div} B^{(\lambda)} \qquad (\lambda = 1, 2, \ldots, n)$$
>
> shall be called "energy relations" since the "conservation theorems" that follow from the variational problem correspond to the "energy theorems" and the $B^{(\lambda)}$ correspond to the "energy components." Then it follows: *If I*

[15] "Aus dem Vorhergehenden ergibt sich schließlich noch der Beweis einer Hilbertschen Behauptung über den Zusammenhang des Versagens eigentlicher Energiesätze mit "allgemeiner Relativität" (erste Kleinsche Note, Göttinger Nachr. 1917, Antwort, 1. Absatz), und zwar in verallgemeinerter gruppentheoretischer Fassung." *Noether, E. 1918*, 256–257.

> *allows for the translation group, then the energy relations become improper ones, if and only if I is invariant under an infinite group that contains the translation group as a subgroup.*[16]

In a footnote, Noether explains

> The energy theorems of classical mechanics and also those of the old "relativity theory" (where $\sum dx^2$ transforms into itself) are "proper" ones, since here no infinite groups occur.[17]

The justification for Hilbert's "assertion" given by Noether at the end of her paper is then the following:

> Hilbert phrases his theorem in such a way that the failure of proper energy theorems is a characteristic feature of the "general theory of relativity." For this statement to be literally true, one needs to understand the term "general relativity" in a more general sense as is usually done, thus that it includes also the previous groups that depend on $n$ arbitrary functions.[18]

How did Noether come to consider the question of the distinction between proper and improper conservation laws? She herself points to Klein's "First Note," and in particular to a remark of Hilbert's to Klein, repeated in that Note, which was an excerpt from the correspondence between Klein and Hilbert about the latter's "First Communication."

In his Note, Klein reproduces the following passage he had written to Hilbert:

> If we now assume the validity of the 14 field equations [...], then the $e^\nu$ [i.e., Hilbert's energy vector of his published note] reduce to this extra term and the claim of your note that
>
> $$\sum_\nu \frac{\partial \sqrt{g} e^\nu}{\partial w^\nu} = 0$$

---

[16]"Die Invarianz gegenüber der Verschiebungsgruppe sagt bekanntlich aus, daß in $I = \int \ldots \int f\left(x, u, \frac{\partial u}{\partial x}, \ldots\right) dx$ die $x$ nicht explizit in $f$ auftreten. Die zugehörigen $n$ Divergenzrelationen

$$\sum \Psi_i \frac{\partial u_i}{\partial x_\lambda} = \mathrm{Div} B^{(\lambda)} \qquad (\lambda = 1, 2, \ldots, n)$$

seien als „Energierelationen" bezeichnet, da die dem Variationsproblem entsprechenden „Erhaltungssätze" $\mathrm{Div}\, B^{(\lambda)} = 0$ den „Energiesätzen", die $B^{(\lambda)}$ den „Energiekomponenten" entsprechen. Dann gilt also: *Gestattet I die Verschiebungsgruppe, so werden die Energierelationen dann und nur dann uneigentliche, wenn I invariant ist gegenüber einer unendlichen Gruppe, die die Verschiebungsgruppe als Untergruppe enthält.*" *Noether, E. 1918*, p. 255.

[17]"Die Energiesätze der klassischen Mechanik und ebenso die der alten „Relativitätstheorie" (wo $\sum dx^2$ in sich übergeht), sind „eigentliche", da hier keine unendlichen Gruppen auftreten." (Ibid.)

[18]"Hilbert spricht seine Behauptung so aus, daß das Versagen eigentlicher Energiesätze ein charakteristisches Merkmal der „allgemeinen Relativitätstheorie" sei. Damit diese Behauptung wörtlich gilt, ist also die Bezeichnung „allgemeine Relativität" weiter als gewöhnlich zu fassen, auch auf die vorangehenden von $n$ willkürlichen Funktionen abhängenden Gruppen auszudehnen." *Noether, E. 1918*, 256–257.

> holds, appears as a mathematical identity. That claim, therefore, cannot be well regarded as analogon to the conservation theorems of energy as they hold in the usual mechanics. For, if we write in the latter
>
> $$\frac{d(T+U)}{dt} = 0,$$
>
> then this differential relation does not hold identically but only as a consequence of the differential equations of mechanics.[19]

It should be noted that Klein explicitly extends the same observation to Einstein's treatment of the energy problem, for he says

> [...] I want to point out that, of course, the same that holds for your theorem (19) also holds for the "conservation theorems" as they were formulated by Einstein in 1916.[20]

In his response to Klein's letter, Hilbert writes

> As regards the matter of the fact, I completely agree with your argument: *Emmy Noether*, whose assistance I called upon a year ago for the clarification of such analytical questions concerning my energy theorem, found at the time that the energy components that I had constructed—like Einstein's—can formally be transformed by means of the Lagrangian differential equations (4), (5) of my first communication into expressions whose divergence vanishes identically, that is without assuming the validity of the Lagrangian equations (4), (5). On the other hand, since the energy equations of classical mechanics, elasticity theory, and electrodynamics are only satisfied as a consequence of the Lagrangian differential equations of the problem at hand, it is justified if you do not regard my energy equations as the analogon of the ones in those theories. Of course, then I claim that for *general* relativity, i.e., for the case of *general* invariance of Hamilton's function, energy equations that correspond to the energy equations of the orthogonal-invariant theories in your sense, do not

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[19]"Nehmen wir jetzt die 14 Feldgleichungen [...] hinzu, so reduziert sich $e^\nu$ [i.e., Hilbert's energy vector of his published note] auf diesen Zusatzterm und die Angabe [...] Ihrer Note, daß

$$\sum_\nu \frac{\partial \sqrt{g} e^\nu}{\partial w^\nu} = 0$$

statt hat, erscheint als mathematische Identität. Besagte Angabe kann also wohl nicht als Analogie zum Erhaltungssatze der Energie, wie er in der gewöhnlichen Mechanik herrscht, angesehen werden. Denn wenn wir in letzterer schreiben:

$$\frac{d(T+U)}{dt} = 0,$$

so besteht diese Differentialbeziehung doch nicht identisch, sondern erst in Folge der Differentialgleichungen der Mechanik." *Klein 1917*, p. 475.

[20]"[...] will ich noch darauf aufmerksam machen, daß für die „Erhaltungssätze", wie sie Einstein 1916 formuliert hat, selbstverständlich das gleiche gilt, wie für Ihren Satz (19)." *Klein 1917*, 476.

> exist at all; yes, I would like to view this circumstance as a characteristic feature of the general theory of relativity. For my assertion one could provide the mathematical proof.[21]

Thus, this *Behauptung* of Hilbert's originated in his "First Communication" on the "Foundations of Physics." It was finally established by Emmy Noether on the basis of her two theorems on invariant variational problems.

## *4 Hilbert's "Second Communication"*

Hilbert's "Second Communication" on the "Foundations of Physics" is much less familiar, and consequently much less discussed than his first one.[22] On 4 December 1915, two weeks after the submission of his "First Communication," Hilbert presented for publication in the *Nachrichten* of the Mathematical-Physical Class of the Göttingen Academy of Sciences a "Second Communication" on the "Foundation of Physics."[23] However, further processing of this "Second Communication" was withheld. No doubt part of the reason for this was a desire to prepare the "First Communication" for the press. As discussed above, this involved a rewrite of the section that dealt with the energy concept in Hilbert's theory, as well as the other revisions that had to be introduced following the proof stage. Furthermore, the "Second Communication" was submitted for publication at a time of intense activity. Some days before, on 30 November, Hilbert and Carathéodory had presented a talk "on invariant theory" to the Göttingen Mathematical Society. On the very same day, Hilbert wrote to the Prussian Ministry on behalf of

[21] "Mit Ihren Ausführungen über den Energiesatz stimme ich sachlich völlig überein: *Emmy Noether*, deren Hülfe ich zur Klärung derartiger analytischer meinen Energiesatz betreffenden Fragen vor mehr als Jahresfrist anrief, fand damals, daß die von mir aufgestellten Energiekomponenten — ebenso wie die *Einstein*schen — formal mittelst der *Lagrange*schen Differentialgleichungen (4), (5) in meiner ersten Mitteilung in Ausdrücke verwandelt werden können, deren Divergenz *identisch* d.h. ohne Benutzung der *Lagrange*schen Gleichungen (4), (5) verschwindet. Da andererseits die Energiegleichungen der klassischen Mechanik, der Elastizitätstheorie und Elektrodynamik nur als Folge der *Lagrange*schen Differentialgleichungen des Problems erfüllt sind, so ist es gerechtfertigt, wenn Sie deswegen in meinen Energiegleichungen nicht das Analogon zu denen jener Theorien erblicken. Freilich behaupte ich dann, daß für die *allgemeine* Relativität, d.h. im Falle der *allgemeinen* Invarianz der *Hamilton*schen Funktion, Energiegleichungen, die in Ihrem Sinne den Energiegleichungen der orthogonalinvarianten Theorien entsprechen, überhaupt nicht existieren; ja ich möchte diesen Umstand sogar als ein charakteristisches Merkmal der allgemeinen Relativitätstheorie bezeichnen. Für meine Behauptung wäre der mathematische Beweis erbringbar." *Klein 1917*, 477.

[22] See, however, *Corry 2004*, ch. 8.4, *Renn and Stachel 2007* and *Brading and Ryckman 2008*.

[23] See entry 739 of the *Journal* for the *Nachrichten* (Archives of the Göttingen Academy of Sciences (GAA), Scient 66, Nr. 2), and *Geschäftliche Mitteilungen* (GM) of the *Königliche Gesellschaft der Wissenschaften, Göttingen* 1916, 6; cp also *Sauer 1999*, 560–561.

Emmy Noether's *Habilitation*. And the following day, i.e., on the 5 December, Hilbert and four of his Göttingen colleagues wrote a formal proposal for the election of Einstein as a Corresponding Member of the Göttingen Academy of Sciences. And three days after, on 7 December, Hilbert and Carathéodory continued their lecture to the Mathematical Society on invariant theory.

In late December, the hectic activities abated somewhat. Einstein was elected as a Corresponding Member of the Göttingen Academy, and conciliatory letters were exchanged between him and Hilbert. Early in 1916, Hilbert gave two more talks to the Göttingen Mathematical Society, the first on 25 January entitled "Invariantentheorie und allgemeiner Energiesatz," and the second three weeks later (15 February) entitled "Zur Integrationstheorie gewöhnlicher Differentialgleichungen." The titles are taken from the reports in Volume 25 (1917) of the *Jahresbericht der Deutschen Mathematiker-Vereinigung* (p. 31 of the *2. Abteilung*). Although nothing is known about the precise contents of these talks, it seems clear that the first lecture concerns Hilbert's revisions of his "First Communication" regarding the omission of the third axiom. As far as the second lecture is concerned, what is known is that Hilbert gave a lecture course on differential equations in that semester and then a lecture course on partial differential equations in the following summer semester.[24] Hilbert may well have addressed a particular topic eventually treated in the published version of his "Second Communication," i.e., the problem of the uniqueness of solutions to differential equations arising from a generally invariant variational principle.

In this context, an important event for the prehistory of Hilbert's "Second Communication" is the publication of Schwarzschild's paper "The gravitational field of a material point according to Einstein's theory".[25] This paper took up Einstein's approximate integration of the gravitational field equations of his paper from 18 November on the anomalous perihelion advance of Mercury (*Einstein 1915c*) and presented the exact solution of the gravitational field equations for the static and spherically symmetric case. In addition to its astronomical implications for the classical tests of general relativity, Schwarzschild's solution directly concerned Hilbert's program of finding a particle solution that would represent the electron.

By mid-February, offprints of Hilbert's "First Communication" were finally available.[26] It may be assumed that Hilbert sent a copy to Einstein,

[24] *Ausarbeitungen* of these courses are extant in the Mathematics Institute of Göttingen University (*Hilbert 1915** and *Hilbert 1916b**). See the list of Hilbert's lectures, p. 719, below.

[25] *Schwarzschild 1916a*; the paper was presented to the Prussian Academy by Einstein on 13 January, read at the joint Academy meeting of both Classes on 3 February, and the published version was issued on 10 February. We do not know exactly when Hilbert learnt about Schwarzschild's solution.

[26] For details, see *Sauer 1999*, 543–544.

presumably with an invitation to visit Göttingen to attend three lectures on the history of mechanics until Galileo to be delivered on behalf of the Wolfskehl foundation by Conrad Müller.[27] Einstein's response from 18 February did not mention Hilbert's paper but did state his intention to visit Göttingen. Hilbert's "Second Communication" was presented to the Göttingen Academy for the second time on 26 February 1916, and the Journal for the Proceedings records that it was forwarded to the printer on 2 March 1916.[28] Probably on the same day, Einstein arrived in Göttingen for a second visit and remained for a few days.[29]

We know little about the precise contents of the first version of the "Second Communication," and similarly we can only guess about the contents and modifications of this second version, since it, too, was subsequently withdrawn. There are indications that Hilbert believed he had found a solution to the electron problem, or was just about to do so. In a letter to Hilbert, dated 29 February 1916, Mie wrote:

> That you have already succeeded in obtaining electrons theoretically surpasses all my expectations, and I am very curious to see your next publications.[30]

Mie's letter was written in response to a letter from Hilbert that in all probability had been written around 26 February when Hilbert had resubmitted his "Second Communication." Hilbert had also sent a brief postcard to Schwarzschild on the same day, asking him whether he had received an offprint of his "First Communication" and alerting him to the fact that the "postulate of $g = 1$ would be unnecessary." A few days later, Hecke sent a postcard to Hilbert, thanking him for offprints and asking:

> When will you publish the electron?[31]

Whether or not this is related to discussions between Hilbert and Einstein in Göttingen, Hilbert decided to withdraw the second version of his paper

[27] See the report in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* for 1917, Volume 25, *2te Abteilung*, p. 31.

[28] *Journal*; see note 23 above.

[29] In the letter from 18 February announcing his visit, Einstein writes that he will arrive on a Thursday; this would be 2 March, since he was still in Berlin on Monday, 28 February, and he was definitely in Göttingen on Friday, 3 March, for there is a postcard from Göttingen to his son postmarked on that day. Einstein was back in Berlin on 10 March, and in a letter to Hilbert from 30 March, he expresses gratitude for the "hoch interessanten und behaglichen Tage meines Aufenthaltes in Ihrem gastlichen Hause." See *CPAE8-A 1998*, Docs. 193, 196–198, 207.

[30] "Dass es Ihnen schon gelungen ist, Elektronen theoretisch zu erhalten, geht über all mein Erwarten, und ich bin sehr gespannt auf Ihre nächsten Veröffentlichungen." Mie to Hilbert, 29 February 1916, SUB Göttingen *Cod. Ms. Hilbert* 254/3.

[31] "Wann publizieren Sie das Elektron?"; see SUB Göttingen *Cod. Ms. Hilbert* 141/9.

from print at the beginning of March, during or immediately following Einstein's visit.[32] With his "Second Communication" withheld again, the ensuing months were apparently devoted to a more thoroughgoing investigation of the issues at hand.

The winter semester in Göttingen officially ended on 15 March 1916.[33] In April, Hilbert spent four weeks in Lugano, Switzerland.[34] When he came back at the beginning of May and began his summer lectures[35], he also had a new assistant for his work on physics, the mathematician Richard Bär from Zurich. For the summer semester, Hilbert had announced a lecture course on partial differential equations, in continuation of his lecture course on ordinary differential equations from the preceding winter semester. He also announced a course entitled "Introduction to the Principles of Physics," the *Ausarbeitung* of which was prepared by Bär under the title "Foundations of Physics." (This *Ausarbeitung* is presented in Chapter 2 of the present Volume.) In the Preface to this *Ausarbeitung*, a parallel set of lectures on invariant theory by Emmy Noether also was announced.[36] In addition to this, Hilbert gave his weekly seminar on the "Structure of Matter" together with Debye.[37]

From Bär's *Ausarbeitung* of the course on the "Foundations of Physics," we see that in May of 1916 Hilbert actually went back to the beginning. He gives considerable attention to an axiomatic discussion of Newtonian and special relativistic kinematics. He gives a brief account of three- and four-dimensional vector calculus. He then devotes some time to special relativistic dynamics, presenting Mie's electrodynamics as an alternative to the Abraham-Born theory based on the rigid electron hypothesis. Only in the last lecture,

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[32] *Journal* (see note 23). The date "2.III.16" in the column "To the printer" was crossed out in pencil, and no dates were entered in the column for corrections.

[33] *Verzeichnis der Vorlesungen auf der Georg-August-Universität zu Göttingen* (VV), Göttingen: Kästner, 1916, p. 1.

[34] Hilbert wrote to Einstein on 27 May 1916: "Ostern war ich vier Wochen in Lugano und habe dort Fleisch u. Schlagsahne geschlemmt etc." See *CPAE8-A 1998*, pp. 290f. See also Königsberger to Hilbert, 8 April 1916 (SUB Göttingen *Cod. Ms. Hilbert* 187/14); Hilbert to Russell, 14 April 1916 (Russell Archives, McMaster University, Hamilton, quoted in *Sieg 1999*, 37ff); and M. Coehn to Hilbert, 21 April 1916 (SUB Göttingen *Cod. Ms. Hilbert* 453/3). Easter Sunday in 1916 was on 23 April.

[35] The summer term officially began on 16 April. However, note that "im allgemeinen finden die ersten Vorlesungen etwa 8 Tage nach offiziellem Semesteranfang statt" (VV, pp. 1, 35). The first lecture of Hilbert's course "Foundations of Physics" was given on 4 May 1916, *Hilbert 1916a**; p. 1 (this Volume, p. 81).

[36] *Hilbert 1916a**; see p. 1, this Volume, p. 81). Since the first attempt to obtain the *venia legendi* for Noether had failed (women were barred from obtaining the *habilitation* at Prussian universities), she was not allowed to announce lectures on her own. (See *Tollmien 1991*.)

[37] The course "Partielle Differentialgleichungen" was given on Monday mornings from 9:00 to 11:00; the course "Einleitung in die Prinzipien der Physik" was given on Thursday mornings, also from 9:00 to 11:00; and the seminar with Debye took place on Wednesday afternoons from 4:00 to 6:00. See the (VV) for Summer 1916, pp. 14–16.