Marco Bramanti · Giancarlo Travaglini

Studying Mathematics

The Beauty, the Toil and the Method

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to our friends

Contents

Why This Book

> Mathematics is the science that yields the best opportunity to observe the working of the mind. Its study is the best training of our abilities as it develops both the power and the precision of our thinking. Mathematics is valuable on account of the number and variety of its applications. And it is equally valuable in another respect: by cultivating it, we acquire the habit of a method of reasoning which can be applied afterwards to the study of any subject and can guide us in life's great and little problems [3].

It is always true that many students encounter some difficulties during the study of mathematics, both in school and in college. According to us this problem is often due to two deficiencies and it is useful to distinguish them.

1. Prerequisites. On the one hand the study of mathematics is organized into a logical path which, from primary school to university, proceeds sequentially (in a more marked way than in other disciplines). Then a possible "difficult time" in the study of mathematics, at any stage of the student's school life, may cause a "gap in the prerequisites" which may be hard to fill later. Among freshmen this problem is quite common and whoever is in this situation needs to spend some time and effort acquiring the missing arguments. In order to do this one can go through her/his high school math textbooks, or look for specific texts which offer a reasonable and sufficiently condensed presentation of the subject.

2. Method. On the other hand the difficulties encountered by many students while studying math courses are also (and sometimes mainly) due to the particular method of study that math requires: one has to understand the need of a precise language and learn how to use it when it is necessary; he/she has to familiarize with logical tools and mathematical symbols, and understand the justifications of results (exercises, theorems, computational techniques), checking every line, exemplifying the arguments and learning how to apply them in similar situations. Briefly speaking: *one needs to* *acquire a suitable kind of mind which systematically prefers logical reasoning to a blind memorization and to the application of routines.*

Many students, although aware of these deficiencies, do not know how to remedy them and gradually set for minimum results in mathematics by doing many math exercises and appealing to the “blind memorization and application of routines.” Several teachers, at every level of the school system, notice this behavior in their students, but the opportunities of carrying out a work explicitly aimed at cultivating this mentality are rare. Students are usually expected to learn the method by seriously studying the subject matter and looking at their teacher’s good example of “method in action.” As a result most college students do not have serious difficulties in math exams, while many of their colleagues are subject to a sort of “natural selection.”

It is false that all the high school graduates possess the skills necessary for attending college math courses, while it is true that several students who drop out because of their failure in math could have been rescued by a specific and timely work.

This book aims at cultivating a “mathematical mind” for the prospective college students. We propose this work with the ambition to set a meeting between two relatives who rarely speak to each other: the “Mathematics of Beauty” (which shows up in popular books, conferences, exhibitions, movies...) and the “Mathematics of Toil” (which a lot of students know pretty well).

Toil can be partially overcome by acquiring an appropriate method of work. Even better if this happens thanks to the use of nice and interesting examples. In the end beauty will not be found only in the admiration of other people’s achievements, but mostly in gaining a way of thinking which enriches the potential of our mind.

This book can be used during the last two years of high school or at the beginning of college. It has been written both to offer a tool for personal use and for preparing ad hoc courses.

Content

The first part of this book provides the necessary precision of language: what is the meaning and the use of the expressions “for all”, “there exists”, “implies”, “is false”,...; then what is a proof by contradiction; how to work with indices; what is a summation symbol; how to use induction. This part is therefore devoted to the “mathematical mind” in its logical-syntactical aspects which are directly involved in college mathematics.

The second part aims to be a sort of "gymnastics" to get accustomed with the specific difficulties of a college math textbook. For instance it shows how to study a definition, first by looking for relevant examples, then by studying it in its general meaning and checking or disproving several implications. Then we discuss how to study a proof (or any other mathematical argument), first by visualizing it with the aid of suitable examples, then by splitting the reasoning into different steps and reconstructing it with no help but a "blank sheet," cleaning the proof from any imperfections and checking the student's understanding by trying to apply the same kind of reasoning to other problems. This is always done by means of several exercises that are chosen not because of their subject (algebra, geometry, arithmetic, etc.) but for the task we have just described. These exercises are demanding, sometimes difficult, but almost always with no prerequisites. We hope the students will find them pleasant and interesting: a fair challenge for those who have already worked out the first part of the book.

After the second part the work proposed in this book is somewhat finished. The reader who so far has worked seriously should be ready to study a "standard" university math textbook. However we have added a third part, shorter than the previous ones, in order to give the curious reader the opportunity of trying something even closer to university mathematics and meet a few mathematical topics. We put it another way: after all this training, let's play the game. We believe that in each chapter it is possible to find a point of view and a whole set of original ideas which the student will probably meet in her/his future studies.

The three parts of the book have a modular structure, so that the student can benefit from each of them at different levels of depth, according to her/his goals: the kind of courses he/she will attend, the time available, her/his interest in mathematics. See the "Instructions for use" below.

The Reader's Job and the Story of Our Work

This book must be read with an active mind, pen and paper always at hand: we are asking the reader a hard, continuous, and essentially individual work (but also, in our experience, an exciting and amusing one). Talking about it with her/his math teacher would be a good idea. We have written "in our experience" because this book has a long history: back to the late 1970s, groups of math students, helped and encouraged by a few professors, particularly by C. F. Manara (Università di Milano), used to give preparatory math courses for freshmen. These courses were based on an idea close to the one of this book and later developed and shaped with the help of many friends, to whom this book is dedicated. Hundreds of students took these informal courses, enriching the experience of two authors who, in 1985, wrote a book [1] that has been the genesis of the present text.

In 2009 we decided to rethink these ideas and propose them again, reshaped by our university teaching experience. This led us to the publication of another book [2] that we have now decided to translate into English, in a revised form, and propose to the international readers. We hope that this work will be helpful and stimulating for many other people, today as in the past, with the idea of joining the study (and its toil) with the enrichment that mathematics, as every field of knowledge does, offers to those who can listen and wish to do so.

We wish to thank Casa Editrice Zanichelli who generously released the rights to this English version. In particular we wish to thank Isabella Nenci for her faith in this work.

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Milano, Italy
December 2017

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Giancarlo Travaglini

Instructions for Use

The level of mathematical preparation one wants to obtain may depend on the depth of the mathematics he/she needs/wishes to study. For this reason we propose three different levels of use of this book.

Level A For those who will use mathematical tools and reasonings inside other scientific disciplines. This level is designed for people who will attend one or few undergraduate math courses.

Level B More demanding than the previous one: for those who must fully acquire a critical mathematical attitude towards subjects such as Computer Science, Engineering, Mathematics, Physics, Statistics, etc.

Level C *Really demanding*, for those who want to face more mathematical challenges.

The readers of level A do exercises A, the readers of Level B do A and B, and the readers of Level C do A, B, C. In Levels A and B we assume standard mathematical skills usually learned in high school. In Level C some combinatorics or basic calculus may be required.

To avoid discouragement we point out that the exercises of Level C are sometimes really difficult and a preparation at this level is by no means *necessary* to start the study of college math. Observe that Level C is not present in Part I of the book.

References

1. Andreini, M., & Travaglini, G. (1985). *Una introduzione allo studio della matematica nell'università*. Milano: Edizioni CUSL.

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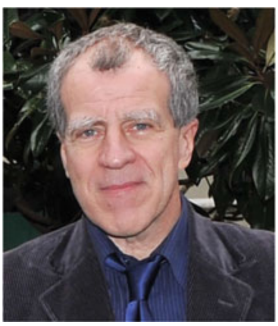
3. Condorcet, M. (1783). *Eloge de M. Euler*, pp. 37–68. Paris: Histoire de l'Académie Royale des Sciences.

About the Authors

Marco Bramanti is a full professor of Mathematical Analysis at the Politecnico di Milano. His field of research comprises partial differential equations, Hörmander vector fields, and real analysis. He is the author of several university textbooks.



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Part I

The Language of Mathematics

> Mathematicians create by acts of insight and intuition. Logic then sanctions the conquests of intuition. It is the hygiene that mathematics practices to keep its ideas healthy and strong [2].

> The strange thing about physics is that for the fundamental laws we still need mathematics (...). The more we investigate, the more laws we find, and the deeper we penetrate nature, the more this disease persists. Every one of our laws is a purely mathematical statement in rather complex and abstruse mathematics (...). You might say, "All right, then if there is no explanation of the law, at least tell me what the law is. Why not tell me in words instead of in symbols? Mathematics is just a language, and I want to be able to translate the language" (...). But I do not think it is possible, because mathematics is not just another language. Mathematics is a language plus reasoning; it is like a language plus logic. Mathematics is a tool for reasoning. It is in fact a big collection of the results of some person's careful thought and reasoning. By mathematics it is possible to connect one statement to another [1].

A curious student approaching or deepening a new discipline wishes to go immediately "at the heart" of that subject. Hence going through a long preliminary part related to the *language* could be frustrating, but this may be due to a prejudice. What is, really, the language? Usually, the language is seen as a tool to communicate a certain content, so that "language" and "content" are thought as independent terms: what you say in English could be translated into French (the language changes, the content remains), what you say in the technical language of a specific field can be explained in popular language, and so on. *But is this really true?* Can Shakespeare be *faithfully* translated into Italian or Dante into English? Can the Relativity Theory be explained in the everyday language, without technical terms? Can we explain mathematics "in a nutshell," without using mathematical symbols and terminology? Is the content really independent of the language we use to communicate it? Or does the content require/prefer a proper language? And does the language influence the content? The above passage by Feynman suggests that in mathematics the language has an essential relation with the content, because the mathematical language is tailored for reasoning, and mathematics, after all, *is* a way of reasoning.

In Part I we will learn, first of all, how to use correctly a few terms and phrases which are typical of the mathematical language. We will learn, as Kline writes, "the hygiene that mathematics practices to keep its ideas healthy and strong." We will also become familiar with some common objects of mathematics (sets, numbers, indices, etc.) and with a few typical forms of reasoning (proof of an implication, counterexample, proof by contradiction, proof by induction, etc.).

We ask the reader to try the following introductory test, which presents a variety of problems related to the topics dealt with in this first part.

Summary and Overview of Part I

In the first chapter we will reflect upon some of the basic logical terms that are commonly used in mathematics: the quantifiers and their terminology, the disjunction "or," the conjunction "and." In Chap. 2 we will introduce the *language of sets* and we will discuss the words "all" and "only." In Chap. 3 we will talk about the concepts of *proposition*, *property*, *variable*. We will devote Chap. 4 to *implications*, which will lead us to discuss the meaning of "if…then" and "any," and we will devote Chap. 5 to the *negation of a proposition*, an operation which corresponds to the basic word "not." This will complete our overview of *logical connectives and quantifiers*. This first portion of Part I is related to the logical language in a broad sense. The last three chapters of Part I will focus on topics with a stronger mathematical content, such as indices, formulas, numerical variables, etc.

References

1. Feynman, R. (1967). *The character of the physical law*. Cambridge, MA: The MIT Press.

2. Kline, M. (1964). *Mathematics in the western culture*. New York, NY: Oxford University Press.

Chapter 1

An Introductory Test (Level A)

This test deals with mathematical language, syntactic aspects of proofs, and the use of indices in set operations and numerical computations. According to her/his result, the student will be able to decide whether to study Part I or skip it. Solutions are in the subsequent section.

Exercises

Exercise 1 *Assume that in a certain family, there are two brothers and two sisters having pairwise different heights. The sentence*

"The brothers are taller than the sisters"

can have different meanings, for instance, the following ones:

(a) *The tallest brother is taller than the tallest sister.*
(b) *Every brother is taller than every sister.*
(c) *Every sister is shorter than some brother.*
(d) *The average of the brothers' heights is greater than the average of the heights of the sisters.*

Establish the existing implications between the previous statements. That is, answer the following questions:

$$a \underset{?}{\Longrightarrow} b\,,\quad a \underset{?}{\Longrightarrow} c\,,\quad a \underset{?}{\Longrightarrow} d\,,\quad b \underset{?}{\Longrightarrow} a\,,\quad b \underset{?}{\Longrightarrow} c\,,\quad b \underset{?}{\Longrightarrow} d\,,$$
$$c \underset{?}{\Longrightarrow} a\,,\quad c \underset{?}{\Longrightarrow} b\,,\quad c \underset{?}{\Longrightarrow} d\,,\quad d \underset{?}{\Longrightarrow} a\,,\quad d \underset{?}{\Longrightarrow} b\,,\quad d \underset{?}{\Longrightarrow} c$$

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(for instance, $a \Longrightarrow b$ means that if (a) is true, then necessarily (b) is true).

Exercise 2 *Given a quadrilateral Q, determine the mutual implications among the following statements:*

(a) *Q has an obtuse angle.*
(b) *Q has three acute angles.*
(c) *Q has no right angle.*

Exercise 3 *Let T be a triangle. Which of the following conditions are necessary in order for T to be isosceles? Which are sufficient?*

(a) *T is equilateral.*
(b) *T has two equal angles.*
(c) *T is right-angled.*
(d) *T has two equal angles of amplitude less than $60°$.*
(e) *There exist two sides of T having lengths of integer quotient.*

Exercise 4 *He says to her:* I am handsome and rich. *She replies:* This is not true. *What does it mean? (we identify "not handsome" with "ugly" and "not rich" with "poor"). Find the correct answer.*

(a) *He is ugly and poor.*
(b) *He is ugly or poor, but not both.*
(c) *He is ugly, or poor, or both.*

Exercise 5 *Given a proposition p, let us recall that the negation of p is a proposition that is true when p is false and is false if p is true. Given the proposition*

$$p: \quad \textit{All men have a tail},$$

discuss the validity of the following tentative negations of p.

(a) *Not all men have a tail.*
(b) *No man has a tail.*
(c) *There exists a man who does not have a tail.*
(d) *There exists a man who does not have a long tail.*

Exercise 6 *Let Q be a quadrilateral. The following three statements:*

(a) *Q has three equal sides*
(b) *Q has two unequal sides*
(c) *Q has three equal angles*

do not imply each other in any way, that is

$$a \not\Rightarrow b\,, \quad a \not\Rightarrow c\,, \quad b \not\Rightarrow a\,, \quad b \not\Rightarrow c\,, \quad c \not\Rightarrow a\,, \quad c \not\Rightarrow b\,.$$

To prove the falsity of each implication, several counterexamples are proposed below, not all of them being correct. Identify the correct ones.

$a \nRightarrow b$ *proposed counterexamples:*

a square,

a rhombus,

an isosceles trapezoid with unequal bases, such that the smallest basis is as long as the oblique sides.

$a \nRightarrow c$ *proposed counterexamples:*

a square,

a rectangle,

an isosceles trapezoid with unequal bases, such that the smallest basis is as long as the oblique sides.

$b \nRightarrow a$ *proposed counterexamples:*

a trapezoid which is not a rhombus,

a rectangle which is not a square,

a rhombus.

$b \nRightarrow c$ *proposed counterexamples:*

a right-angled trapezoid which is not a rectangle,

a rectangle,

an isosceles trapezoid with unequal bases, such that the smallest basis is as long as the oblique sides.

$c \nRightarrow a$ *proposed counterexamples:*

a rectangle which is not a square,

a rhombus which is not a square,

a right-angled trapezoid.

$c \nRightarrow b$ *proposed counterexamples:*

a square,

a rhombus,

a trapezoid which is not a rectangle.

Exercise 7 *Negate the following statements.*

(a) *There exists a point which does not belong to the straight line* p *and does not belong to the straight line* q.

(b) *For every real number* x*, we have* $f(x) \geq 5$.

(c) *There exists a circle which is tangent to the straight lines* p *and* q*, but not to the straight line* r.

(d) *The quadrilateral* Q *and the pentagon* P *have at least two common vertices.*

(e) *The equation* $(*)$ *has precisely three solutions.*

(f) p *is a prime odd number less than* 10.

Exercise 8 *Decide whether the following relations are correct.*

(*a*) $\bigcup_{n=1}^{3} A_n = A_1 \cup A_3$.

(*b*) $\left(\bigcup_{n=1}^{5} A_n\right) \bigcap \left(\bigcup_{n=3}^{8} A_n\right) \supseteq A_4$.

(*c*) $\left(\bigcup_{n=1}^{5} A_n\right)^{\complement} = \bigcap_{n=1}^{5} A_n^{\complement}$, *where* $A^{\complement}$ *is the complement of* A.

(*d*) $\bigcup_{n=1}^{5} \left(\bigcup_{k=1}^{n} A_k\right) = \bigcup_{n=1}^{5} A_n$.

(*e*) *If* $\bigcap_{n=1}^{5} A_n = \varnothing$, *then there exist* A_i *and* A_j *such that* $A_i \cap A_j = \varnothing$.

(*f*) *If* $\bigcup_{n=1}^{5} A_n = \bigcap_{n=1}^{5} A_n$, *then* $A_1 = A_2 = A_3 = A_4 = A_5$.

Exercise 9 *Let us denote by* $\sum$ *and* $\prod$, *respectively, the symbols of summation and product. Therefore, for instance,*

$$\sum_{n=-2}^{5} n^2 = 4+1+0+1+4+9+16+25 , \qquad \prod_{k=2}^{6} \frac{k}{k+1} = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} .$$

Compute

(*a*) $\sum_{n=1}^{N} (-1)^n$.

(*b*) $\sum_{n=0}^{10} a_k$.

(*c*) $\sum_{n=0}^{5} (100n+4)$.

(*d*) $\sum_{n=0}^{0} 2^{n+2}$.

(*e*) $\dfrac{\left(\prod_{n=1}^{N} a_n\right)\left(\prod_{n=2}^{N+1} a_n\right)}{\left(\prod_{n=0}^{N-1} a_n\right)\left(\prod_{n=3}^{N+2} a_n\right)}$.

(*f*) $\sum_{n=1}^{5} \sum_{j=1}^{n} j$

Exercise 10 *Let* $f(x) = x^2 - 2^{x+1}$. *Write*

(*a*) $f(x+1)$

(*b*) $(f(x))^2$

(*c*) $f(2x) + f(x^2)$

(*d*) $f(f(x))$

(*e*) $3f(2) - 2f(3)$

(*f*) $\sum_{n=1}^{3} f(-n)$

Solutions of the Test

Below are the solutions of the previous exercises, with the only aim to let the student decide whether to study Part I or skip it.

Solution of Exercise 1

$a \not\Rightarrow b$ we denote by $1B$ the first brother's height (in centimeters) and in a similar way we write $2B$, $1S$, $2S$. To prove that $a \not\Rightarrow b$, we consider the counterexample $1B = 180$, $2B = 170$, $1S = 178$, $2S = 176$, which satisfies (a) but not (b). Hence the truth of (a) does not imply the truth of (b) (we write $a \not\Rightarrow b$).

$a \Rightarrow c$ because if (a) holds, then all the sisters are shorter than the tallest brother; hence (c) holds.

$a \not\Rightarrow d$ see the counterexample for $a \not\Rightarrow b$.

$b \Rightarrow a$ if every brother is taller than every sister, then, in particular, the tallest brother is taller than the tallest sister.

$b \Rightarrow c$ because we have just proved that $b \Longrightarrow a$ and that $a \Longrightarrow c$.

$b \Rightarrow d$ because the average height of the brothers is larger than the height of the shorter brother, which, by (b) is larger than the height of the tallest sister, which, in turn, is larger than the average height of the sisters.

$c \Rightarrow a$ because if (c) holds, then the tallest sister is shorter than some brother; hence she is shorter than the tallest brother (therefore (a) and (c) are equivalent ; we write $a \Longleftrightarrow c$).

$c \not\Rightarrow b$ otherwise we would have $a \Rightarrow c \Rightarrow b$ (false).

$c \not\Rightarrow d$ as above.

$d \not\Rightarrow a$ let us consider the counterexample $1B = 180$, $2B = 176$, $1S = 182$, $2S = 168$, satisfying (d) but not (a).

$d \not\Rightarrow b$ otherwise we would have $d \Rightarrow b \Rightarrow a$ (false).

$d \not\Rightarrow c$ as above.

Solution of Exercise 2

$a \not\Rightarrow b$ counterexample: a rhombus which is not a square.

$a \not\Rightarrow c$ counterexample: a right-angled trapezoid which is not a rectangle.

$b \Rightarrow a$ because the sum of the interior angles in a quadrilateral is $360°$.

$b \Rightarrow c$ by the previous argument.

$c \Rightarrow a$ the angles cannot be all acute.

$c \not\Rightarrow b$ counterexample: a rhombus which is not a square.

Solution of Exercise 3

(a) is sufficient, but not necessary.
(b) is necessary and sufficient.
(c) is neither necessary nor sufficient.
(d) is sufficient but not necessary.
(e) is necessary but not sufficient.

Solution of Exercise 4

The correct answer is (c). We can represent the problem by means of sets. If B is the set of handsome men and R is the set of rich men, he claims to be in $B \cap R$, and saying that this is false (i.e., that he, as an element, does not belong to $B \cap R$) amounts to saying that he can belong to $B \setminus R$ (i.e., he can be handsome but poor) or belong to $R \setminus B$ (i.e., he can be rich but ugly) or belong to the complement of $B \cup R$ (i.e., he can be ugly and poor).

Solution of Exercise 5

(a) is formally a correct negation of (p). Anyway, it is scarcely useful, because a sentence beginning with "not" is usually not very clear and, in particular, is not well-suited to draw consequences from it, which instead is exactly what we need to do in the course of a proof by contradiction.

(b) is not a correct negation of (p). Actually if (p) is true, then (b) is false, but if (p) is false, then (b) is not necessarily true (for instance, some men do have a tail and some do not).

(c) is a correct negation of (p). Actually if (p) is true, then there does not exist a man without a tail, that is, (c) is false. If instead (p) is false, then it is not true that all men have a tail; hence there exists a man without a tail, that is, (c) is true.

(d) is not a correct negation of (p). Actually if (p) is false, then (d) is true, because a man without a tail is also a man without a long tail. Nevertheless, if (p) is true, then (d) is not necessarily false (some tail could be short).

Solution of Exercise 6

Let us point out, for each false implication, the correct counterexamples.

$a \not\Rightarrow b$

a square,

a rhombus.

$a \not\Rightarrow c$

an isosceles trapezoid with unequal bases, such that the smallest basis is as long as the oblique sides.

$b \not\Rightarrow a$

a rectangle which is not a square.

$b \not\Rightarrow c$

a right-angled trapezoid which is not a rectangle,

an isosceles trapezoid with unequal bases, such that the smallest basis is as long as the oblique sides.

$c \not\Rightarrow a$

a rectangle which is not a square,

$c \not\Rightarrow b$

a square.

Solution of Exercise 7

(a) All the points belong to at least one of the two straight lines p and q.

(b) There exists a real number x_0 such that $f(x_0) < 5$.

(c) If a circle is tangent to the straight lines p and q, then it is also tangent to the straight line r.

(d) The quadrilateral Q and the pentagon P have at most one common vertex.

(e) The equation $(*)$ either has at most two solutions, or it has at least four.

(f) p satisfies at least one of the three following conditions: it is not prime, is even, and is greater than or equal to 10.

Solution of Exercise 8

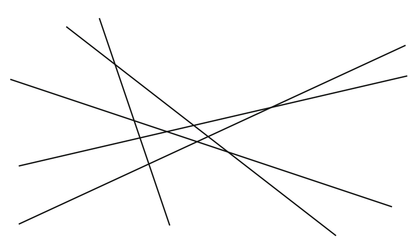
(a) False.

(b) True.

(c) True.

(d) True.

(e) False. As a counterexample we can consider the five straight lines as in the following picture (seeing the lines as sets of points in the plane).



The intersection of the five straight lines is empty, but each of them meets all the others.

(f) True.

Solution of Exercise 9

(a) $\begin{cases} 0 & \text{if } N \text{ is even} \\ -1 & \text{if } N \text{ is odd} \end{cases}$.

(b) $11a_k$.

(c) 1524 .

(d) 4 .

(e) $\frac{a_N a_2}{a_0 a_{N+2}}$.

(f) 35 .

Solution of Exercise 10

(a) $(x+1)^2 - 2^{x+2}$.

(b) $\left(x^2 - 2^{x+1}\right)^2 = x^4 - x^2 2^{x+2} + 2^{2x+2}$.

(c) $(2x)^2 - 2^{2x+1} + x^4 - 2^{x^2+1} = 4x^2 - 2^{2x+1} + x^4 - 2^{x^2+1}$.

(d) $\{f(x)\}^2 - 2^{f(x)+1} = \left(x^2 - 2^{x+1}\right)^2 - 2^{x^2 - 2^{x+1} + 1}$
$= x^4 - x^2 2^{x+2} + 2^{2x+2} - 2^{x^2 - 2^{x+1} + 1}$.

(e) $3\left(2^2 - 2^3\right) - 2\left(3^2 - 2^4\right) = 2$.

(f) $49/4$.

Chapter 2

Quantifying (Level A)

> Mathematics as a science commenced when first someone, probably a Greek, proved propositions about *any* things or about *some* things, without specification of definite particular things [1].

In the everyday language, our sentences often deal with a single, *individual* object or person (my car, that bag, your friend Paul, etc.). A mathematical statement instead very often refers to some *class* of objects (natural numbers, regular polygons, etc.) or to some (unspecified) object inside some class ("Let n be a natural number, etc."). Sometimes we state the existence of at least one object with a specific property, sometimes we state that every object of that kind possesses that property (two quite different statements!). In general it is very important to *properly quantify* the objects we are talking about. This may not be obvious, as the following example shows.

Example 2.0.1 (`How Many?`) *We ask ourselves what is the meaning of the statement:*

"A computer has a problem"

*Perhaps it means that "*A specific *computer has a problem" or that "*At least one *of the computers we are considering has a problem." Moreover, by saying "a problem," do we mean "a single problem," "at least one problem," or what else?*

Let us reflect upon the above example. Every option that we have formulated is realistic, and the reader can easily describe several contexts where the sentence "a computer has a problem" takes different meanings. This is due to the fact that in everyday language, the indefinite article a/an can assume different meanings, according to the context. In mathematics the meaning of the basic words should not depend "on the context" and should be defined

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specifically. Therefore it is necessary to reformulate the above sentence in a way that makes clear whether we are talking about a specific computer or some computer (and if we are talking about one and only one problem, or at least one problem, or at most one problem).

The mathematical language does not need to be formal; it must be univocal.

Hence the language of mathematics needs some conventions. For instance,

"there exists a/one" means "there exists *at least* a/one

(i.e., one or more than one). If we want to say "exactly one," we have to make it explicit. Analogously, "there exist two" means "there exist at least two" and so on.

We can *quantify* the computers we are talking about, for instance, choosing between the expressions:

- "There exists one and only one computer such that…"
- "There exists a computer such that…" (i.e., "at least one computer")

We shall also consider the expression:

- "For every computer we have that…"

The phrases "there exists one and only one," "there exists," and "for every" are called *quantifiers*, and they are associated with some symbols that the reader has to familiarize with:

Phrase	Symbol
There exists one and only one	$\exists!$
There exists	$\exists$
For every, for all	$\forall$

The first two quantifiers are often completed by the phrase *such that*, like in the sentence "there exists a computer x such that x has a problem":

Phrase	Symbol
Such that	$:$

We can quantify the expression *a problem*, for instance, choosing among:

- "...has exactly one problem"
- "...has a problem" (i.e., "at least one problem")

We shall also consider the expression:

- "...has at most one problem".

We have said that in mathematics the phrase "there exists an x such that..." must always be intended as "there exists *at least* an x such that...." An analogous convention applies also to other phrases. Again, this can be occasionally in contrast with everyday language:

Example 2.0.2 *(To Have One) Let us consider the following sentences:*

(a) *I will take the bus: I should have one ticket in my pocket.*

(b) *I cannot buy this book now: I have 1 dollar in my pocket.*

The phrase to have one (ticket/dollar) is used in the two sentences with two different meanings:

In (a), *the person does not (implicitly) exclude to have more than one ticket: What we understand is that he/she can take the bus because he/she has* at least one *ticket in her/his pocket. In* (b), *instead, the person is complaining about having* only 1 *dollar, as this does not allow her/him to buy the book; we understand that he/she has exactly 1 dollar in her/his pocket, in particular, that he/she does not have* more than 1 *dollar in her/his pocket.*

Again, in the everyday language, the same phrase can take different meanings in different contexts. On the contrary:

> In mathematics a sentence like
> "I have 1 dollar in my pocket" or "In my pocket there is 1 dollar"
> is always meant as
> "In my pocket there is *at least* 1 dollar."

In order to express statements different from this, one can use phrases like:

"In my pocket there is *at most* 1 dollar," that is, there is one, or less, or nothing,

or:

"In my pocket there is *exactly* 1 dollar"

or also (with the same meaning)

"In my pocket there is *1 and only 1* dollar," if one wants to emphasize that he possesses neither more nor less than 1 dollar.

Example 2.0.3 *(`Some`) Let us consider the sentences:*

(*a*) *Some boys really love pizza*

(*b*) *Some student has left a bag on the table*

(*c*) *The following theorem will show that some solution of the equation (*) actually exists.*

Again, we are interested in quantifying the objects that these sentences deal with. In (*a*) *the plural says that the boys who love pizza are surely more than one. In* (*b*) *we are talking about one student who has left her/his bag on the table. We do not know this student, but since there is one (and only one) bag on the table, one (and only one) student must have left it there. Finally* (*c*) *is a typical sentence that we could read in a math book. Here "some" means "at least one."*

In mathematics "Some x has this property" always means "There exists at least one x having this property."

Example 2.0.4 *(`The Indefinite Article as an Implicit Quantifier`) Let us consider the sentences:*

(*a*) *"A multiple of* 3 *is a multiple of* 6*."*

(*b*) *"A turtle born before 1980 is still alive."*

What is the meaning of (*a*)*? "Every multiple of* 3 *is also a multiple of* 6*" (false, nine is a multiple of three but not of six) or "There exists a multiple of* 3 *that is also a multiple of* 6*" (true, for instance, six)? Probably in this sentence "A multiple of* 3*" means "A generic multiple of* 3*," that is, "Every multiple of* 3*," hence* (*a*) *is false.*

The sentence (*b*) *instead means "Some turtle born before 1980 is still alive," that is, "There is a turtle born before 1980 which is still alive" (true).*