Sergio Macías

Topics on Continua

Second Edition

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To Elsa

León Felipe escribió un
tributo, no al héroe de la
historia, sino a su fiel
caballo Rocinante,
quien lo llevó en su lomo
por las tierras de España.

El héroe es, por supuesto,
Don Quijote de la Mancha:

"El Caballero de la Triste Figura"

¡Yo quería ese nombre!
pero me lo ganaron,
llegué a este mundo casi
trescientos cincuenta
años tarde...

Ya sólo me queda ser:

"El Caballero de la Triste Locura..."

S. M.

Preface to the Second Edition

After 12 years of the publication of *Topics on Continua* many things have happened. As it is well known, it is impossible to include everything. This Second Edition contains two new chapters which appear for the first time in a book, namely: *n-fold Hyperspace Suspensions* and *Induced Maps on n-fold Hyperspaces*. We include recent developments.

The first two chapters have very few modifications. In the first one, we prepare the way to prove the monotone-light factorization theorem, which appears later in chapter eight. We also add the notions of freely decomposable continuum and more concepts of aposyndesis. We include the notions of arc-smoothness of continua and arcwise decomposable continua too. For the second chapter we have not included much because of the two books on inverse limits and generalized inverse limits that appeared in 2012, namely: the book by Professors W. T. Ingram and William S. Mahavier *Inverse Limits: From Chaos to Continua*, Developments in Mathematics, Vol. 25, Springer, 2012 and the book by Professor W. T. Ingram *An Introduction to Inverse Limits with Set-valued Functions*, Springer Briefs in Mathematics, 2012. If the reader is interested in such topics, please refer to the mentioned books. We add the notions of confluent and weakly confluent maps to show that the bonding maps of an inverse limit are confluent if and only if the projection maps are confluent and the fact that each surjective map onto a chainable continuum is weakly confluent. By using inverse limits, it can be shown that the Cantor set is a topological group.

Chapter 3 has four new sections, namely: *Idempotency of* $\mathcal{T}$, *Three Decomposition Theorems*, *Examples*, and $\mathcal{T}$*-closed sets*. Throughout the chapter, we present characterizations of locally connected continua using the distinct forms of aposyndesis added in the first chapter. A sufficient condition for the idempotency on closed sets is given we also present an example showing that the condition is not necessary. We present a study of the relation between arc-smoothness and strict point $\mathcal{T}$-asymmetry. In particular, we show that Question 9.2.9 has a negative answer. We include results about the idempotency of $\mathcal{T}$ on products, cones, and suspensions. In particular, we prove that the first part of Question 9.2.3 has always a negative answer. We present three decomposition theorems using $\mathcal{T}$. In the strongest of the theorems, we obtain a continuous decomposition of the continuum with a locally connected quotient space and many of the elements of the decomposition are indecomposable continua. We present several classes of continua for which $\mathcal{T}$ is continuous and we study the family of $\mathcal{T}$-closed sets.

Chapter 4 remains essentially the same; we add three more consequences of the Property of Effros. The same happens with Chap. 5 where we include a few characterizations of the continuity of the set function $\mathcal{T}$ for homogeneous continua.

Chapter 6 has two new sections, namely: *Z-sets* and *Strong Size Maps*. Throughout the chapter, we include several bounds for the dimension of the n-fold hyperspace of certain classes of continua. We show that the n-fold hyperspaces are zero-dimensional aposyndetic. We give the correct statement and proof of Theorem 6.5.14. We give basic properties of Z-sets and sufficient conditions in order to show that the n-fold symmetric product of a continuum is a Z-set of the n-hyperspace of such continuum. We add several results that indicate when the n-fold symmetric product is a strong deformation retract of the m-fold hyperspace or of the hyperspace of closed sets. Also, we include properties of the continuum and the n-fold symmetric product when this is a retract of the m-fold hyperspace. We add a characterization of the graphs for which their n-fold hyperspace is a Cantor manifold. We also characterize the class of continua for which its n-fold hyperspace is a k-cell. We include results about suspensions and products related to the ones already given for cones. We end the chapter with a study of strong size maps, which are a nice generalization of Whitney maps to n-fold hyperspaces.

Chapter 7 is new. It is about hyperspace suspensions. We present most of what is known about n-fold hyperspace suspensions. We prove several properties of these spaces. We give sufficient conditions in order to obtain that n-fold hyperspace suspensions are contractible. We show that they are zero-dimensional aposyndetic. We study these hyperspaces when the continuum is locally connected. In particular, we give a sufficient condition to obtain that the n-fold hyperspace suspension of a locally connected continuum is the Hilbert cube. We characterize indecomposable continua by showing that their n-fold hyperspace suspensions are arcwise disconnected by removing two points. We present a description of the arc components of arcwise disconnected n-fold hyperspace suspensions when those two points are removed. We study properties of the n-fold hyperspace suspensions when they are homeomorphic to cones, suspensions, or products of continua. We present several results about the fixed point property of these hyperspaces. We study absolute n-fold hyperspace suspensions. We end this chapter by proving that hereditarily indecomposable continua have unique n-fold hyperspace suspensions.

Chapter 8 is also new. It is about induced maps between n-fold hyperspaces; these include hyperspace suspensions. We start with the definition of all the classes of maps that we study. Then we continue with general properties about the induced maps and present results about homeomorphisms, atomic maps, ε-maps, refinable maps, and almost monotone maps. We continue with results about confluent, monotone, open, light, and freely decomposable maps.

Chapter 9 (former Chap. 7), the last chapter, which is about questions, has two new sections (one for each of the new chapters), with questions on n-fold hyperspace suspensions and induced maps between n-fold hyperspaces.

I thank Javier Camargo for letting me include part of his dissertation in the second edition of the book.

I thank Ms. Elsa Arroyo for preparing all the pictures of the second edition of the book.

I thank the people at Springer, especially Professor Dr. Jan Holland, Ms. Anne Comment, Mr. Tilton Edward Stanley, Ms. Uma Periasamy and Ms. Kathleen Moriarty, for all their help.

Ciudad de México, México Sergio Macías

Preface to the First Edition

My aim is to present four of my favorite topics in continuum theory: inverse limits, Professor Jones's set function $\mathcal{T}$, homogeneous continua, and n-fold hyperspaces.

Most topics treated in this book are not covered in Professor Sam B. Nadler Jr.'s book: *Continuum Theory: An Introduction*, Monographs and Textbooks in Pure and Applied Math., Vol. 158, Marcel Dekker, New York, Basel, Hong Kong, 1992.

The reader is assumed to have taken a one-year course on general topology.

The book has seven chapters. In Chap. 1, we include the basic background to be used in the rest of the book. The experienced readers may prefer to skip this chapter and jump right to the study of their favorite subject. This can be done without any problem. The topics of Chap. 1 are essentially independent of one another and can be read at any time.

Chapter 2 is for the most part about inverse limits of continua. We present the basic results on inverse limits. Some theorems are stated without proof in Professor W. Tom Ingram's book: *Inverse Limits*, Aportaciones Matemáticas, Textos # 18, Sociedad Matemática Mexicana, 2000. We show that the operation of taking inverse limits commutes with the operations of taking finite products, cones, and hyperspaces. We also include some applications of inverse limits.

In Chap. 3 we discuss Professor F. Burton Jones's set function $\mathcal{T}$. After giving the basic properties of this function, we present properties of continua in terms of $\mathcal{T}$, such as connectedness im kleinen, local connectedness, and semi-local connectedness. We also study continua for which the set function $\mathcal{T}$ is continuous. In the last section we present some applications of $\mathcal{T}$.

In Chap. 4 we start our study of homogeneous continua. We present a topological proof of a Theorem of Professor E. G. Effros given by F. D. Ancel. We include a brief introduction to topological groups and group actions.

Chapter 5 contains our main study of homogeneous continua. We present two Decomposition Theorems of such continua, whose proofs are applications of Professor Jones's set function $\mathcal{T}$ and Professor Effros's Theorem. These theorems have narrowed the study of homogeneous continua in such a way that they may hopefully be eventually classified. We also give examples of nontrivial homogeneous continua and their covering spaces.

In Chap. 6 we present most of what is known about n-fold hyperspaces. This chapter is slightly different from the other chapters because the proofs of many of the theorems are based on results in the literature that we do not prove; however, we give references to the appropriate places where proofs can be found. This chapter is a complement of the two existing books—Sam B. Nadler, Jr., *Hyperspaces of Sets: A Text with Research Questions*, Monographs and Textbooks in Pure and Applied Math., Vol. 49, Marcel Dekker, New York, Basel, 1978[1] and Alejandro Illanes and Sam B. Nadler, Jr., *Hyperspaces: Fundamentals and Recent Advances*, Monographs and Textbooks in Pure and Applied Math., Vol. 216, Marcel Dekker, New York, Basel, 1999, in which a thorough study of hyperspaces is done.

In Chap. 6, we also prove general properties of n-fold hyperspaces. In particular, we show that n-fold hyperspaces are unicoherent and finitely aposyndetic. We study the arcwise accessibility of points of the n-fold symmetric products from their complement in n-fold hyperspaces. We give a treatment of the points that arcwise disconnect n-fold hyperspaces of indecomposable continua. Then we study continua for which the operation of taking n-fold hyperspaces is continuous ($\mathcal{C}_n^*$-smoothness). We also investigate continua for which there exist retractions between their various hyperspaces. Next, we present some results about the n-fold hyperspaces of graphs. We end Chap. 6 by studying the relation between n-fold hyperspaces and cones over continua.

We end the book with a chapter (Chap. 7) containing open questions on each of the subjects presented in the book.

We include figures to illustrate definitions and aspects of proofs.

The book originates from two sources—class notes I took from the course on continuum theory given by Professor James T. Rogers, Jr. at Tulane University in the Fall Semester of 1988 and the one-year courses on continuum theory I have taught in the graduate program of mathematics at the Facultad de Ciencias of the Universidad Nacional Autónoma de México, since the spring of 1993. I thank all the students who have taken such courses.

I thank María Antonieta Molina and Juan Carlos Macías for letting me include part of their thesis in the book. Ms. Molina's thesis was based on two talks on the set function $\mathcal{T}$ given by Professor David P. Bellamy in the *IV Research Workshop on Topology*, celebrated in Oaxaca City, Oaxaca, México, November 14 through 16, 1996.

I thank Professors Sam B. Nadler, Jr. and James T. Rogers, Jr. for reading parts of the manuscript and making valuable suggestions. I also thank Ms. Gabriela Sanginés and Mr. Leonardo Espinosa for answering my questions about LaTeX, while I was typing this book.

I thank Professor Charles Hagopian and Marvi Hagopian for letting me use their living room to work on the book during my visit to California State University, Sacramento.

[1] This book has been reprinted in: Aportaciones Matemáticas de la Sociedad Matemática Mexicana, Serie Textos # 33, 2006.

I thank the Instituto de Matemáticas of the Universidad Nacional Autónoma de México and the Mathematics Department of West Virginia University, for the use of resources during the preparation of the book.

Finally, I thank the people at Marcel Dekker, Inc., especially Ms. Maria Allegra and Mr. Kevin Sequeira, who were always patient and helpful.

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Contents

Chapter 1
Preliminaries

We gather some of the results of topology of metric spaces which will be useful for the rest of the book. We assume the reader is familiar with the notion of metric space and its elementary properties. We present the proofs of most of the results; we give an appropriate reference otherwise.

The topics reviewed in this chapter are: product topology, continuous decompositions, homotopy, fundamental group, geometric complexes, polyhedra, complete metric spaces, compacta, continua and hyperspaces.

1.1 Product Topology

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the positive integers, integers, rational numbers, real numbers and complex numbers, respectively. The material of this section is taken from [10, 13, 17, 18, 25, 28].

The word *map* means a continuous function. A *compactum* is a compact metric space.

1.1.1 Definition Given a sequence, $\{X_n\}_{n=1}^{\infty}$, of nonempty sets, we define its *Cartesian product*, denoted by $\prod_{n=1}^{\infty} X_n$, as the set:

$$\prod_{n=1}^{\infty} X_n = \{(x_n)_{n=1}^{\infty} \mid x_n \in X_n \text{ for each } n \in \mathbb{N}\}.$$

For each $m \in \mathbb{N}$, there exists a function

$$\pi_m : \prod_{n=1}^{\infty} X_n \twoheadrightarrow X_m$$

defined by $\pi_m((x_n)_{n=1}^{\infty}) = x_m$. This function π_m is called the *mth-projection map*.

S. Macías, *Topics on Continua*, https://doi.org/10.1007/978-3-319-90902-8_1

1.1.2 Remark Given a metric space (X, d'), there exists a metric, d, which generates the same topology as d', with the property that $d(x, x') \leq 1$ for each pair of points x and x' of X. This metric d is called *bounded metric*. An example of such metric is given by $d(x, x') = \min\{1, d'(x, x')\}$.

1.1.3 Notation Given a metric space (X, d) and a subset A of X, $Cl_X(A)$, $Int_X(A)$ and $Bd_X(A)$ denote the closure, interior and boundary of A, respectively. We omit the subindex if there is no confusion. If ε is a positive real number, then the symbol $\mathcal{V}^d_\varepsilon(A)$ denotes the *open ball of radius ε about A*. If $A = \{x\}$, for some $x \in X$, we write $\mathcal{V}^d_\varepsilon(x)$ instead of $\mathcal{V}^d_\varepsilon(\{x\})$.

1.1.4 Definition If $\{(X_n, d_n)\}_{n=1}^\infty$ is a sequence of metric spaces, with bounded metrics, we define a metric ρ, for its Cartesian product as follows:

$$\rho((x_n)_{n=1}^\infty, (x'_n)_{n=1}^\infty) = \sum_{n=1}^\infty \frac{1}{2^n} d_n(x_n, x'_n).$$

1.1.5 Remark Since the metrics, d_n, in Definition 1.1.4 are bounded, ρ is well defined.

1.1.6 Lemma *If $\{(X_n, d_n)\}_{n=1}^\infty$ is a sequence of metric spaces, with bounded metrics, then ρ (Definition 1.1.4) is a metric and for each $m \in \mathbb{N}$, π_m is a continuous function.*

Proof The proof of the fact that ρ is, in fact, a metric is left to the reader.

Let $m \in \mathbb{N}$ be given. We show that π_m is continuous. Let $\varepsilon > 0$ and let $\delta = \frac{1}{2^m}\varepsilon$. If $(x_n)_{n=1}^\infty$ and $(x'_n)_{n=1}^\infty$ are two points of $\prod_{n=1}^\infty X_n$ such that $\rho((x_n)_{n=1}^\infty, (x'_n)_{n=1}^\infty) < \delta$, then, since $\frac{1}{2^m} d_m(x_m, x'_m) \leq \sum_{n=1}^\infty \frac{1}{2^n} d_n(x_n, x'_n)$, we have that $\frac{1}{2^m} d_m(x_m, x'_m) < \delta$. Hence,

$$d_m(x_m, x'_m) < 2^m\delta = \varepsilon.$$

Therefore, π_m is continuous.

Q.E.D.

1.1.7 Lemma *If $\{(X_n, d_n)\}_{n=1}^\infty$ is a sequence of metric spaces, with bounded metrics, then given $\varepsilon > 0$ and a point $(x_n)_{n=1}^\infty \in \prod_{n=1}^\infty X_n$, there exist $N \in \mathbb{N}$ and N positive real numbers, $\varepsilon_1, \ldots, \varepsilon_N$, such that $\bigcap_{j=1}^N \pi_j^{-1}(\mathcal{V}^{d_j}_{\varepsilon_j}(x_j)) \subset \mathcal{V}^\rho_\varepsilon((x_n)_{n=1}^\infty)$.*

Proof Let $N \in \mathbb{N}$ be such that $\sum_{n=N+1}^\infty \frac{1}{2^n} < \frac{\varepsilon}{2}$. For each $j \in \{1, \ldots, N\}$, let $\varepsilon_j = \frac{\varepsilon}{2N}$. We assert that $\bigcap_{j=1}^N \pi_j^{-1}\left(\mathcal{V}^{d_j}_{\varepsilon_j}(x_j)\right) \subset \mathcal{V}^\rho_\varepsilon((x_n)_{n=1}^\infty)$. To see this, let $(y_n)_{n=1}^\infty \in \bigcap_{j=1}^N \pi_j^{-1}\left(\mathcal{V}^{d_j}_{\varepsilon_j}(x_j)\right)$. We want to see that $\rho((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty) < \varepsilon$.

Note that

$$\rho((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n) =$$

$$\sum_{n=1}^{N} \frac{1}{2^n} d_n(x_n, y_n) + \sum_{n=N+1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n) <$$

$$\sum_{n=1}^{N} \frac{1}{2^n} \frac{1}{2^N} \varepsilon + \frac{1}{2}\varepsilon = \left(1 - \frac{1}{2^N}\right) \frac{1}{2^N}\varepsilon + \frac{1}{2}\varepsilon \leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

Q.E.D.

1.1.8 Lemma *If $\{(X_n, d_n)\}_{n=1}^{\infty}$ is a sequence of metric spaces, with bounded metrics, then given a finite number of positive real numbers $\varepsilon_1, \ldots, \varepsilon_k$ and a point $(x_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} X_n$, there exists $\varepsilon > 0$ such that $\mathcal{V}_{\varepsilon}^{\rho}((x_n)_{n=1}^{\infty}) \subset \bigcap_{j=1}^{k} \pi_j^{-1}(\mathcal{V}_{\varepsilon_j}^{d_j}(x_j))$.*

Proof Let $(x_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} X_n$, and let $U = \bigcap_{j=1}^{k} \pi_j^{-1}\left(\mathcal{V}_{\varepsilon_j}^{d_j}(x_j)\right)$. Take

$$\varepsilon = \min\left\{\frac{1}{2}\varepsilon_1, \ldots, \frac{1}{2^k}\varepsilon_k\right\}.$$

We show $\mathcal{V}_{\varepsilon}^{\rho}((x_n)_{n=1}^{\infty}) \subset U$. Let $(y_n)_{n=1}^{\infty} \in \mathcal{V}_{\varepsilon}^{\rho}((x_n)_{n=1}^{\infty})$. Then

$$\rho((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}) < \varepsilon, \text{ i.e., } \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n) < \varepsilon.$$

Hence, $\frac{1}{2^j} d_j(x_j, y_j) < \varepsilon \leq \frac{1}{2^j}\varepsilon_j$ for each $j \in \{1, \ldots, k\}$. Thus, if $j \in \{1, \ldots, k\}$, then $d_j(x_j, y_j) < \varepsilon_j$. Therefore, $\mathcal{V}_{\varepsilon}^{\rho}((x_n)_{n=1}^{\infty}) \subset U$.

Q.E.D.

1.1.9 Theorem *Let Z be a metric space. If $\{(X_n, d_n)\}_{n=1}^{\infty}$ is a sequence of metric spaces, then a function $f : Z \to \prod_{n=1}^{\infty} X_n$ is continuous if and only if $\pi_n \circ f$ is continuous for each $n \in \mathbb{N}$.*

Proof Clearly, if f is continuous, then $\pi_n \circ f$ is continuous for each $n \in \mathbb{N}$.

Suppose $\pi_n \circ f$ is continuous for each $n \in \mathbb{N}$. Let $\bigcap_{j=1}^{k} \pi_j^{-1}(U_j)$ be a basic open subset of $\prod_{n=1}^{\infty} X_n$. Since

$$f^{-1}\left(\bigcap_{j=1}^{k} \pi_j^{-1}(U_j)\right) = \bigcap_{j=1}^{k} f^{-1}(\pi_j^{-1}(U_j))$$
$$= \bigcap_{j=1}^{k} (\pi_j \circ f)^{-1}(U_j),$$

we have that $f^{-1}\left(\bigcap_{j=1}^{k} \pi_j^{-1}(U_j)\right)$ is open in Z. Hence, f is continuous.

Q.E.D.

1.1.10 Theorem *Let $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ be two countable collections of metric spaces. Suppose that for each $n \in \mathbb{N}$, there exists a map $f_n \colon X_n \to Y_n$. Then the function $\prod_{n=1}^{\infty} f_n \colon \prod_{n=1}^{\infty} X_n \to \prod_{n=1}^{\infty} Y_n$ given by $\prod_{n=1}^{\infty} f_n((x_n)_{n=1}^{\infty}) = (f_n(x_n))_{n=1}^{\infty}$ is continuous.*

Proof For each $m \in \mathbb{N}$, let $\pi_m \colon \prod_{n=1}^{\infty} X_n \twoheadrightarrow X_m$ and $\pi'_m \colon \prod_{n=1}^{\infty} Y_n \twoheadrightarrow Y_m$ be the projection maps.

Let $(x_n)_{n=1}^{\infty}$ be a point of $\prod_{n=1}^{\infty} X_n$, and let $m \in \mathbb{N}$. Then $\pi'_m \circ \prod_{n=1}^{\infty} f_n((x_n)_{n=1}^{\infty}) = \pi'_m\left((f_n(x_n))_{n=1}^{\infty}\right) = f_m(x_m) = f_m \circ \pi_m((x_n)_{n=1}^{\infty})$. Hence, by Theorem 1.1.9, $\prod_{n=1}^{\infty} f_n$ is continuous.

Q.E.D.

The following result is a particular case of *Tychonoff's Theorem,* which says that the Cartesian product of any family of compact topological spaces is compact. The proof of this theorem uses the *Axiom of Choice*. However, the case we show only uses the fact that compactness and sequential compactness are equivalent in metric spaces [17, Remark 3, p. 3].

1.1.11 Theorem *If $\{(X_n, d_n)\}_{n=1}^{\infty}$ is a sequence of compacta, then $\prod_{n=1}^{\infty} X_n$ is compact.*

Proof By Lemma 1.1.6, $\prod_{n=1}^{\infty} X_n$ is a metric space. We show that any sequence of points of $\prod_{n=1}^{\infty} X_n$ has a convergent subsequence.

Let $\{p^k\}_{k=1}^{\infty}$ be a sequence of points of $\prod_{n=1}^{\infty} X_n$, where $p^k = (p_n^k)_{n=1}^{\infty}$ for each $k \in \mathbb{N}$ (in this way, if we keep n fixed, $\{p_n^k\}_{k=1}^{\infty}$ is a sequence of points of X_n). Since (X_1, d_1) is sequentially compact, $\{p_1^k\}_{k=1}^{\infty}$ has a convergent subsequence $\{p_1^{k_j}\}_{j=1}^{\infty}$ converging to a point q_1 of X_1. Let us note that, implicitly, we have defined a subsequence $\{p^{k_j}\}_{j=1}^{\infty}$ of $\{p^k\}_{k=1}^{\infty}$.

Now, suppose, inductively, that for some $m \in \mathbb{N}$, we have defined a subsequence $\{p^{k_i}\}_{i=1}^{\infty}$ of $\{p^k\}_{k=1}^{\infty}$ such that $\{p_m^{k_i}\}_{i=1}^{\infty}$ converges to a point q_m of X_m. Since (X_{m+1}, d_{m+1}) is sequentially compact, $\{p_{m+1}^{k_i}\}_{i=1}^{\infty}$ has a convergent subsequence

$\{p_{m+1}^{k_{i_j}}\}_{j=1}^{\infty}$ such that it converges to a point q_{m+1} of X_{m+1}. Hence, by the Induction Principle, we have defined a sequence of subsequences of $\{p^k\}_{k=1}^{\infty}$ in such a way that each subsequence is a subsequence of the preceding one. Now, let $\Sigma = \{p^1, p^{k_2}, p^{k_{j_3}}, p^{k_{j_{i_4}}}, \ldots\}$. Clearly, Σ is a subsequence of $\{p^k\}_{k=1}^{\infty}$ which converges to the point $(q_n)_{n=1}^{\infty}$. Therefore, $\prod_{n=1}^{\infty} X_n$ is compact.

Q.E.D.

1.1.12 Definition Let $\mathcal{Q} = \prod_{n=1}^{\infty}[0,1]_n$, where $[0,1]_n = [0,1]$, for each $n \in \mathbb{N}$. Then $\mathcal{Q}$ is called the *Hilbert cube*.

1.1.13 Theorem *The Hilbert cube is a connected compactum.*

Proof By Lemma 1.1.6, $\mathcal{Q}$ is a metric space. By Theorem 1.1.11, $\mathcal{Q}$ is compact. By [17, Theorem 11, p. 137], $\mathcal{Q}$ is connected.

Q.E.D.

1.1.14 Definition Let $f: X \to Y$ be a map between metric spaces. We say that f is an *embedding* if f is a homeomorphism onto $f(X)$.

1.1.15 Definition A map $f: X \to Y$ between metric spaces is said to be *closed* provided that for each closed subset K of X, $f(K)$ is closed in Y.

The next theorem says that there exists a “copy” of every compactum inside the Hilbert cube.

1.1.16 Theorem *If X is a compactum, then X can be embedded in the Hilbert cube $\mathcal{Q}$.*

Proof Let d be the metric of X. Without loss of generality, we assume that $\operatorname{diam}(X) \leq 1$. Since X is a compactum, it contains a countable dense subset, $\{x_n\}_{n=1}^{\infty}$. Let $h: X \to \mathcal{Q}$ be given by $h(x) = (d(x, x_n))_{n=1}^{\infty}$. By Theorem 1.1.9, h is continuous. Clearly, h is one-to-one. Since X is compact and $\mathcal{Q}$ is metric, h is a closed map. Therefore, h is an embedding.

Q.E.D.

1.2 Continuous Decompositions

We present a method to construct “new” spaces from “old” ones by “shrinking” certain subsets to points. The preparation of this section is based on [10, 13, 18, 27].

1.2.1 Definition A *decomposition* of a set X is a collection of nonempty, pairwise disjoint sets whose union is X. The decomposition is said to be *closed* if each of its element is a closed subset of X.

1.2.2 Definition Let $\mathcal{G}$ be a decomposition of a metric space X. We define $X/\mathcal{G}$ as the set whose elements are the elements of the decomposition $\mathcal{G}$. $X/\mathcal{G}$ is called the *quotient space*. The function $q\colon X \twoheadrightarrow X/\mathcal{G}$, which sends each point x of X to the unique element G of $\mathcal{G}$ such that $x \in G$, is called the *quotient map*.

1.2.3 Remark Given a decomposition of a metric space X, note that $q(x) = q(y)$ if and only if x and y belong to the same element of $\mathcal{G}$. We give a topology to $X/\mathcal{G}$ in such a way that the function q is continuous and it is the biggest with this property.

1.2.4 Definition Let X be a metric space, let $\mathcal{G}$ be a decomposition of X and let $q\colon X \twoheadrightarrow X/\mathcal{G}$ be the quotient map. Then the topology

$$\mathcal{U} = \{U \subset X/\mathcal{G} \mid q^{-1}(U) \text{ is open in } X\}$$

is called the *quotient topology for* $X/\mathcal{G}$.

1.2.5 Remark Let $\mathcal{G}$ be a decomposition of a metric space X, and let $q\colon X \twoheadrightarrow X/\mathcal{G}$ be the quotient map. Then a subset U of $X/\mathcal{G}$ is open (closed, respectively) if and only if $q^{-1}(U)$ is an open (closed, respectively) subset of X.

1.2.6 Definition Let $f\colon X \twoheadrightarrow Y$ be a surjective map between metric spaces. Since f is a function, $\mathcal{G}_f = \{f^{-1}(y) \mid y \in Y\}$ is a decomposition of X. The function $\varphi_f\colon X/\mathcal{G}_f \to Y$ given by $\varphi_f(q(x)) = f(x)$ is of special interest. Note that φ_f is well defined; in fact, it is a bijection and the following diagram:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y \\ & X/\mathcal{G}_f \arrow[ur, "\varphi_f"'] & \end{tikzcd}$$

is commutative.

The next lemma is a special case of the *Transgression Lemma* [27, 3.22].

1.2.7 Lemma *Let $f\colon X \twoheadrightarrow Y$ be a surjective map between metric spaces. If $X/\mathcal{G}_f$ has the quotient topology, then the function φ_f is continuous.*

Proof If U is an open subset of Y, then $\varphi_f^{-1}(U) = qf^{-1}(U)$. Since $q^{-1}\varphi_f^{-1}(U) = q^{-1}qf^{-1}(U) = f^{-1}(U)$ and $f^{-1}(U)$ is an open subset of X, we have, by the definition of quotient topology, that $\varphi_f^{-1}(U)$ is an open subset of $X/\mathcal{G}_f$. Therefore, φ_f is continuous.

Q.E.D.

1.2.8 Example Let $X = [0, 2\pi)$ and let $f\colon X \twoheadrightarrow \mathcal{S}^1$, where $\mathcal{S}^1$ is the unit circle, be given by $f(t) = \exp(t) = e^{it}$. Then f is a continuous bijection. Since $\mathcal{G}_f$ is, "essentially," X, it follows that $X/\mathcal{G}_f$ is homeomorphic to X. On the other hand, X is not homeomorphic to $\mathcal{S}^1$, since X is not compact and $\mathcal{S}^1$ is. Therefore, φ_f is not a homeomorphism.

1.2.9 Definition A map $f\colon X \to Y$ between metric spaces is said to be *open* provided that for each open subset K of X, $f(K)$ is open in Y.

The following theorem gives sufficient conditions to ensure that φ_f is a homeomorphism:

1.2.10 Theorem *Let $f: X \twoheadrightarrow Y$ be a surjective map between metric spaces. If f is open or closed, then $\varphi_f : X/\mathcal{G}_f \twoheadrightarrow Y$ is a homeomorphism.*

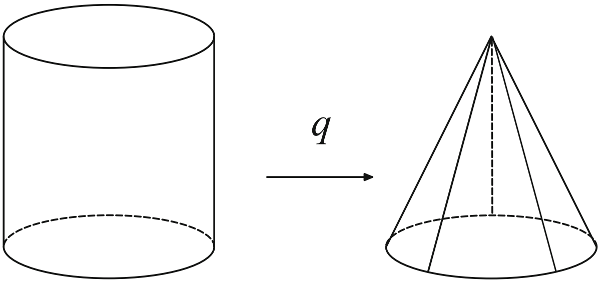
Proof Suppose f is an open map. Since φ is a bijective map, it is enough to show that φ_f is open. Let A be an open subset of $X/\mathcal{G}_f$. Since $\varphi_f(A) = fq^{-1}(A)$, $\varphi_f(A)$ is an open subset of Y. Therefore, φ_f is an open map.

The proof of the case when f is closed is similar.

Q.E.D.

Decompositions are also used to construct the cone and suspension over a given space.

1.2.11 Definition Let X be a metric space and let $\mathcal{G} = \{\{(x,t)\} \mid x \in X \text{ and } t \in [0,1)\} \cup \{(X \times \{1\})\}$. Then $\mathcal{G}$ is a decomposition of $X \times [0,1]$. The *cone over* X, denoted by $K(X)$, is the quotient space $(X \times [0,1])/\mathcal{G}$. The element $\{X \times \{1\}\}$ of $(X \times [0,1])/\mathcal{G}$ is called the *vertex* of the cone and it is denoted by ν_X.

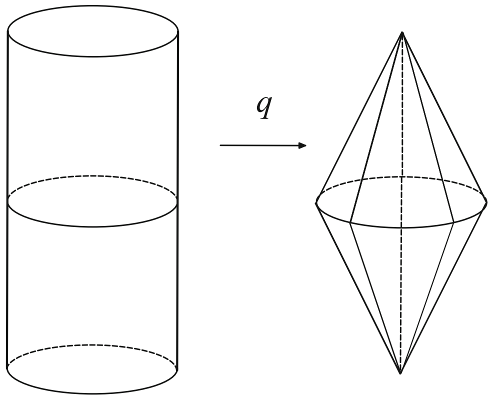


A proof of the following proposition may be found in [10, 5.2, p. 127].

1.2.12 Proposition *Let $f: X \to Y$ be a map between metric spaces. Then f induces a map $K(f): K(X) \to K(Y)$ by*

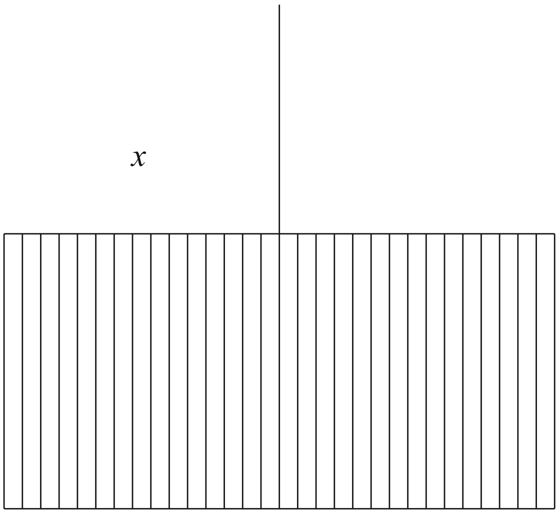
$$K(f)(\omega) = \begin{cases} \nu_Y, & \text{if } \omega = \nu_X \in K(X); \\ (f(x), t), & \text{if } \omega = (x,t) \in K(X) \setminus \{\nu_X\}. \end{cases}$$

1.2.13 Definition Let X be a metric space and let $\mathcal{G} = \{\{(x,t)\} \mid x \in X \text{ and } t \in (0,1)\} \cup \{(X \times \{0\}), (X \times \{1\})\}$. Then $\mathcal{G}$ is a decomposition of $X \times [0,1]$. The *suspension over* X, denoted by $\Sigma(X)$, is the quotient space $(X \times [0,1])/\mathcal{G}$. The elements $\{X \times \{0\}\}$ and $\{X \times \{1\}\}$ of $(X \times [0,1])/\mathcal{G}$ are called the *vertexes* of the suspension and are denoted by ν^- and ν^+, respectively.



1.2.14 Definition Let X be a metric space and let $\mathcal{G}$ be a decomposition of X. We say that $\mathcal{G}$ is *upper semicontinuous* if for each $G \in \mathcal{G}$ and each open subset U of X such that $G \subset U$, there exists an open subset V of X such that $G \subset V$ and such that if $G' \in \mathcal{G}$ and $G' \cap V \neq \emptyset$, then $G' \subset U$. We say that $\mathcal{G}$ is *lower semicontinuous* provided that for each $G \in \mathcal{G}$ any two points x and y of G and each open set U of X such that $x \in U$, there exists an open set V of X such that $y \in V$ and such that if $G' \in \mathcal{G}$ and $G' \cap V \neq \emptyset$, then $G \cap U \neq \emptyset$. Finally, we say that $\mathcal{G}$ is *continuous* if $\mathcal{G}$ is both upper and lower semicontinuous.

1.2.15 Example Let $X = ([-1, 1] \times [0, 1]) \cup (\{0\} \times [0, 2])$. For each $t \in [-1, 1] \setminus \{0\}$, let $G_t = \{t\} \times [0, 1]$, and for $t = 0$, let $G_0 = \{0\} \times [0, 2]$. Let $\mathcal{G} = \{G_t \mid t \in [0, 1]\}$. Then $\mathcal{G}$ is an upper semicontinuous decomposition of X.

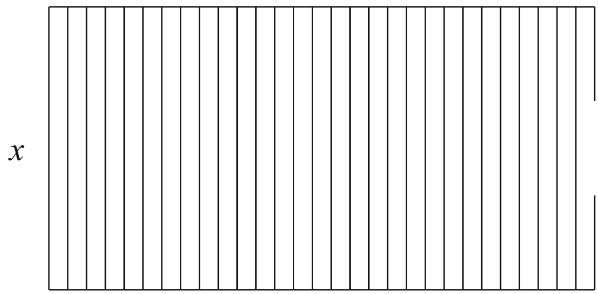


Upper semicontinuous decomposition

1.2.16 Example Let

$$X = ([0,1] \times [0,1)) \cup \left(\{1\} \times \left[0, \frac{1}{3}\right] \cup \{1\} \times \left[\frac{2}{3}, 1\right] \right).$$

For each $t \in [0,1)$, let $G_t = \{t\} \times [0,1]$, let $G_1 = \{1\} \times \left[0, \frac{1}{3}\right]$ and let $G_1' = \{1\} \times \left[\frac{2}{3}, 1\right]$. Let $\mathcal{G} = \{G_t \mid t \in [0,1]\} \cup \{G_1'\}$. Then $\mathcal{G}$ is a lower semicontinuous decomposition of X.



Lower semicontinuous decomposition

The following theorem gives us a way to obtain upper semicontinuous decompositions of compacta.

1.2.17 Theorem *Let $f : X \twoheadrightarrow Y$ be a surjective map between compacta. If $\mathcal{G}_f = \{f^{-1}(y) \mid y \in Y\}$, then $\mathcal{G}_f$ is an upper semicontinuous decomposition of X.*

Proof Let U be an open subset of X such that $f^{-1}(y) \subset U$. Note that $X \setminus U$ is a closed subset; hence, compact, of X. Then $f(X \setminus U)$ is a compact subset; hence, closed, of Y such that $y \notin f(X \setminus U)$. Thus, $Y \setminus f(X \setminus U)$ is an open subset of Y containing y.

If $V = f^{-1}(Y \setminus f(X \setminus U))$, then V is an open subset of X such that $f^{-1}(y) \subset V \subset U$. Since $V = \bigcup\{f^{-1}(y) \mid y \in Y \setminus f(X \setminus U)\}$, clearly V satisfies the required property of the definition of upper semicontinuous decomposition.

Q.E.D.

1.2.18 Remark Let us note that Theorem 1.2.17 is not true without the compactness of X. Let X be the Euclidean plane $\mathbb{R}^2$ and let $\pi : \mathbb{R}^2 \twoheadrightarrow \mathbb{R}$ be given by $\pi((x, y)) = x$. Then $\mathcal{G}_\pi$ is a decomposition of X which is not upper semicontinuous. To see this, let $U = \left\{(x, y) \in X \;\middle|\; x \neq 0 \text{ and } y < \frac{1}{x}\right\} \cup \{0\} \times \mathbb{R}$. Then U is an open set of X such that $\pi^{-1}(0) \subset U$, whose boundary is asymptotic to $\pi^{-1}(0)$. Hence, for each $t \in \mathbb{R} \setminus \{0\}$, $\pi^{-1}(t) \cap (X \setminus U) \neq \emptyset$.

The next theorem gives three other ways to think about upper semicontinuous decompositions.

1.2.19 Theorem *If X is a metric space and $\mathcal{G}$ is a decomposition of X, then the following conditions are equivalent:*

(a) $\mathcal{G}$ is an upper semicontinuous decomposition;
(b) the quotient map $q : X \twoheadrightarrow X/\mathcal{G}$ is closed;
(c) if U is an open subset of X, then $W_U = \bigcup\{G \in \mathcal{G} \mid G \subset U\}$ is an open subset of X;
(d) if D is a closed subset of X, then $K_D = \bigcup\{G \in \mathcal{G} \mid G \cap D \neq \emptyset\}$ is a closed subset of X.

Proof Suppose $\mathcal{G}$ is an upper semicontinuous decomposition. Let D be a closed subset of X. By Remark 1.2.5, we have that $q(D)$ is closed in $X/\mathcal{G}$ if and only if $q^{-1}(q(D))$ is closed in X. We show that $X \setminus q^{-1}(q(D))$ is open in X. Let $x \in X \setminus q^{-1}(q(D))$. Then $q(x) \in X/\mathcal{G} \setminus q(D)$ and, hence, $q^{-1}(q(x)) \subset X \setminus D$. Therefore, since $X \setminus D$ is open, by Definition 1.2.14, there exists an open set V of X such that $q^{-1}(q(x)) \subset V$ and for each $y \in V$, $q^{-1}(q(y)) \subset X \setminus D$. Clearly, $x \in V$ and $q(V) \subset X/\mathcal{G} \setminus q(D)$. Thus, $V \subset X \setminus q^{-1}(q(D))$. Therefore, $X \setminus q^{-1}(q(D))$ is open, since $x \in V \subset X \setminus q^{-1}(q(D))$.

Now, suppose q is a closed map. Let U be an open subset of X. Since q is a closed map, we have that $q^{-1}(X/\mathcal{G} \setminus q(X \setminus U))$ is an open subset of X such that $q^{-1}(X/\mathcal{G} \setminus q(X \setminus U)) = W_U$. (If $x \in q^{-1}(X/\mathcal{G} \setminus q(X \setminus U))$, then $q(x) \in X/\mathcal{G} \setminus q(X \setminus U)$. Hence, $q^{-1}(q(x)) \subset X \setminus q^{-1}(q(X \setminus U)) \subset X \setminus (X \setminus U)) = U$. Thus, $x \in W_U$. The other inclusion is obvious.)

Next, suppose W_U is open for each open subset U of X. Let D be a closed subset of X. Then $X \setminus D$ is open in X. Hence, $W_{X \setminus D}$ is open in X. Since, clearly, $K_D = X \setminus W_{X \setminus D}$, we have that K_D is closed.

Finally, suppose K_D is closed for each closed subset D of X. To see $\mathcal{G}$ is upper semicontinuous, let $G \in \mathcal{G}$ and let U be an open subset of X such that $G \subset U$. Note that $X \setminus U$ is a closed subset of X. Hence, $K_{X \setminus U}$ is a closed subset of X. Let $V = X \setminus K_{X \setminus U}$. Then V is open, $G \subset V \subset U$ and if $G' \in \mathcal{G}$ and $G' \cap V \neq \emptyset$, then $G' \subset V$. Therefore, $\mathcal{G}$ is upper semicontinuous.

Q.E.D.

1.2.20 Corollary *Let X be a metric space. If $\mathcal{G}$ is an upper semicontinuous decomposition of X, then the elements of $\mathcal{G}$ are closed.*

Proof Let $G \in \mathcal{G}$. Take $x \in G$ and let $q : X \twoheadrightarrow X/\mathcal{G}$ be the quotient map. Since X is a metric space, $\{x\}$ is closed in X. By Theorem 1.2.19, $q(\{x\})$ is closed in $X/\mathcal{G}$. Since q is continuous and $q^{-1}(q(\{x\})) = G$, G is a closed subset of X.

Q.E.D.

1.2.21 Theorem *If X is a compactum and $\mathcal{G}$ is an upper semicontinuous decomposition of X, then $X/\mathcal{G}$ has a countable basis.*

Proof Let $q: X \twoheadrightarrow X/\mathcal{G}$ be the quotient map. Since X is a compactum, it has a countable basis $\mathcal{U}$. Let

$$\mathcal{B} = \left\{ \bigcup_{j=1}^{n} U_j \;\middle|\; U_1, \ldots, U_n \in \mathcal{U} \text{ and } n \in \mathbb{N} \right\}.$$

Note that $\mathcal{B}$ is a countable family of open subsets of X.

Let

$$\boldsymbol{\mathcal{B}} = \{X/\mathcal{G} \setminus q(X \setminus U) \mid U \in \mathcal{B}\}.$$

We see that $\boldsymbol{\mathcal{B}}$ is a countable basis for $X/\mathcal{G}$. Clearly, $\boldsymbol{\mathcal{B}}$ is a countable family of open subsets of $X/\mathcal{G}$. Let $\boldsymbol{U}$ be an open subset of $X/\mathcal{G}$ and let $\boldsymbol{x} \in \boldsymbol{U}$. Then $q^{-1}(\boldsymbol{U})$ is an open subset of X and $q^{-1}(\boldsymbol{x}) \subset q^{-1}(\boldsymbol{U})$. Since $q^{-1}(\boldsymbol{x})$ is compact, there exist $U_1, \ldots, U_k \in \mathcal{U}$ such that $q^{-1}(\boldsymbol{x}) \subset \bigcup_{j=1}^{k} U_j \subset q^{-1}(\boldsymbol{U})$. Let $U = \bigcup_{j=1}^{k} U_j$. Then $U \in \mathcal{B}$. Hence, $X/\mathcal{G} \setminus q(X \setminus U) \in \boldsymbol{\mathcal{B}}$. Also, $\boldsymbol{x} \in X/\mathcal{G} \setminus q(X \setminus U) \subset \boldsymbol{U}$. Therefore, $\boldsymbol{\mathcal{B}}$ is a countable basis for $X/\mathcal{G}$.

Q.E.D.

1.2.22 Corollary *If X is a compactum and $\mathcal{G}$ is an upper semicontinuous decomposition of X, then $X/\mathcal{G}$ is metrizable.*

Proof By Theorem 1.2.21, we have that $X/\mathcal{G}$ has a countable basis. By [16, Theorem 1, p. 241], it suffices to show that $X/\mathcal{G}$ is a Hausdorff space. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two distinct points of $X/\mathcal{G}$. Then $q^{-1}(\boldsymbol{x})$ and $q^{-1}(\boldsymbol{y})$ are two disjoint closed subsets of X. Since X is a metric space, there exist two disjoint open subsets, U_1 and U_2, of X such that $q^{-1}(\boldsymbol{x}) \subset U_1$ and $q^{-1}(\boldsymbol{y}) \subset U_2$. Note that, by Theorem 1.2.19 (c), W_{U_1} and W_{U_2} are open subsets of X such that $q^{-1}(\boldsymbol{x}) \subset W_{U_1} \subset U_1$, $q^{-1}(\boldsymbol{y}) \subset W_{U_2} \subset U_2$, and $q(W_{U_1})$ and $q(W_{U_2})$ are open subsets of $X/\mathcal{G}$. Since $U_1 \cap U_2 = \emptyset$, $q(W_{U_1}) \cap q(W_{U_2}) = \emptyset$. Therefore, $X/\mathcal{G}$ is a Hausdorff space.

Q.E.D.

The next theorem gives a characterization of lower semicontinuous decompositions.

1.2.23 Theorem *Let X be a metric space and let $\mathcal{G}$ be a decomposition of X. Then $\mathcal{G}$ is lower semicontinuous if and only if the quotient map $q: X \twoheadrightarrow X/\mathcal{G}$ is open.*

Proof Suppose $\mathcal{G}$ is lower semicontinuous. Let U be an open subset of X. We show $q(U)$ is an open subset of $X/\mathcal{G}$. To this end, by Remark 1.2.5, we only need to prove that $q^{-1}(q(U))$ is an open subset of X.

Let $y \in q^{-1}(q(U))$. Then $q(y) \in q(U)$, and there exists a point x in U such that $q(x) = q(y)$. Since $\mathcal{G}$ is a lower semicontinuous decomposition, there exists an open subset V of X containing y such that if $G \in \mathcal{G}$ and $G \cap V \neq \emptyset$, then $G \cap U \neq \emptyset$. Hence, $V \subset q^{-1}(q(U))$. Therefore, q is open.

Now, suppose q is open. Let $G \in \mathcal{G}$. Take $x, y \in G$ and let U be an open subset of X such that $x \in U$. Since q is open, $V = q^{-1}(q(U))$ is an open subset of X such that $G \subset V$. In particular, $y \in V$. Let $G' \in \mathcal{G}$ be such that $G' \cap V \neq \emptyset$. Then $G' \subset V$. Thus, $q(G') \in q(U)$. Hence, there exists a point $u \in U$ such that $q(u) = q(G')$. Since $q^{-1}(q(G')) = G'$, $u \in G'$. Thus, $G' \cap U \neq \emptyset$. Therefore, $\mathcal{G}$ is lower semicontinuous.

Q.E.D.

The following corollary is a consequence of Theorems 1.2.19 and 1.2.23:

1.2.24 Corollary *Let X be a metric space and let $\mathcal{G}$ be a decomposition of X. Then $\mathcal{G}$ is continuous if and only if the quotient map is both open and closed.*

The following theorem gives us a necessary and sufficient condition on a map $f\colon X \twoheadrightarrow Y$ between compacta, to have that $\mathcal{G}_f = \{f^{-1}(y) \mid y \in Y\}$ is a continuous decomposition.

1.2.25 Theorem *Let X and Y be compacta and let $f\colon X \twoheadrightarrow Y$ be a surjective map. Then $\mathcal{G}_f = \{f^{-1}(y) \mid y \in Y\}$ is continuous if and only if f is open.*

Proof If $\mathcal{G}_f$ is a continuous decomposition of X, by Theorem 1.2.23, the quotient map $q\colon X \twoheadrightarrow X/\mathcal{G}_f$ is open. By Theorem 1.2.10, $\varphi_f\colon X/\mathcal{G}_f \twoheadrightarrow Y$ is a homeomorphism. Hence, $f = \varphi_f \circ q$ is an open map.

Now, suppose f is open. By Theorem 1.2.17, $\mathcal{G}_f$ is upper semicontinuous. Since $q = \varphi_f^{-1} \circ f$ and f is open, q is open. By Theorem 1.2.23, $\mathcal{G}_f$ is a lower semicontinuous decomposition. Therefore, $\mathcal{G}_f$ is continuous.

Q.E.D.

In the following definition a notion of convergence of sets is introduced.

1.2.26 Definition Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of subsets of the metric space X. Then:

(1) the *limit inferior* of the sequence $\{X_n\}_{n=1}^{\infty}$ is defined as follows:

$$\liminf X_n = \{x \in X \mid \text{for each open subset } U \text{ of } X \text{ such that}$$

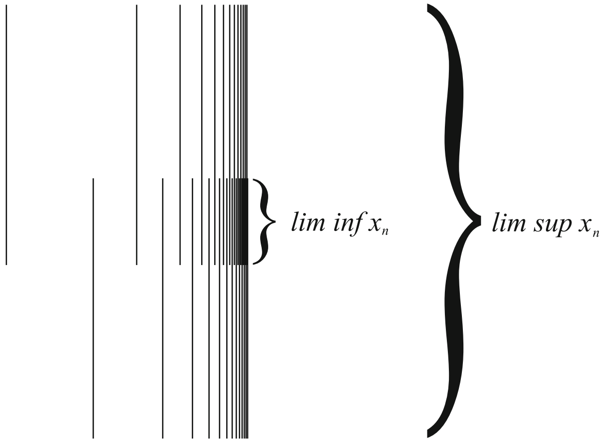
$$x \in U,\ U \cap X_n \neq \emptyset \text{ for each } n \in \mathbb{N},\ \text{save, possibly, finitely many}\}.$$

(2) the *limit superior* of the sequence $\{X_n\}_{n=1}^{\infty}$ is defined as follows:

$$\limsup X_n = \{x \in X \mid \text{for each open subset } U \text{ of } X \text{ such that}$$

$$x \in U,\ U \cap X_n \neq \emptyset \text{ for infinitely many indices } n \in \mathbb{N}\}.$$

Clearly, $\liminf X_n \subset \limsup X_n$. If $\liminf X_n = \limsup X_n = L$, then we say that the sequence $\{X_n\}_{n=1}^{\infty}$ is a *convergent sequence* with limit $L = \lim\limits_{n\to\infty} X_n$.



1.2.27 Lemma *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of subsets of the metric space X. Then* $\liminf X_n$ *and* $\limsup X_n$ *are both closed subsets of X.*

Proof Let $x \in Cl(\liminf X_n)$. Let U be an open subset of X such that $x \in U$. Since $x \in Cl(\liminf X_n) \cap U$, we have that $\liminf X_n \cap U \neq \emptyset$. Hence, $U \cap X_n \neq \emptyset$ for each $n \in \mathbb{N}$, save, possibly, finitely many. Therefore, $x \in \liminf X_n$. The proof for $\limsup$ is similar.

Q.E.D.

The next theorem tells us that separable metric spaces behave like sequentially compact spaces using the notion of convergence just introduced.

1.2.28 Theorem *Each sequence $\{X_n\}_{n=1}^{\infty}$ of closed subsets of a separable metric space X has a convergent subsequence.*

Proof Let $\{U_m\}_{m=1}^{\infty}$ be a countable basis for X. Let $\{X_n^1\}_{n=1}^{\infty} = \{X_n\}_{n=1}^{\infty}$. Suppose, inductively, that we have defined the sequence $\{X_n^m\}_{n=1}^{\infty}$. We define the sequence $\{X_n^{m+1}\}_{n=1}^{\infty}$ as follows:

(1) If $\{X_n^m\}_{n=1}^{\infty}$ has a subsequence $\{X_{n_k}^m\}_{k=1}^{\infty}$ such that $\limsup X_{n_k}^m \cap U_m = \emptyset$, then let $\{X_n^{m+1}\}_{n=1}^{\infty}$ be such subsequence of $\{X_n^m\}_{n=1}^{\infty}$.
(2) If for each subsequence $\{X_{n_k}^m\}_{k=1}^{\infty}$ of $\{X_n^m\}_{n=1}^{\infty}$, we have that $\limsup X_{n_k}^m \cap U_m \neq \emptyset$, we define $\{X_n^{m+1}\}_{n=1}^{\infty}$ as $\{X_n^m\}_{n=1}^{\infty}$.

Since we have the subsequences $\{X_n^m\}_{n=1}^{\infty}$, let us consider the "diagonal subsequence" $\{X_m^m\}_{m=1}^{\infty}$. By construction, $\{X_m^m\}_{m=1}^{\infty}$ is a subsequence of $\{X_n\}_{n=1}^{\infty}$. We see that $\{X_m^m\}_{m=1}^{\infty}$ converges.

Let us assume that $\{X_m^m\}_{m=1}^{\infty}$ does not converge. Hence, there exists $p \in \limsup X_m^m \setminus \liminf X_m^m$. Let U_k be a basic open set such that $p \in U_k$ and $U_k \cap X_{m_\ell}^{m_\ell} = \emptyset$ for some subsequence $\{X_{m_\ell}^{m_\ell}\}_{\ell=1}^{\infty}$ of $\{X_m^m\}_{m=1}^{\infty}$ ($\liminf X_n$ is a closed subset of X by Lemma 1.2.27). Clearly, $\{X_{m_\ell}^{m_\ell}\}_{\ell=k}^{\infty}$ is a subsequence of $\{X_n^k\}_{n=1}^{\infty}$. Thus, $\{X_n^k\}_{n=1}^{\infty}$ satisfies condition (1), with k in place of m. Hence, $\limsup X_n^{k+1} \cap U_k = \emptyset$. Since

$\{X_m^m\}_{m=k+1}^{\infty}$ is a subsequence of $\{X_n^{k+1}\}_{n=1}^{\infty}$ and $\limsup X_m^m \subset \limsup X_n^{k+1}$, it follows that $\limsup X_m^m \cap U_k = \emptyset$. Now, recall that $p \in \limsup X_m^m \cap U_k$. Thus, we obtain a contradiction. Therefore, $\{X_m^m\}_{m=1}^{\infty}$ converges.

Q.E.D.

1.2.29 Theorem *Let X be a compactum. If $\{X_n\}_{n=1}^{\infty}$ is a sequence of connected subsets of X and $\liminf X_n \neq \emptyset$, then $\limsup X_n$ is connected.*

Proof Suppose, to the contrary, that $\limsup X_n$ is not connected. Since $\limsup X_n$ is closed, by Lemma 1.2.27, we assume, without loss of generality, that there exist two disjoint closed subsets A and B of X such that $\limsup X_n = A \cup B$. Since X is a metric space, there exist two disjoint open subsets U and V of X such that $A \subset U$ and $B \subset V$. Then there exists $N' \in \mathbb{N}$ such that if $n \geq N'$, then $X_n \subset U \cup V$. To show this, suppose it is not true. Then for each $n \in \mathbb{N}$, there exists $m_n > n$ such that $X_{m_n} \setminus (U \cup V) \neq \emptyset$. Let $x_{m_n} \in X_{m_n} \setminus (U \cup V)$ for each $n \in \mathbb{N}$. Since X is compact, without loss of generality, we assume that the sequence $\{x_{m_n}\}_{n=1}^{\infty}$ converges to a point x of X. Note that $x \in X \setminus (U \cup V)$ and, by construction, $x \in \limsup X_n$, a contradiction. Therefore, there exists $N' \in \mathbb{N}$ such that if $n \geq N'$, then $X_n \subset U \cup V$.

Since $\liminf X_n \neq \emptyset$ and $\liminf X_n \subset \limsup X_n$, we assume, without loss of generality, that $\liminf X_n \cap U \neq \emptyset$. Then there exists $N'' \in \mathbb{N}$ such that if $n \geq N''$, $U \cap X_n \neq \emptyset$. Let $N = \max\{N', N''\}$. Hence, if $n \geq N$, then $X_n \subset U \cup V$ and $U \cap X_n \neq \emptyset$. Since X_n is connected for every $n \in \mathbb{N}$, $X_n \cap V = \emptyset$ for each $n \geq N$, a contradiction. Therefore, $\limsup X_n$ is connected.

Q.E.D.

The following theorem gives us a characterization of an upper semicontinuous decomposition of a compactum in terms of limits inferior and superior.

1.2.30 Theorem *Let X be a compactum, with metric d. Then a decomposition $\mathcal{G}$ of X is upper semicontinuous if and only if $\mathcal{G}$ is a closed decomposition and for each sequence $\{X_n\}_{n=1}^{\infty}$ of elements of $\mathcal{G}$ and each element Y of $\mathcal{G}$ such that $\liminf X_n \cap Y \neq \emptyset$, then $\limsup X_n \subset Y$.*

Proof Suppose $\mathcal{G}$ is an upper semicontinuous decomposition of X. By Corollary 1.2.20, $\mathcal{G}$ is a closed decomposition. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of elements of $\mathcal{G}$ and let Y be an element of $\mathcal{G}$ such that $\liminf X_n \cap Y \neq \emptyset$.

Suppose there exists $p \in \limsup X_n \setminus Y$. Since Y is closed and p is not an element of Y, there exists an open set W of X such that $p \in W$ and $Cl(W) \cap Y = \emptyset$. Let $U = X \setminus Cl(W)$. Since $\mathcal{G}$ is upper semicontinuous, there exists an open set V of X such that $Y \subset V$ and if $G \in \mathcal{G}$ such that $G \cap V \neq \emptyset$, $G \subset U$.

Let $q \in \liminf X_n \cap Y$. Then $q \in \liminf X_n \cap V$. Hence, $V \cap X_n \neq \emptyset$ for each $n \in \mathbb{N}$, save, possibly, finitely many. Thus, $W \cap X_n = \emptyset$ for each $n \in \mathbb{N}$, save, possibly, finitely many. This contradicts the fact that $p \in W \cap \limsup X_n$. Therefore, $\limsup X_n \subset Y$.

Now, suppose $\mathcal{G}$ is a closed decomposition and let Y be an element of $\mathcal{G}$. Suppose that if $\{X_n\}_{n=1}^{\infty}$ is a sequence of elements of $\mathcal{G}$ such that $\liminf X_n \cap Y \neq \emptyset$, then $\limsup X_n \subset Y$.

To see $\mathcal{G}$ is upper semicontinuous, let U be an open subset of X such that $Y \subset U$. For each $n \in \mathbb{N}$, let $V_n = \mathcal{V}^d_{\frac{1}{n}}(Y)$. Suppose that for each $n \in \mathbb{N}$, there exists an element X_n of $\mathcal{G}$ such that $X_n \cap V_n \neq \emptyset$ and $X_n \not\subset U$. For each $n \in \mathbb{N}$, let $p_n \in X_n \cap V_n$. Since X is compact, $\{p_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{p_{n_k}\}_{k=1}^{\infty}$. Let p be the point of convergence of $\{p_{n_k}\}_{k=1}^{\infty}$. Note that $p \in \liminf X_{n_k} \cap Y$. Hence, $\limsup X_{n_k} \subset Y$.

For each $k \in \mathbb{N}$, let $q_k \in X_{n_k} \setminus U$. Since X is compact, the sequence $\{q_{n_k}\}_{k=1}^{\infty}$ has a convergent subsequence $\{q_{n_{k_\ell}}\}_{\ell=1}^{\infty}$. Let q be the point of convergence of $\{q_{n_{k_\ell}}\}_{\ell=1}^{\infty}$. Note that $q \notin Y$ and $q \in \limsup X_{n_{k_\ell}} \subset \limsup X_{n_k}$, a contradiction. Therefore, $\mathcal{G}$ is upper semicontinuous.

Q.E.D.

As an application of Theorem 1.2.30, we have the following result which says that the components of the elements of an upper semicontinuous decomposition form another upper semicontinuous decomposition.

1.2.31 Theorem *Let X be a compactum and let $\mathcal{G}$ be an upper semicontinuous decomposition of X. If $\mathcal{D} = \{D \mid D$ is a component of G, for some $G \in \mathcal{G}\}$, then $\mathcal{D}$ is an upper semicontinuous decomposition of X.*

Proof Clearly, the elements of $\mathcal{D}$ are closed in X. Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of elements of $\mathcal{D}$ and let $D \in \mathcal{D}$ be such that $\liminf D_n \cap D \neq \emptyset$. Let $G \in \mathcal{G}$ be such that $D \subset G$ and for each $n \in \mathbb{N}$, let $G_n \in \mathcal{G}$ be such that $D_n \subset G_n$. Since $\liminf D_n \subset \liminf G_n$ and $\liminf D_n \cap D \neq \emptyset$, $\liminf G_n \cap G \neq \emptyset$. Hence, by the upper semicontinuity of $\mathcal{G}$, $\limsup G_n \subset G$ (Theorem 1.2.30). Thus, since $\limsup D_n \subset \limsup G_n$, $\limsup D_n \subset G$.

By hypothesis, each D_n is connected. Then, by Theorem 1.2.29, $\limsup D_n$ is connected. Since D is a component of G, $\limsup D_n \subset G$ and $\limsup D_n \cap D \neq \emptyset$, $\limsup D_n \subset D$. Therefore, by Theorem 1.2.30, $\mathcal{D}$ is an upper semicontinuous decomposition.

Q.E.D.

The following theorem gives us a characterization of a continuous decomposition of a compactum in terms of limits inferior and superior.

1.2.32 Theorem *Let X be a compactum, with metric d. Then a decomposition $\mathcal{G}$ of X is continuous if and only if $\mathcal{G}$ is a closed decomposition and for each sequence $\{X_n\}_{n=1}^{\infty}$ of elements of $\mathcal{G}$ and each element Y of $\mathcal{G}$ such that* $\liminf X_n \cap Y \neq \emptyset$, *then* $\limsup X_n = Y$.

Proof Suppose $\mathcal{G}$ is a continuous decomposition of X. By Corollary 1.2.20, $\mathcal{G}$ is a closed decomposition. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of elements of $\mathcal{G}$ and let Y be an element of $\mathcal{G}$ such that $\liminf X_n \cap Y \neq \emptyset$. By Theorem 1.2.30, $\limsup X_n \subset Y$. Suppose there exists $p \in Y \setminus \limsup X_n$. Let U be an open subset of X such that $p \in U$ and $U \cap \limsup X_n = \emptyset$ (by Lemma 1.2.27, $\limsup X_n$ is closed). Let $q \in \limsup X_n \subset Y$. Since $\mathcal{G}$ is a lower semicontinuous decomposition, there exists an open set V of X such that $q \in V$ and if $G \in \mathcal{G}$ and $G \cap V \neq \emptyset$, then $G \cap U \neq \emptyset$.