CLASSICS IN MATHEMATICS

Lars Hörmander

The Analysis of Linear Partial Differential Operators III

Pseudo-Differential Operators

Springer

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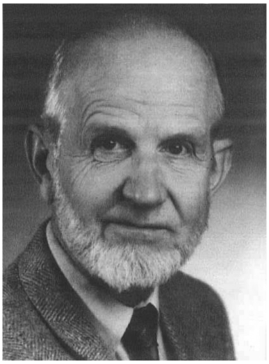

Born on January 24, 1931, on the southern coast of Sweden, Lars Hörmander did his secondary schooling as well as his undergraduate and doctoral studies in Lund. His principal teacher and adviser at the University of Lund was Marcel Riesz until he retired, then Lars Gårding. In 1956 he worked in the USA, at the universities of Chicago, Kansas, Minnesota and New York, before returning to a chair at the University of Stockholm. He remained a frequent visitor to the US, particularly to Stanford and was Professor at the IAS, Princeton from 1964 to 1968. In 1968 he accepted a chair at the University of Lund, Sweden, where, today, he is Emeritus Professor.

Hörmander's lifetime work has been devoted to the study of partial differential equations and its applications in complex analysis. In 1962 he was awarded the Fields Medal for his contributions to the general theory of linear partial differential operators. His book *Linear Partial Differential Operators* published 1963 by Springer in the Grundlehren series was the first major account of this theory. His four volume text *The Analysis of Linear Partial Differential Operators*, published in the same series 20 years later, illustrates the vast expansion of the subject in that period.

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The Analysis of Linear Partial Differential Operators III

Pseudo-Differential Operators

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Lars Hörmander

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Pseudo-Differential Operators

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Preface

to Volumes III and IV

The first two volumes of this monograph can be regarded as an expansion and updating of my book "Linear partial differential operators" published in the Grundlehren series in 1963. However, volumes III and IV are almost entirely new. In fact they are mainly devoted to the theory of linear differential operators as it has developed after 1963. Thus the main topics are pseudo-differential and Fourier integral operators with the underlying symplectic geometry. The contents will be discussed in greater detail in the introduction.

I wish to express here my gratitude to many friends and colleagues who have contributed to this work in various ways. First I wish to mention Richard Melrose. For a while we planned to write these volumes together, and we spent a week in December 1980 discussing what they should contain. Although the plan to write the books jointly was abandoned and the contents have been modified and somewhat contracted, much remains of our discussions then. Shmuel Agmon visited Lund in the fall of 1981 and generously explained to me all the details of his work on long range scattering outlined in the Goulaouic-Schwartz seminars 1978/79. His ideas are crucial in Chapter XXX. When the amount of work involved in writing this book was getting overwhelming Anders Melin lifted my spirits by offering to go through the entire manuscript. His detailed and constructive criticism has been invaluable to me; I as well as the readers of the book owe him a great debt. Bogdan Ziemian's careful proofreading has eliminated numerous typographical flaws. Many others have also helped me in my work, and I thank them all.

Some material intended for this monograph has already been included in various papers of mine. Usually it has been necessary to rewrite these papers completely for the book, but selected passages have been kept from a few of them. I wish to thank the following publishers holding the copyright for granting permission to do so, namely:
Marcel Dekker, Inc. for parts of [41] included in Section 17.2;
Princeton University Press for parts of [38] included in Chapter XXVII;
D. Reidel Publishing Company for parts of [40] included in Section 26.4;
John Wiley & Sons Inc. for parts of [39] included in Chapter XVIII.
(Here [N] refers to Hörmander [N] in the bibliography.)

Finally I wish to thank the Springer-Verlag for all the support I have received during my work on this monograph.

Djursholm in November, 1984 — Lars Hörmander

Contents

Introduction

to Volumes III and IV

A great variety of techniques have been developed during the long history of the theory of linear differential equations with variable coefficients. In this book we shall concentrate on those which have dominated during the latest phase. As a reminder that other earlier techniques are sometimes available and that they may occasionally be preferable, we have devoted the introductory Chapter XVII mainly to such methods in the theory of second order differential equations. Apart from that Volumes III and IV are intended to develop systematically, with typical applications, the three basic tools in the recent theory. These are the theory of pseudo-differential operators (Chapter XVIII), Fourier integral operators and Lagrangian distributions (Chapter XXV), and the underlying symplectic geometry (Chapter XXI). In the choice of applications we have been motivated mainly by the historical development. In addition we have devoted considerable space and effort to questions where these tools have proved their worth by giving fairly complete answers.

Pseudo-differential operators developed from the theory of singular integral operators. In spite of a long tradition these played a very modest role in the theory of differential equations until the appearance of Calderón's uniqueness theorem at the end of the 1950's and the Atiyah-Singer-Bott index theorems in the early 1960's. Thus we have devoted Chapter XXVIII and Chapters XIX, XX to these topics. The early work of Petrowsky on hyperbolic operators might be considered as a precursor of pseudo-differential operator theory. In Chapter XXIII we discuss the Cauchy problem using the improvements of the even older energy integral method given by the calculus of pseudo-differential operators.

The connections between geometrical and wave optics, classical mechanics and quantum mechanics, have a long tradition consisting in part of heuristic arguments. These ideas were developed more systematically by a number of people in the 1960's and early 1970's. Chapter XXV is devoted to the theory of Fourier integral operators which emerged from this. One of its first applications was to the study of asymptotic properties of eigenvalues (eigenfunctions) of higher order elliptic operators. It is therefore discussed in Chapter XXIX here together with a number of later developments which give beautiful proofs of the power of the tool. The study by Lax of the propagation of singularities of solutions to the Cauchy problem was one of the forerunners of the theory. We prove such results using only pseudo-differential operators in Chapter XXIII. In Chapter XXVI the propagation of singularities is discussed at great length for operators of principal type. It is the only known approach to general existence theorems for such operators. The completeness of the results obtained has been the reason for the inclusion of this chapter and the following one on subelliptic operators. In addition to Fourier integral operators one needs a fair amount of symplectic geometry then. This topic, discussed in Chapter XXI, has deep roots in classical mechanics but is now equally indispensible in the theory of linear differential operators. Additional symplectic geometry is provided in the discussion of the mixed problem in Chapter XXIV, which is otherwise based only on pseudo-differential operator theory. The same is true of Chapter XXX which is devoted to long range scattering theory. There too the geometry is a perfect guide to the analytical constructs required.

The most conspicuous omission in these books is perhaps the study of analytic singularities and existence theory for hyperfunction solutions. This would have required another volume - and another author. Very little is also included concerning operators with double characteristics apart from a discussion of hypoellipticity in Chapter XXII. The reason for this is in part shortage of space, in part the fact that few questions concerning such operators have so far obtained complete answers although the total volume of results is large. Finally, we have mainly discussed single operators acting on scalar functions or occasionally determined systems. The extensive work done on for example first order systems of vector fields has not been covered at all.

Chapter XVII. Second Order Elliptic Operators

Summary

The study of differential operators with variable coefficients has led to the development of quite elaborate techniques which will be exposed in the following chapters. However, much simpler classical methods will often work in the second order case, and some results are in fact only valid then. Moreover, second order operators (or rather related first order systems) play an important role in many geometrical contexts, so it seems natural to exploit the simplifications which are possible for them. However, the well motivated reader aiming for the most high powered machinery can very well skip this chapter altogether.

Elliptic operators are of constant strength so the results proved in Chapter XIII are applicable to them. The perturbation arguments used in Chapter XIII are recalled in Section 17.1 in the context of elliptic operators with low regularity assumptions on the coefficients and with L^p or Hölder conditions on the solutions. However, we shall not aim for such refinements later on since their main interest comes from the theory of non-linear differential equations which is beyond the scope of this book.

Section 17.2 is mainly devoted to the Aronszajn-Cordes uniqueness theorem stating in particular that if

$$\sum_{|\alpha| \leqq 2} a_\alpha(x) D^\alpha u = 0$$

is an elliptic equation where a_α are real valued Lipschitz continuous functions for $|\alpha|=2$ and a_α are bounded for $|\alpha|<2$, then u vanishes identically if u vanishes of infinite order at some point. No such result is true for operators of higher order than two although there are weaker uniqueness theorems concerning solutions vanishing in an open set (see also Chapter XXVIII). In this context we also return to the uniqueness theorems of Section 14.7 where we now allow first order perturbations.

In Section 17.3 we study the simplest classical boundary problem, the Dirichlet problem, consisting in finding a solution of $\Delta u = f$ with given boundary values. When the coefficients are constant and the boundary is flat, a reduction to the results of Section 17.1 is obtained by a simple reflection argument. As in Section 17.1 we can then use perturbation methods to

handle variable coefficients and a curved boundary. Thus the boundary is flattened, coefficients are frozen at a boundary point, the norm of the error then committed is estimated, and a Neuman series is applied. Obviously no good information on the singularities of solutions can be obtained in that way. In Section 17.4 we therefore present the Hadamard parametrix method which exploits the simple form of a second order operator in geodesic coordinates to describe the singularities of the fundamental solution with arbitrarily high precision. This method is in fact applicable to all second order operators with real non-degenerate principal symbol. It can also be applied to the Dirichlet problem although with considerable limitations due to the possible occurence of tangential or multiply reflected geodesics.

In Section 17.5 we combine the results of Sections 17.3 and 17.4 to a study of the asymptotic properties of eigenfunctions and eigenvalues of the Dirichlet problem. First we prove the precise error estimate of Avakumovič away from the boundary. A fairly precise analogue at the boundary is given, but one component of the proof cannot be completed until Chapter XXIV. Further refinements will be given in Chapter XXIX.

17.1. Interior Regularity and Local Existence Theorems

Despite the title of the chapter we shall here study a differential operator

$$P(x,D)=\sum_{|\alpha|\leqq m} a_\alpha(x)\,D^\alpha$$

of arbitrary order m in an open set $X\subset\mathbb{R}^n$. We assume that for some $p\in(1,\infty)$

(i) a_α is continuous when $|\alpha|=m$;

(ii) $P_m(0,D)=\sum_{|\alpha|=m} a_\alpha(0)\,D^\alpha$ is elliptic;

(iii) $a_\alpha\in L^{n/(m-|\alpha|)}_{\text{loc}}$ if $m-|\alpha|<n/p$, $a_\alpha\in L^{p+\varepsilon}_{\text{loc}}$ for some $\varepsilon>0$ if

$$m-|\alpha|=n/p,\qquad a_\alpha\in L^p_{\text{loc}}\ \text{if}\ m-|\alpha|>n/p.$$

We can then supplement Theorem 13.2.1 as follows:

Theorem 17.1.1. *If* (i)–(iii) *are fulfilled and X is a sufficiently small neighborhood of* 0, *then there is a linear operator E in $L^p(X)$ such that*

(17.1.1) $L^p(X)\ni f\mapsto D^\alpha Ef\in L^q(X)$ *is continuous if* $p\leqq q\leqq\infty$ *and* $1/q\geqq 1/p-(m-|\alpha|)/n$ *with strict inequality if* $q=\infty$;

(17.1.2) $$P(x,D)Ef=f,\quad f\in L^p(X),$$

(17.1.3) $$EP(x,D)u=u\ \text{ if }\ u\in C_0^\infty(X).$$

Proof. Let $p(D)=P_m(0,D)$ and choose $F_0\in\mathscr{S}'$ according to Theorem 7.1.22 so that $\hat F_0(\xi)=1/p(\xi)$ when $|\xi|\geqq 1$ and $\hat F_0\in C^\infty$. Then it follows from Theorems

7.9.5 and 4.5.9 that

$$\|D^\alpha F_0 * g\|_{L^q} \leqq C\|g\|_{L^p} \quad \text{if } g \in L^p \cap \mathscr{E}',\ 1/q = 1/p - (m-|\alpha|)/n,\ q<\infty. \tag{17.1.4}$$

Moreover, $D^\alpha F_0 \in L^r_{\mathrm{loc}}$ if $m-|\alpha|>n(1-1/r)$, for $D^\alpha F_0$ is essentially homogeneous of degree $m-|\alpha|-n>-n/r$.

Let E_0 be a fundamental solution of $p(D)$. Then $F_0-E_0 \in C^\infty$. If $g \in L^p(X)$ we define $g_0=g$ in X and $g_0=0$ in $\complement X$, and set $E_0 g = E_0 * g_{0|X}$. From (17.1.4) and the subsequent observations it follows if X is contained in the unit ball that

$$\|D^\alpha E_0 g\|_{L^q(X)} \leqq C\|g\|_{L^p(X)},\quad g \in L^p(X). \tag{17.1.5}$$

Here $1/q = 1/p - (m-|\alpha|)/n$ when $m-|\alpha|<n/p$, we choose $q=p(p+\varepsilon)/\varepsilon$ with ε as in condition (iii) when $m-|\alpha|=n/p$, and $q=\infty$ when $m-|\alpha|>n/p$ (take $1/r+1/p=1$). Now

$$P(x,D)E_0 g = p(D)E_0 g + (P(x,D)-p(D))E_0 g = g + Rg,$$

$$Rg = \sum_{|\alpha|=m} (a_\alpha(x)-a_\alpha(0))\, D^\alpha E_0 g + \sum_{|\alpha|<m} a_\alpha(x)\, D^\alpha E_0 g.$$

By Hölder's inequality, (17.1.5) and conditions (i) and (iii), we have

$$\|Rg\|_{L^p(X)} \leqq \tfrac{1}{2}\|g\|_{L^p(X)}, \qquad g \in L^p(X),$$

if X is sufficiently small. Thus $I+R$ is then invertible, and $E=E_0(I+R)^{-1}$ has properties (17.1.1) and (17.1.2) by (17.1.5) and the fact that

$$P(x,D)\,Ef = (I+R)(I+R)^{-1} f = f.$$

Finally, if $f=P(x,D)u$, $u \in C_0^\infty(X)$, then the unique solution of the equation $g+Rg=f$ is $g=p(D)u$, for $E_0 g = u$, hence

$$p(D)u + Rp(D)u = p(D)u + \sum_{|\alpha|=m} (a_\alpha(x)-a_\alpha(0)) D^\alpha u + \sum_{|\alpha|<m} a_\alpha(x)\, D^\alpha u$$

is equal to $P(x,D)u$ in X. This completes the proof.

If one replaces the L^p conditions by Hölder conditions one obtains the following theorem instead:

Theorem 17.1.1′. *Assume that for some $\gamma \in (0,1)$ the coefficients of $P(x,D)$ are in C^γ in a neighborhood of 0, and that $P_m(0,D)$ is elliptic. If X is a sufficiently small ball with center at 0 then there exists a linear operator E in $\bar{C}^\gamma(X)$ such that*

$$\bar{C}^\gamma(X) \ni f \mapsto D^\alpha E f \in \bar{C}^\gamma(X) \quad \textit{is continuous if } |\alpha| \leqq m; \tag{17.1.1$'$}$$

$$P(x,D)\,Ef = f, \quad f \in \bar{C}^\gamma(X); \tag{17.1.2$'$}$$

$$EP(x,D)u = u \quad \textit{if } u \in C_0^\infty(X). \tag{17.1.3$'$}$$

Here $\bar{C}^\gamma(X)$ is the set of all continuous functions in X such that the norm

$$\sup_{x\in X}|g(x)|+\sup_{x,y\in X}|g(x)-g(y)|/|x-y|^\gamma$$

is finite. If X has radius r, then a C^γ extension to the whole space is given by

$$\begin{aligned} g_0(x)&=g(x), && x\in X;\\ g_0(x)&=g(rx/|x|)(2-|x|/r), && r\leqq|x|\leqq 2r;\\ g_0(x)&=0, && |x|>2r. \end{aligned}$$

The proof of Theorem 17.1.1′ is identical to that of Theorem 17.1.1 except that g_0 is defined in this way and that (17.1.4) is replaced by the continuity in C^γ when $|\alpha|=m$, which follows from Theorem 7.9.6. We leave the details for the reader since the result will never be used here.

By a slight twist of the proof of Theorem 17.1.1 one can prove a logarithmic convexity theorem for the L^p norms of the derivatives which will be useful later on. To shorten the proofs we exclude lower order terms now. First we prove a lemma.

Lemma 17.1.2. *If $P(D)$ is homogeneous and elliptic of order m, then*

$$\sum_{|\alpha|\leqq m} A^{m-|\alpha|}\|D^\alpha v\|_{L^p}\leqq C(\|P(D)v\|_{L^p}+A^m\|v\|_{L^p}) \tag{17.1.6}$$

if $A>0$ and $D^\alpha v\in L^p$, $|\alpha|\leqq m$.

Proof. Introducing Ax as a new variable instead of x makes A disappear in (17.1.6) so we may assume in the proof that $A=1$. We define F_0 as in the proof of Theorem 17.1.1, thus $P(D)F_0=\delta+\omega$ where $\omega\in\mathscr{S}$. Then we have

$$D^\alpha v=D^\alpha F_0 * P(D)v-(D^\alpha\omega)*v,$$

and (17.1.6) follows since $D^\alpha\omega\in L^1$ and $D^\alpha F_0$ satisfies the hypotheses of Theorem 7.9.5.

Remark. It follows from the proof that C can be taken independent of P if P varies in a compact set of elliptic polynomials of degree m.

Theorem 17.1.3. *Assume that $P_m(x,D)$ satisfies the hypotheses* (i) *and* (ii) *above in a compact neighborhood K of* 0. *Let $X\subset K$ be an open set, and denote by $d(x)$ the distance from $x\in X$ to $\complement X$. If $D^\alpha u\in L^p(X)$, $|\alpha|\leqq m$, it follows then that*

$$\|d(x)^{|\alpha|}D^\alpha u\|_{L^p(X)}\leqq C(\|d(x)^m P_m(x,D)u\|_{L^p(X)}+\|u\|_{L^p(X)})^{|\alpha|/m}\|u\|_{L^p(X)}^{1-|\alpha|/m}, \tag{17.1.7}$$

where C is independent of X.

Proof. Let $B=B(y,R)$ be a ball with radius R and center $y\in X$ with $d(y)\geqq 2R$. Set $\chi_B(x)=\chi((x-y)/R)$ with a fixed $\chi\in C_0^\infty(B(0,1))$ which is equal to 1 in $B(0,\frac{1}{2})$. Applying (17.1.6) to $P(D)=P_m(y,D)$ and $v=\chi_B u$ gives with another C

$$\begin{aligned}\sum_{|\alpha|\leqq m} A^{p(m-|\alpha|)} \int_{B(y,\frac{1}{2}R)} |D^\alpha u|^p\,dx &\leqq C\Big(\int_{B(y,R)} |P_m(x,D)u|^p\,dx+\varepsilon(R)\sum_{|\alpha|=m}\int_{B(y,R)} |D^\alpha u|^p\,dx \\ &\quad+\sum_{|\alpha|<m} R^{-p(m-|\alpha|)}\int_{B(y,R)} |D^\alpha u|^p\,dx+A^{pm}\int_{B(y,R)} |u|^p\,dx\Big).\end{aligned}$$

Here we have expanded $P(D)(\chi_B u)$ by Leibniz' formula and estimated $\chi_B(P_m(y,D)-P_m(x,D))\,u(x)$ by means of the modulus of continuity ε of the coefficients. Thus $\varepsilon(R)\to 0$ when $R\to 0$. Now we take $A=M/R$ where M is a large constant and multiply by R^{pm}. This gives

$$\begin{aligned}\sum_{|\alpha|\leqq m} M^{p(m-|\alpha|)} R^{p|\alpha|} \int_{B(y,\frac{1}{2}R)} |D^\alpha u|^p\,dx &\leqq C\Big(\int_{B(y,R)} |R^m P_m(x,D)u|^p\,dx+\varepsilon(R)\sum_{|\alpha|=m}\int_{B(y,R)} |R^m D^\alpha u|^p\,dx \\ &\quad+\sum_{|\alpha|<m} R^{p|\alpha|}\int_{B(y,R)} |D^\alpha u|^p\,dx+M^{pm}\int_{B(y,R)} |u|^p\,dx\Big).\end{aligned}$$

With some small R_0 to be chosen later we define

$$R(y)=\min(R_0, d(y)/2)$$

and integrate with respect to $R(y)^{-n}\,dy$ over X. Since $|R(x)-R(y)|\leqq|x-y|/2$ it follows if $|x-y|<R(y)$ that $|R(y)-R(x)|<R(y)/2$, hence

$$R(y)/2<R(x)<3R(y)/2.$$

On the other hand, if $|x-y|<2R(x)/5$ then $|R(y)-R(x)|<R(x)/5$ so

$$4R(x)/5<R(y)<6R(x)/5.$$

Hence

$$\begin{aligned}\int_{x\in B(y,R(y))} dy/R(y)^n &\leqq (3/2)^n \int_{|x-y|<2R(x)} dy/R(x)^n = 3^n \int_{|y|<1} dy,\\ \int_{x\in B(y,\frac{1}{2}R(y))} dy/R(y)^n &\geqq (5/6)^n \int_{|x-y|<2R(x)/5} dy/R(x)^n = 3^{-n} \int_{|y|<1} dy.\end{aligned}$$

With a new constant C independent of R_0 it follows that

$$\begin{aligned}\sum_{|\alpha|\leqq m} M^{p(m-|\alpha|)} \int |R(x)^{|\alpha|} D^\alpha u|^p\,dx &\leqq C\Big(\int |R(x)^m P_m(x,D)u|^p\,dx+\varepsilon(R_0)\sum_{|\alpha|=m}\int |R(x)^m D^\alpha u|^p\,dx \\ &\quad+\sum_{|\alpha|<m}\int |R(x)^{|\alpha|} D^\alpha u|^p\,dx+M^{pm}\int |u|^p\,dx\Big).\end{aligned}$$

Choose R_0 so small that $C\varepsilon(R_0)<\frac{1}{2}$. When $M\geqq M_0$, say, we can then cancel the two sums on the right-hand side against half of the left-hand side and obtain

$$M^{m-|\alpha|}\|R(x)^{|\alpha|}D^\alpha u\|_{L^p}\leqq C(\|R(x)^m P_m(x,D)u\|_{L^p}+M^m\|u\|_{L^p}).$$

We choose $M=M_0$ if $\|R(x)^m P_m(x,D)u\|_{L^p}<M_0^m\|u\|_{L^p}$; otherwise we take M so that

$$M^m\|u\|_{L^p}=\|R(x)^m P_m(x,D)u\|_{L^p},$$

which gives (17.1.7).

Corollary 17.1.4. *Assume that* P_m *satisfies the hypotheses* (i), (ii) *in a neighborhood* K *of* 0. *If* $D^\alpha u\in L^p$ *in* $K\setminus\{0\}$ *for* $|\alpha|\leqq m$ *and*

$$\int_{R<|x|<2R}|u|^p\,dx=O(R^N),\qquad R\to 0,\tag{17.1.8}$$

$$|P_m(x,D)u|\leqq C\sum_{|\alpha|<m}|D^\alpha u|\,|x|^{|\alpha|-m}\qquad in\ K\setminus\{0\}\tag{17.1.9}$$

then it follows if $|\alpha|\leqq m$ *that*

$$\int_{R<|x|<2R}|R^{|\alpha|}D^\alpha u|^p\,dx=O(R^N),\qquad R\to 0.\tag{17.1.10}$$

Proof. We can apply Theorem 17.1.3 with $X=B(0,2R)\setminus B(0,R)$ if R is small. Then

$$d(x)^m|P_m(x,D)u|\leqq C\sum_{|\alpha|<m}d(x)^{|\alpha|}|D^\alpha u|$$

because $d(x)\leqq R\leqq|x|$. Hence it follows from (17.1.7) that

$$S=\sum_{|\alpha|<m}\|d^{|\alpha|}D^\alpha u\|_{L^p(X)}\leqq C_1 S^{(m-1)/m}\|u\|^{1/m}_{L^p(X)}.$$

Thus

$$\sum_{|\alpha|<m}\int_{4R<3|x|<5R}|(R/3)^{|\alpha|}D^\alpha u|^p\,dx\leqq\sum_{|\alpha|<m}\|d^{|\alpha|}D^\alpha u\|^p_{L^p(X)}$$

$$\leqq C_1^{mp}\|u\|^p_{L^p(X)}=O(R^N),$$

which proves (17.1.10) for $|\alpha|<m$. Another application of (17.1.7) gives (17.1.10) when $|\alpha|=m$ also.

With applications to global existence theory in mind we shall discuss in Section 17.2 whether a solution u of a differential equation with principal symbol P_m must be zero when (17.1.8) is valid for all N (or, equivalently, if (17.1.10) is valid for all α with $|\alpha|<m$ and all N). We shall then have to assume that the coefficients of P_m are Lipschitz continuous, that is, $|a_\alpha(x)-a_\alpha(y)|\leqq C|x-y|$, $|\alpha|=m$. Then we can define $P_m(x,D)u$ in the distribu-

tion sense if $D^\alpha u\in L^p$, $|\alpha|<m$, and Theorem 17.1.3 as well as Corollary 17.1.4 can be improved by means of Friedrichs' lemma:

Lemma 17.1.5. *Let $v\in L^p(\mathbb{R}^n)$ and let $|a(x)-a(y)|\leqq M|x-y|$ if $x,y\in\mathbb{R}^n$. If $\phi\in C_0^\infty$ and $\phi_\varepsilon(x)=\phi(x/\varepsilon)\varepsilon^{-n}$, then*

$$\|(aD_j v)*\phi_\varepsilon-a(D_j v*\phi_\varepsilon)\|_{L^p}\leqq M\,\|v\|_{L^p}\int(|\phi|+|y|\,|D_j\phi|)\,dy. \tag{17.1.11}$$

For fixed v the left-hand side tends to 0 when $\varepsilon\to 0$.

Proof. Since C_0^∞ is dense in L^p we may assume that $v\in C_0^\infty$, and it suffices to prove (17.1.11) since it is then obvious that the limit is 0. The quantity to estimate is

$$\begin{aligned}&|\int(a(x-y)-a(x))(D_j v)(x-y)\,\phi_\varepsilon(y)\,dy|\\&\quad=|\int(a(x-y)-a(x))\,v(x-y)\,D_j\phi_\varepsilon(y)\,dy-\int(D_j a)(x-y)\,v(x-y)\,\phi_\varepsilon(y)\,dy|\\&\quad\leqq M\int|v(x-y)|\,(|y|\,|D_j\phi_\varepsilon(y)|+|\phi_\varepsilon(y)|)\,dy.\end{aligned}$$

(17.1.11) follows now from Minkowski's inequality since

$$\int(|\phi_\varepsilon(y)|+|y|\,|D_j\phi_\varepsilon(y)|)\,dy$$

is independent of ε.

Let us now return to Theorem 17.1.3 assuming only that $D^\alpha u\in L^p(X)$, $|\alpha|<m$, but that a_α are Lipschitz continuous and that $P_m(x,D)u\in L^p(X)$. Let $\chi_0,\chi_1\in C_0^\infty(X)$, $\chi_1=1$ in a neighborhood of supp χ_0, and set $v=\chi_0 u$. Then $v\in\mathscr{E}'(X)$ and $D^\alpha v\in L^p$, $|\alpha|<m$, $P_m(x,D)v\in L^p$. Choose $\phi\in C_0^\infty$ with $\int\phi\,dx=1$ and set $v_\varepsilon=v*\phi_\varepsilon$ where $\phi_\varepsilon(x)=\phi(x/\varepsilon)/\varepsilon^n$. Then $v_\varepsilon\in C_0^\infty$ and if $b_\alpha=\chi_1 a_\alpha$ we have for small ε

$$P_m(x,D)\,v_\varepsilon=\sum_{|\alpha|=m}b_\alpha D^\alpha v_\varepsilon\to P_m(x,D)\,v \text{ in } L^p$$

by Lemma 17.1.5 since $P_m(x,D)\,v=\sum b_\alpha D^\alpha v$. Hence we can apply (17.1.7) to $v_\varepsilon-v_\delta$ and conclude that $D^\alpha v_\varepsilon$ has a limit in L^p when $\varepsilon\to 0$ if $|\alpha|=m$. Hence $D^\alpha u\in L^p_{\mathrm{loc}}(X)$ when $|\alpha|\leqq m$. The estimate (17.1.7) is therefore true if X is replaced by $\{x\in X;\,d(x)>\rho\}$. Letting $\rho\to 0$ we obtain (17.1.7) as it stands. Thus Theorem 17.1.3 and Corollary 17.1.4 are valid when a_α are Lipschitz continuous and $D^\alpha u\in L^p$, $|\alpha|<m$.

17.2. Unique Continuation Theorems

We shall begin with a unique continuation theorem similar to Theorem 8.6.5 where operators of higher order are allowed. Let

$$P_m(x,D)=\sum_{|\alpha|=m}a_\alpha(x)\,D^\alpha$$

be defined in an open set $X \subset \mathbb{R}^n$ and assume

(i) a_α is Lipschitz continuous in X,

(ii) P_m is elliptic in X.

By Σ we denote the closed conic set

$$\Sigma = \{(x,N) \in T^*(X) \setminus 0;\ P_m(x,\xi+\tau N) \text{ has a zero } \tau \text{ of multiplicity} \geqq 2 \text{ with } \xi+\tau N \neq 0 \text{ for some } \xi \in \mathbb{R}^n\}. \tag{17.2.1}$$

Of course τ cannot be real then.

Theorem 17.2.1. *If $D^\alpha u \in L^2_{loc}(X)$, $|\alpha|<m$, and $P_m(x,D)u \in L^2_{loc}(X)$,*

$$|P_m(x,D)u| \leqq C \sum_{|\alpha|<m} |D^\alpha u| \quad \text{in } X \tag{17.2.2}$$

then $\bar{N}(\operatorname{supp} u) \subset \Sigma$, where Σ is defined by (17.2.1).

For the notation $\bar{N}$ and the global uniqueness results which follow from Theorem 17.2.1 we refer to Sections 8.5 and 8.6. The definitions of Σ and of $\bar{N}$ are both local and invariant under local diffeomorphisms so it is sufficient to prove that if $0 \in X$ and $(0,N) \notin \Sigma$, $N=(0,\dots,0,1)$ then $u=0$ in a neighborhood of 0 if $\operatorname{supp} u \cap \{x; x_n \geqq 0\} \subset \{0\}$. This will be done by means of estimates with respect to high powers of a weight function with maximum in the support of u taken at 0 only.

Set $p(\xi)=P_m(0,\xi)$. Then the hypothesis $(0,N)\notin\Sigma$ means that $p(\xi+i\tau N)$ and $p^{(n)}(\xi+i\tau N)=\partial p(\xi+i\tau N)/\partial\xi_n$ have no common zero $(\xi,\tau)\in\mathbb{R}^{n+1}\setminus\{0\}$. Thus

$$\sum_{|\alpha|\leqq m} \tau^{2(m-|\alpha|)}|\xi^\alpha|^2 \leqq C(|p(\xi+i\tau N)|^2+\tau^2|p^{(n)}(\xi+i\tau N)|^2); \quad (\xi,\tau)\in\mathbb{R}^{n+1}; \tag{17.2.3}$$

for both sides are homogeneous of degree $2m$ and can only vanish if $\tau=0$ and $p(\xi)=0$, that is $\xi=0$. Next we need an identity of Treves which is closely related to the commutation relations.

Lemma 17.2.2. *Let $Q(x)=\sum a_j x_j+\sum b_j x_j^2/2$ be a real quadratic polynomial in $\mathbb{R}^n$ and let $P(D)$ be a differential operator with constant coefficients. If $u\in C_0^\infty(\mathbb{R}^n)$ and $v=u\,e^{Q/2}$ then*

$$\int |P(D)u|^2 e^Q dx = \int |P(D+iQ'/2)v|^2 dx = \int \sum_a \left|\bar{P}^{(\alpha)}(D-iQ'/2)v\right|^2 b^\alpha/\alpha!\,dx. \tag{17.2.4}$$

Proof. The first equality is obvious since $D_j u=e^{-Q/2}(D_j+i\partial_j Q/2)v$. The adjoint of $D_j+i\partial_j Q/2$ is $D_j-i\partial_j Q/2$ so we must show that

$$\bar{P}(D-iQ'/2)\,P(D+iQ'/2)=\sum P^{(\alpha)}(D+iQ'/2)\,\bar{P}^{(\alpha)}(D-iQ'/2)\,b^\alpha/\alpha!. \tag{17.2.5}$$

Now the commutators

$$[D_j - i\partial_j Q/2, D_k + i\partial_k Q/2] = \partial_j \partial_k Q = b_j \delta_{jk}$$

are the same as the commutators of ∂_j and $b_k x_k$. Since as operators

$$\bar{P}(\partial) P(bx) = \sum (\partial^\alpha P(bx)) \bar{P}^{(\alpha)}(\partial)/\alpha!$$

by Leibniz' rule and this is a purely algebraic consequence of the commutation relations, it follows that (17.2.5) holds.

The following is the crucial estimate in the proof of Theorem 17.2.1.

Proposition 17.2.3. *Let* $P_m(x, D)$ *satisfy conditions* (i) *and* (ii) *above in a neighborhood of* 0 *and assume that* $(0, N) \notin \Sigma$. *Then there is a neighborhood* $X_0 \subset X$ *of* 0 *such that with* $\phi(x) = x_n + x_n^2/2$ *we have for small* $\varepsilon > 0$ *and large* $\tau > 0$

$$(17.2.6) \qquad \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\phi} dx \\ \leq C \int |P_m(\varepsilon x, D) u|^2 e^{2\tau\phi} dx, \quad u \in C_0^\infty(X_0).$$

Proof. If we write $v(x) = u(x) e^{\tau\phi(x)}$ then

$$Du = e^{-\tau\phi}(D + i\tau\phi')v \quad \text{and} \quad Dv = e^{\tau\phi}(D - i\tau\phi')u.$$

Apart from the size of the constant, (17.2.6) is therefore equivalent to

$$(17.2.6)' \qquad \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha v|^2 dx \\ \leq C \int |P_m(\varepsilon x, D + i\tau\phi')v|^2 dx, \quad v \in C_0^\infty(X_0).$$

Assume first that the coefficients of P_m are *constant*, thus $P_m = p$. If we apply (17.2.4) with $P = p$ and $Q = 2\tau\phi$ it follows that

$$(17.2.7) \qquad \int |\bar{p}(D - i\tau\phi')v|^2 dx + 2\tau \int |\bar{p}^{(n)}(D - i\tau\phi')v|^2 dx \\ \leq \int |p(D + i\tau\phi')v|^2 dx.$$

By (17.2.3) and Parseval's formula we have for all $v \in C_0^\infty(\mathbb{R}^n)$

$$(17.2.3)' \qquad \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} \int |D^\alpha v|^2 dx \leq C(\int |\bar{p}(D - i\tau N)v|^2 dx \\ + \tau^2 \int |\bar{p}^{(n)}(D - i\tau N)v|^2 dx)$$

If $v \in C_0^\infty(X_0)$ and $|x| < \delta$ in X_0, it follows from (17.2.3)′ that

$$(17.2.8) \qquad \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} \int |D^\alpha v|^2 dx \leq 2C(\int |\bar{p}(D - i\tau\phi')v|^2 dx \\ + \tau^2 \int |\bar{p}^{(n)}(D - i\tau\phi')v|^2 dx) + C'(1 + \delta^2\tau^2) \sum_{|\alpha| \leq m} \tau^{2(m-1-|\alpha|)} \int |D^\alpha v|^2 dx.$$

When δ is small and τ is large we have $C'(1+\delta^2\tau^2)<\tau^2/2$ which allows us to cancel the last sum against half of the left hand side. (17.2.6) is then a consequence of (17.2.8) and (17.2.7).

To complete the proof we need an elementary lemma which allows us to handle variable coefficients. We denote the L^2 norm simply by $\| \ \|$.

Lemma 17.2.4. *Let $X\subset\mathbb{R}^n$ be an open set, and let A be a Lipschitz continuous function with $|A(x)-A(y)|\leqq L|x-y|$ for $x, y\in X$. Then*

$$\left|\int A(x)(D^\alpha u(x)\overline{D^\beta v(x)}-D^\beta u(x)\overline{D^\alpha v(x)})\,dx\right|\leqq|\alpha+\beta|\,LM$$

if $u, v\in C_0^\infty(X)$ and

$$\|D^{\alpha'}u\|\,\|D^{\beta'}v\|\leqq M \qquad \text{when } |\alpha'+\beta'|<|\alpha+\beta|, \quad \max(|\alpha'|,|\beta'|)\leqq\max(|\alpha|,|\beta|).$$

Also the last inequality can be taken strict when $|\alpha|\neq|\beta|$.

Proof. This is obvious when $\alpha+\beta=0$. If $|\alpha+\beta|=1$ we just have to note that

$$\int A(x)(D_j u(x)\overline{v(x)}-u(x)\overline{D_j v(x)})\,dx=-\int D_j A(x)u(x)\overline{v(x)}\,dx.$$

An integration by parts also gives the statement when $|\alpha|=|\beta|=1$,

$$\int A(x)(D_j u(x)\overline{D_k v(x)}-D_k u(x)\overline{D_j v(x)})\,dx=$$
$$-\int u(x)(D_jA(x)\overline{D_k v(x)}-D_kA(x)\overline{D_j v(x)})\,dx.$$

These two identities allow us to exchange indices between α and β and transfer excess derivatives at a cost of LM for each index affected.

End of Proof of Proposition 17.2.3. Writing $P_m(0,D)=p(D)$ and $r(x,D)=P_m(x,D)-p(D)$ now, we know by hypothesis that the coefficients of $r(\varepsilon x,D)$ and their Lipschitz constants are $O(\varepsilon)$ in X. With the notation in the first part of the proof we form

$$\int|P_m(\varepsilon x,D+i\tau\phi')v|^2\,dx-\int|\bar{P}_m(\varepsilon x,D-i\tau\phi')v|^2\,dx.$$

Inserting $P_m=p+r$ we first obtain the terms

$$\int|p(D+i\tau\phi')v|^2\,dx-\int|\bar{p}(D-i\tau\phi')v|^2\,dx\geqq2\tau\int|\bar{p}^{(n)}(D-i\tau\phi')v|^2\,dx.$$

The other terms where no derivative falls on ϕ' are of the form

$$\tau^{2m-|\alpha|-|\beta|}\int A(x)(D^\alpha v(x)\overline{D^\beta v(x)}-D^\beta v(x)\overline{D^\alpha v(x)})\,dx; \qquad |\alpha|\leqq m,\ |\beta|\leqq m;$$

where the Lipschitz constant of A is $O(\varepsilon)$. These terms can be estimated by means of Lemma 17.2.4. In addition there are terms of the form

$$\tau^\nu\int A(x)\,D^\alpha v(x)\overline{D^\beta v(x)}\,dx; \qquad \nu+|\alpha|+|\beta|<2m, \qquad |\alpha|\leqq m,\ |\beta|\leqq m;$$

where $\sup|A|=O(\varepsilon)$. Thus

$$\|\bar{P}_m(\varepsilon x, D-i\tau\phi')v\|^2+\tau\,\|\bar{p}^{(n)}(D-i\tau\phi')v\|^2 \leqq \|P_m(\varepsilon x, D+i\tau\phi')v\|^2+C\varepsilon\sum_{|\alpha|\leqq m}\tau^{2(m-|\alpha|)-1}\|D^\alpha v\|^2.$$

If we observe that (17.2.8) remains valid with $\bar{p}(D-i\tau\phi')$ replaced by $\bar{P}_m(\varepsilon x, D-i\tau\phi')$ provided that $\varepsilon<\delta$, we complete the proof of (17.2.6) just as in the constant coefficient case.

Proof of Theorem 17.2.1. We recall that it suffices to prove that if $0\in X$ and $(0,N)\notin\Sigma$, $N=(0,\dots,0,1)$ then $u=0$ in a neighborhood of 0 if $\operatorname{supp} u\cap\{x; x_n\geqq 0\}\subset\{0\}$. In doing so we set $u_\varepsilon(x)=u(\varepsilon x)$ where ε is chosen so small that (17.2.6) is valid for a neighborhood $X_0\subset X/\varepsilon$ of 0. Let $\chi\in C_0^\infty(X_0)$ be equal to 1 in a neighborhood V of 0, and set $U=\chi u_\varepsilon$. If $P_m(x,D)u=f$ then

$$P_m(\varepsilon x, D)\,U=\varepsilon^m\chi(x)f(\varepsilon x)+\sum_{0<|\alpha|\leqq m} D^\alpha\chi\,P_m^{(\alpha)}(\varepsilon x, D)\,u_\varepsilon/\alpha!$$

which implies that $P_m(\varepsilon x, D)\,U\in L^2$ and that, by (17.2.2),

$$|P_m(\varepsilon x, D)\,U|\leqq C\sum_{|\alpha|<m}\varepsilon^{m-|\alpha|}|D^\alpha U| \quad \text{in } V.$$

By the remarks at the end of Section 17.1 we have $D^\alpha U\in L^2$ when $|\alpha|\leqq m$, so it is clear that (17.2.6) may be applied to U. If $\operatorname{supp}\chi$ is small enough we have $\phi\leqq -c$ for some $c>0$ in $\operatorname{supp} U\smallsetminus V$. Hence we obtain using (17.2.6)

$$\tau^{\frac{1}{2}}\sum_{|\alpha|<m}\|e^{\tau\phi}D^\alpha U\|\leqq C\,\|e^{\tau\phi}P_m(\varepsilon x, D)\,U\|\leqq C'\sum_{|\alpha|<m}\|e^{\tau\phi}D^\alpha U\|+C''e^{-c\tau}.$$

For large τ it follows that

$$\tau^{\frac{1}{2}}\sum_{|\alpha|<m}\|e^{\tau\phi}D^\alpha U\|\leqq 2\,C''e^{-c\tau}.$$

Hence $U=0$ when $\phi>-c$, which proves the theorem.

In the second order case the following lemma shows that the set Σ has a very simple description:

Lemma 17.2.5. *Let p be a quadratic form in $\mathbb{R}^n$ with complex coefficients which is elliptic, that is, $p(\xi)\neq 0$ when $0\neq\xi\in\mathbb{R}^n$. If $N\in\mathbb{R}^n\smallsetminus 0$ and $\xi\in\mathbb{R}^n\smallsetminus\mathbb{R}N$, $n\neq 2$, it follows that the equation $p(\xi+\tau N)=0$ has one root with* $\operatorname{Im}\tau>0$ *and one with* $\operatorname{Im}\tau<0$. *When $n=2$ the roots are distinct unless p is the square of a linear form.*

Proof. $\mathbb{R}^n\smallsetminus\mathbb{R}N$ is connected if $n>2$. Since $p(\xi+\tau N)$ has no real zero if $\xi\in\mathbb{R}^n\smallsetminus\mathbb{R}N$ it follows that the number of zeros with $\operatorname{Im}\tau>0$ is independent of ξ. Replacing ξ by $-\xi$ changes the sign of τ also so there must be one zero in each half plane. When $n=2$ there is a factorization $p(\xi)=L_1(\xi)L_2(\xi)$ with linear factors L_1 and L_2. They must be proportional if they have a common zero; and then they can be chosen equal.

If $m=2$ it follows that Σ is empty when $n>2$ and that $\Sigma=\bigcup(T_x^*\setminus 0)$ for all x such that $P_m(x,\xi)$ is the square of a linear form when $n=2$. If X is connected and $P_m(x,\xi)$ is real for some x then Σ is empty, for the two zeros of $P_m(x,\xi+\tau N)$ must remain in different half planes for reasons of continuity.

In what follows we shall only consider the second order case and shall then use the notation $p(x,D)$ instead of $P_m(x,D)$. We shall prove that if u satisfies a weakened form of (17.2.2) and vanishes of infinite order at a point where the coefficients are real, then u is equal to 0.

Theorem 17.2.6. *Let $p(x,D)=\sum a_{jk}(x)D_jD_k$ be an elliptic operator in a connected neighborhood X of 0 such that $a_{jk}(0)$ is real, a_{jk} is continuous in X, Lipschitz continuous in $X\setminus\{0\}$, and $|a'_{jk}|\leqq C|x|^{\delta-1}$ for some $\delta>0$. If $D^\alpha u\in L^2_{\text{loc}}$, $|\alpha|\leqq 1$, and*

$$|p(x,D)u|\leqq C\sum_{|\alpha|\leqq 1}|x|^{\delta+|\alpha|-2}|D^\alpha u|, \tag{17.2.2)$'$}$$

$$\int_{|x|<\varepsilon}|u|^2\,dx=O(\varepsilon^N),\qquad \varepsilon\to 0, \tag{17.2.9}$$

for every N, then $u=0$ in X.

Proof. Since (17.2.2)′ implies (17.1.9) it follows from Corollary 17.1.4 in the extended form discussed at the end of Section 17.1 that for $|\alpha|\leqq 2$ and all N

$$\int_{\varepsilon<|x|<2\varepsilon}|D^\alpha u|^2\,dx=O(\varepsilon^N),\qquad \varepsilon\to 0. \tag{17.2.9)$'$}$$

Hence u is the sum of a function in $H^{\text{loc}}_{(2)}(X)$ and a distribution with support at 0. However, no distribution with support at 0 is in L^2_{loc} so it follows that $u\in H^{\text{loc}}_{(2)}(X)$. By Theorem 17.2.1 it suffices to show that $u=0$ in a neighborhood of 0. Without restriction we may assume that $p(0,D)=\sum D_j^2$.

As in the proof of Proposition 14.7.1 we introduce polar coordinates in $\mathbb{R}^n\setminus\{0\}$ by writing $x=e^t\omega$ where $t\in\mathbb{R}$ and $\omega\in S^{n-1}$. Then we have

$$\partial/\partial x_j=e^{-t}(\omega_j\partial/\partial t+\Omega_j)$$

where Ω_j is a vector field in S^{n-1}. With the notation $p(x,D)=\sum a_{jk}(x)D_jD_k$ it follows that

$$p(x,D)=-e^{-2t}\sum a_{jk}(e^t\omega)(\omega_j(\partial/\partial t-1)+\Omega_j)(\omega_k\partial/\partial t+\Omega_k).$$

With $U(t,\omega)=u(e^t\omega)$ the inequality (17.2.2)′ can be written

$$|\sum a_{jk}(e^t\omega)(\omega_j(\partial/\partial t-1)+\Omega_j)(\omega_k\partial/\partial t+\Omega_k)U|\leqq C\sum_{|\alpha|\leqq 1}e^{\delta t}|U_\alpha| \tag{17.2.2)$''$}$$

where $U_\alpha=(\omega\,\partial/\partial t+\Omega)^\alpha U$. By assumption we have $a_{jk}(e^t\omega)=\delta_{jk}+O(e^{\delta t})$ as $t\to-\infty$, first order derivatives are $O(e^{\delta t})$, and

$$\sum(\omega_j(\partial/\partial t-1)+\Omega_j)(\omega_j\,\partial/\partial t+\Omega_j)=\partial^2/\partial t^2+(n-2)\,\partial/\partial t+\sum\Omega_j^2$$

since $\sum\omega_j\Omega_j=0$ and $\sum\Omega_j\omega_j=\sum r\,\partial\omega_j/\partial x_j=\sum r\,\partial(x_j/r)/\partial x_j=n-1$. The operator $\sum\Omega_j^2$ is the Laplace-Beltrami operator Δ_ω in the unit sphere. The adjoint of Ω_j as an operator in $L^2(S^{n-1})$ is $(n-1)\,\omega_j-\Omega_j$. In fact,

$$\begin{aligned}\int(\Omega_j u)\,v\,dx+\int u\,\Omega_j v\,dx&=\int\Omega_j(u\,v)\,dx=\int|x|\,\partial(u\,v)/\partial x_j\,dx-\iint\omega_j\,\partial(u\,v)/\partial r\,r^n\,d\omega\,dr\\&=-\int\omega_j u\,v\,dx+n\int\omega_j u\,v\,r^{n-1}\,d\omega\,dr\\&=(n-1)\int\omega_j u\,v\,dx.\end{aligned}$$

In spite of this Δ_ω is of course self-adjoint; indeed, we have

$$\sum((n-1)\,\omega_j-\Omega_j)^2=(n-1)^2-(n-1)\sum\Omega_j\omega_j+\sum\Omega_j^2=\sum\Omega_j^2.$$

In the proof of Theorem 17.2.1 the essential estimate (17.2.7) was obtained from (17.2.4) thanks to the positivity of b_j, that is, the convexity of the exponent ϕ. To obtain a similar effect we introduce for some ε with $0<\varepsilon<\delta$ a new variable T instead of t,

$$t=T+e^{\varepsilon T};\qquad dt/dT=1+\varepsilon\,e^{\varepsilon T}>0.$$

Note that $T<t<T+1<T/2$ if $T<-2$. After multiplication by $(1+\varepsilon e^{\varepsilon T})^2$ the operator in the left-hand side of (17.2.2)″ becomes

$$Q=\partial^2/\partial T^2+c(T)\,\partial/\partial T+(1+\varepsilon\,e^{\varepsilon T})^2\sum\Omega_j^2+\sum_{|\alpha|+j\leqq 2}c_{\alpha,j}(T,\omega)(\partial/\partial T)^j\,\Omega^\alpha.$$

Here $c(T)=(n-2)(1+\varepsilon\,e^{\varepsilon T})-\varepsilon^2\,e^{\varepsilon T}/(1+\varepsilon\,e^{\varepsilon T})$ is close to $n-2$ at $-\infty$, and

(17.2.10) $$c_{\alpha,j}=O(e^{\delta t}),\qquad dc_{\alpha,j}=O(e^{\delta t})\ \text{as}\ T\to-\infty.$$

(Note that this change of variables is not smooth in the original variables.) We shall prove that for some T_0

(17.2.11) $$\begin{aligned}&\sum_{j+|\alpha|\leqq 2}\tau^{3-2(j+|\alpha|)}\iint|(\partial/\partial T)^j\,\Omega^\alpha\,U|^2\,e^{-(2\tau-\varepsilon)T}\,d\omega\,dT\\&\leqq C\iint|QU|^2\,e^{-2\tau T}\,d\omega\,dT,\qquad U\in C_0^\infty((-\infty,T_0)\times S^{n-1}).\end{aligned}$$

(When $|\alpha|=2$ we define Ω^α for example as a product $\Omega_j\Omega_k$ with $j\leqq k$.) This will serve the same purpose as (17.2.6) did in the proof of Theorem 17.2.1.

Proof of (17.2.11). Set $U=e^{\tau T}\,V$ and

$$Q_\tau\,V=e^{-\tau T}\,Q(e^{\tau T}\,V).$$

Thus Q_τ is obtained from Q when $\partial/\partial T$ is replaced by $\partial/\partial T+\tau$. Then (17.2.11) is equivalent to

(17.2.11)′ $$\begin{aligned}&\sum_{j+|\alpha|\leqq 2}\tau^{3-2(j+|\alpha|)}\iint|(\partial/\partial T)^j\,\Omega^\alpha\,V|^2\,e^{\varepsilon T}\,d\omega\,dT\\&\leqq C\iint|Q_\tau\,V|^2\,d\omega\,dT,\qquad V\in C_0^\infty((-\infty,T_0)\times S^{n-1}).\end{aligned}$$