Structural Analysis with Finite Elements

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With 408 Figures and 26 Tables

Springer

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Preface

The finite element method has become an indispensible tool in structural analysis, and tells an unparalleled success story. With success, however, came criticism, because it was noticeable that knowledge of the method among practitioners did not keep up with success. Reviewing engineers complain that the method is increasingly applied without an understanding of structural behavior. Often a critical evaluation of computed results is missing, and frequently a basic understanding of the limitations and possibilities of the method are nonexistent.

But a working knowledge of the fundamentals of the finite element method *and* classical structural mechanics is a prerequisite for any sound finite element analysis. Only a well trained engineer will have the skills to critically examine the computed results.

Finite element modeling is more than preparing a mesh connecting the elements at the nodes and replacing the load by nodal forces. This is a popular model but this model downgrades the complex structural reality in such a way that—instead of being helpful—it misleads an engineer who is not well acquainted with finite element techniques.

The object of this book is therefore to provide a foundation for the finite element method from the standpoint of structural analysis, and to discuss questions that arise in modeling structures with finite elements.

What encouraged us in writing this book was that—thanks to the intensive research that is still going on in the finite element community—we can explain the principles of finite element methods in a new way and from a new perspective by making ample use of influence functions. This approach should appeal in particular to structural engineers, because influence functions are a genuine engineering concept and are thus deeply rooted in classical structural mechanics, so that the structural engineer can use his engineering knowledge and insight to assess the accuracy of finite element results or to discuss the modeling of structures with finite elements.

Just as a change in the elastic properties of a structure changes the Green's functions or influence functions of the structure so a finite element mesh effects a shift of the Green's functions.

We have tried to concentrate on ideas, because we considered these and not necessarily the technical details to be important. The emphasis should be on structural mechanics and not on programming the finite elements, and therefore we have also provided many illustrative examples.

Finite element technology was not developed by mathematicians, but by engineers (Argyris, Clough, Zienkiewicz). They relied on heuristics, their intuition and their engineering expertise, when in the tradition of medieval craftsmen they designed and tested elements without fully understanding the exact background. The results were empirically useful and engineers were grateful because they could suddenly tackle questions which were previously unanswerable. After these early achievements self-confidence grew, and a second epoch followed that could be called baroque: the elements became more and more complex (some finite element programs offered 50 or more elements) and enthusiasm prevailed. In the third phase, the epoch of "enlightment" mathematicians became interested in the method and tried to analyze the method with mathematical rigor. To some extent their efforts were futile or extremely difficult, because engineers employed "techniques" (reduced integration, nonconforming elements, discrete Kirchhoff elements) which had no analogy in the calculus of variations. But little by little knowledge increased, the gap closed, and mathematicians felt secure enough with the method that they could provide reliable estimates about the behavior of some elements. We thus recognize that mathematics is an essential ingredient of finite element technology.

One of the aims of this book is to teach structural engineers the theoretical foundations of the finite element method, because this knowledge is invaluable in the design of safe structures.

This book is an extended and revised version of the original German version. We have dedicated the web page `http://www.winfem.de` to the book. From this page the programs WINFEM (finite element program with focus on influence functions and adaptive techniques), BE-SLABS (boundary element analysis of slabs) and BE-PLATES (boundary element analysis of plates) can be downloaded by readers who want to experiment with the methods. Additional information can also be found on `http://www.sofistik.com`.

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Munich August 2003 *Casimir Katz*

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Preface to the second edition

One of the joys of writing a book is that the authors learn more about a subject. This does not stop after a book is finished. So we have added additional sections to the text

- The Dirac energy
- How to predict changes
- The influence of a single element
- Retrofitting structures
- Generalized finite element methods (X-FEM)
- Cables
- Hierarchical elements
- Sensitivity analysis
- Weak form of influence functions

in the hope that these additional topics will also attract the readers' interest.

Kassel
Munich October 2006

Friedel Hartmann
Casimir Katz

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1. What are finite elements?

1.1 Introduction

In this introductory chapter various aspects of the FE method are studied, initially highlighting the key points.

1.2 Key points of the FE method

- FE method = restriction

Analyzing a structure with finite elements essentially amounts to constraining the structure (see Fig. 1.1), because the structure can only assume those shapes that can be represented by shape functions.

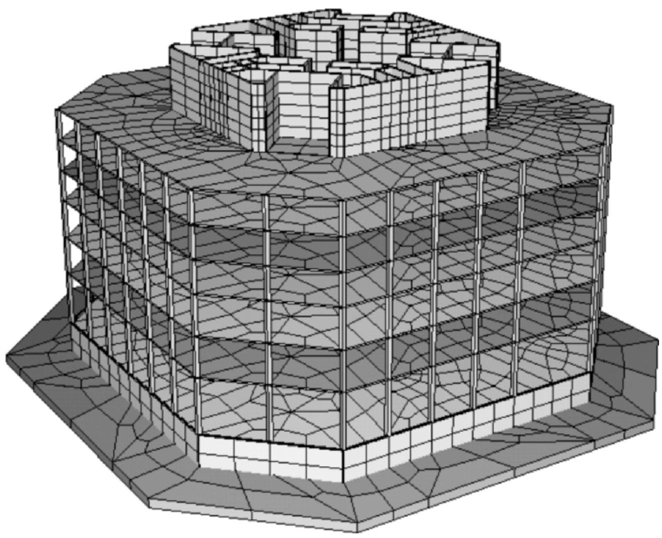


Fig. 1.1. The building can only execute movements that can be represented by shape functions

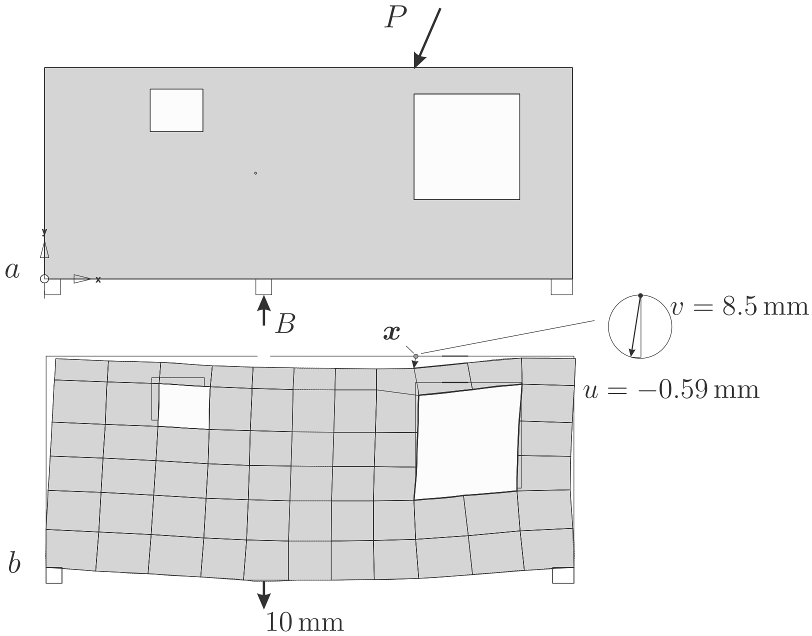


Fig. 1.2. Shear wall: **a)** support reaction B; **b)** the displacements observed at $\boldsymbol{x}$ if the support B moves in the vertical direction are a direct measure of the influence a (nearly concentrated point) load $\boldsymbol{P} = [P_x, P_y]^T$ has on the support B. About 85% of P_y and 6% of P_x will contribute to B. The better an FE program can model the movement of the support B, the better the accuracy

This is an important observation, because the accuracy of an FE solution depends fundamentally on how accurately a program can approximate the influence functions for stresses or displacements. Influence functions are displacements: they are the response of a structure to certain point loads. The more flexible an FE structure is, the better it can react to such point loads, and hence the better the accuracy of the FE solution; see Fig. 1.2.

- FE method = method of substitute load cases

It is possible to interpret the FE method as a method of substitute loadings or load cases, because in some sense all an FE program does is to replace the original load with a work-equivalent load, and solve that load case exactly. The important point is that structures are designed for these substitute loads not for the original loads.

- FE method = projection method

The shadow of a 3-D vector is that vector in the plane with the shortest distance to the tip of the vector.

The FE method is also a projection method, because the FE solution is the shadow of the exact solution when it is projected onto the trial space V_h, where

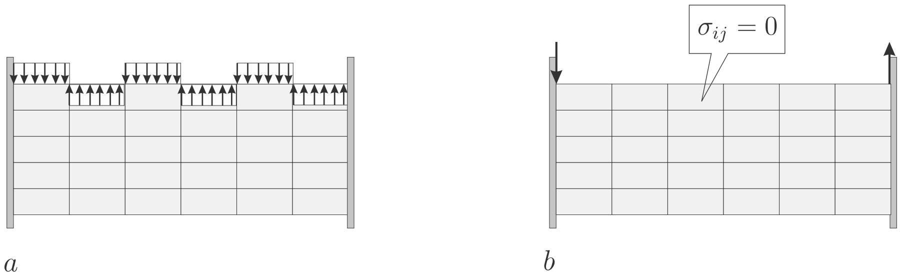


Fig. 1.3. Plate with alternating edge load: **a)** system and load; **b)** equivalent nodal forces

V_h contains all the deformations the FE structure can undergo. The metric applied in the projection is the strain energy: one chooses that deformation u_h in V_h whose distance to the exact solution u measured in units of strain energy is a minimum.

Let $\boldsymbol{u}$ denote the exact equilibrium position of a plate (subjected to some load), and let $\boldsymbol{u}_h$ be the FE approximation of this position. Now to correct the FE position, that is, to force the plate into the correct shape, a displacement field $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}_h$ must be added to $\boldsymbol{u}_h$.

Let σ^e_{ij} and ε^e_{ij} denote the stresses and strains caused by this displacement field $\boldsymbol{e}$. The FE solution guarantees that the energy needed to correct the FE solution is a minimum

$$a(\boldsymbol{e}, \boldsymbol{e}) = \int (\sigma^e_{xx}\ \varepsilon^e_{xx} + \sigma^e_{xy}\ \gamma^e_{xy} + \sigma^e_{yy}\ \varepsilon^e_{yy})\, d\Omega \quad \rightarrow \quad \text{minimum}\,. \tag{1.1}$$

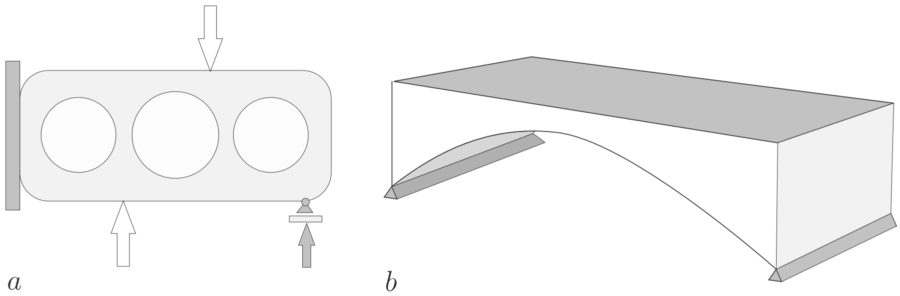
This is equivalent to saying[1] that the work needed to force the plate from its position $\boldsymbol{u}_h$ into the correct position $\boldsymbol{u}$ is a minimum. The effort cannot be made any smaller on the given mesh.

In a vertical projection the length of a shadow is always less than the length of the original vector (see *Bessel's inequality* [232]); this implies that the strain energy of the FE solution is always less than the strain energy of the exact solution. An engineer would say that the FE solution overestimates the stiffness of the structure.

The situation is different if a support of a structure is displaced. Then the FE projection is a skew projection (see Sect. 1.38, p. 187), that is, the shadow is longer than the original vector. This means that a greater effort is needed to displace a support of a more rigid structure than of a more flexible structure. But it will be seen later that even then a minimum principle still applies.

Because the FE solution is the shadow of the true solution, it cannot be improved on the same mesh. This is also why some load cases cannot be solved on an FE mesh. Each projection has a blind spot; see Fig. 1.3. The equivalent nodal forces at the free nodes cancel and so $\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{0}$.

[1] $a(\boldsymbol{u}, \boldsymbol{u}) = a(\boldsymbol{u}_h, \boldsymbol{u}_h) - 2\, a(\boldsymbol{u}_h, \boldsymbol{e}) + a(\boldsymbol{e}, \boldsymbol{e})$ and $a(\boldsymbol{u}_h, \boldsymbol{e}) = 0$



a *b*

Fig. 1.4. Theoretically these load cases cannot be solved with the FE method because the strain energy is infinite: **a)** Concentrated forces acting on a plate; **b)** concrete block placed on line supports

- FE method = energy method

An FE program thinks in terms of work and energy. Loads that contribute no work do not exist for an FE program. Nodal forces represent *equivalence classes* of loads. Loads that contribute the same amount of work are identical for an FE program.

In modern structural analysis, zero is replaced by vanishing work. In classical structural analysis a distributed load $p(x)$ is identical to a second load $p_h(x)$ if at each point $0 < x < l$ of the beam the load is the same:

$$p(x) = p_h(x) \qquad 0 < x < l \qquad \text{strong equal sign}\,. \tag{1.2}$$

In contrast, identity is based on a weaker concept in modern structural analysis. Two loads are considered identical if the virtual work is the same for any virtual displacement $\delta w(x)$:

$$\int_0^l p(x)\,\delta w(x)\,dx = \int_0^l p_h(x)\,\delta w(x)\,dx \qquad \text{for all } \delta w(x)\,. \tag{1.3}$$

This is the *weak equal sign*. If *all* really means *all* then of course the weak equal sign is identical to the strong equal sign. But in all other cases there remains a specific difference, in that equivalence is established only with regard to a finite set of virtual displacements δw.

Because the FE method is an energy method, problems in which the strain energy is infinite—theoretically at least—cannot be solved with this method; see Fig. 1.4.

- FE method = method of approximate influence functions

We will see that a mesh is only as good as the influence functions that can be generated on that mesh. According to Betti's theorem, the displacement $u(x)$ or the stress $\sigma_x(x)$ at a point x is the L_2-scalar product of the applied load p and the corresponding influence function (*Green's function*)

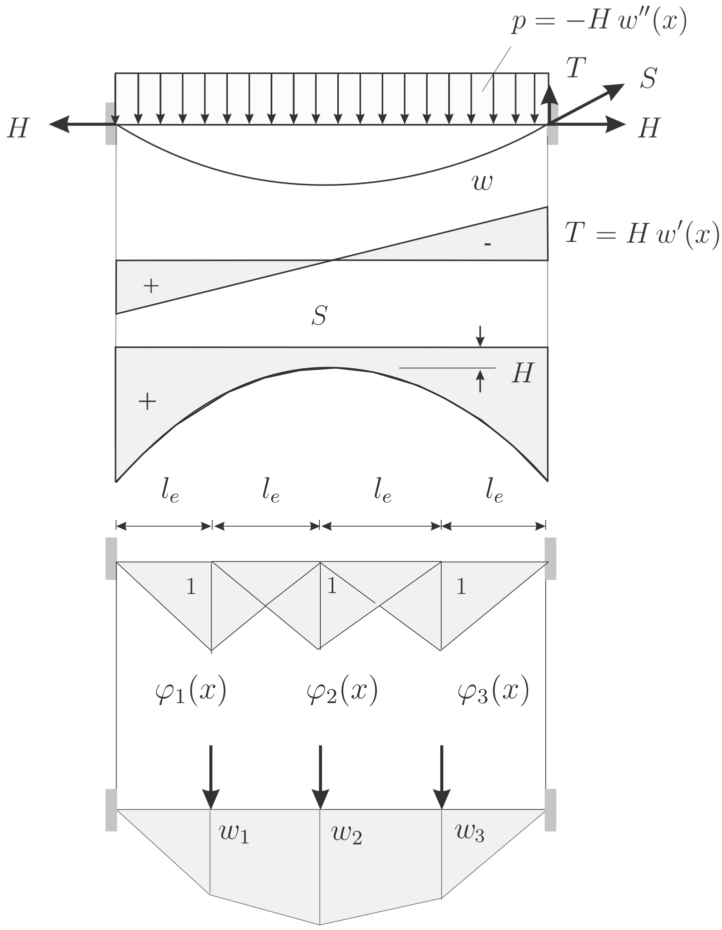


Fig. 1.5. FE analysis of a taut rope

$$u(x) = \int_0^l G_0(y,x)\, p\,(y)\, dy\,, \qquad \sigma_x(x) = \int_0^l G_1(y,x)\, p\,(y)\, dy\,. \tag{1.4}$$

All an FE program does is to replace the exact Green's functions with approximate Green's functions G_0^h and G_1^h, respectively. Therefore the error in an FE solution is proportional to the distance between the approximate and the exact Green's function:

$$u(x) - u_h(x) = \int_0^l \left[G_0(y,x) - G_0^h(y,x)\right]\, p\,(y)\, dy\,, \tag{1.5}$$

$$\sigma_x(x) - \sigma_x^h(x) = \int_0^l \left[G_1(y,x) - G_1^h(y,x)\right]\, p\,(y)\, dy\,. \tag{1.6}$$

1.3 Potential energy

To see these principles applied, we analyze a very simple structure, a taut rope (see Fig. 1.5).

Imagine that the rope is pulled taut by a horizontal force H and that it carries a distributed load p. The distribution of the vertical force V within the rope and the deflection w of the rope are to be calculated. The deflection w is the solution of the boundary value problem

$$-Hw''(x) = p(x) \qquad 0 < x < l \qquad w(0) = w(l) = 0\,. \tag{1.7}$$

The vertical (or transverse) force T is proportional to the slope w'

$$T = Hw'\,, \tag{1.8}$$

and the vector sum of H and T is the tension S in the rope

$$S = \sqrt{H^2 + T^2}\,. \tag{1.9}$$

The potential energy of the rope is the expression

$$\Pi(w) = \frac{1}{2}\int_0^l H(w')^2\,dx - \int_0^l p\,w\,dx = \frac{1}{2}\int_0^l \frac{T^2}{H} - \int_0^l p\,w\,dx\,. \tag{1.10}$$

For completeness we also note Green's first identity for the operator $-H\,w''$:

$$G(w,\hat{w}) = \int_0^l -H\,w''\,\hat{w}\,dx + [T\,\hat{w}]_0^l - \int_0^l \frac{T\,\hat{T}}{H}\,dx = 0 \tag{1.11}$$

because it encapsulates the structural mechanics of the rope.

To approximate the deflection $w(x)$ of the rope, the rope is subdivided into four linear elements: see Fig. 1.5. The first and the last node are fixed so that only the three internal nodes can be moved. Between the nodes the deflection is linear, that is the rope is only allowed to assume shapes that can be expressed in terms of the three unit displacements $\varphi_i(x)$ of the three internal nodes (see Fig. 1.5)

$$w_h(x) = w_1 \cdot \varphi_1(x) + w_2 \cdot \varphi_2(x) + w_3 \cdot \varphi_3(x)\,. \tag{1.12}$$

The nodal deflections, w_1, w_2, w_3, play the role of weights. They signal how much of each unit deflection is contained in w_h.

All these different shapes—let the numbers w_1, w_2, w_3 vary from $-\infty$ to $+\infty$—constitute the so-called *trial space* V_h.

The space V_h itself is a subset of a greater space, the *deformation space* V of the rope. The space V contains all deflection curves $w(x)$ that the rope can possibly assume under different loadings during its lifetime. It is obvious that the piecewise linear functions w_h in the subset V_h represent only a very small fraction of V.

The next question then is: what values should be chosen for the three nodal deflections w_1, w_2, w_3 of the FE solution? What is the optimal choice?

According to the principle of minimum potential energy, the true deflection w results in the lowest potential energy on V

$$\Pi(w) = \frac{1}{2}\int_0^l H(w')^2 dx - \int_0^l p\, w\, dx\,. \tag{1.13}$$

But if the exact solution w wins the competition on the big space V, it seems a good strategy to choose the nodal deflections w_i in such a way that the FE solution

$$w_h(x) = \sum_{i=1}^{3} w_i\, \varphi_i(x) \tag{1.14}$$

wins the competition on the small subset $V_h \subset V$. Then $\Pi(w_h)$ is as close as possible to $\Pi(w)$ on V_h.

Because each function w_h in V_h is uniquely determined by the nodal deflections w_i at the three interior nodes, i.e. the vector $\boldsymbol{w} = [w_1, w_2, w_3]^T$, the potential energy on V_h is a function of these three numbers only

$$\begin{aligned}
\Pi(w_h) = \Pi(\boldsymbol{w}) &= \frac{1}{2}\,\boldsymbol{w}^T\boldsymbol{K}\boldsymbol{w} - \boldsymbol{f}^T\boldsymbol{w}\\
&= \frac{1}{2}\,[w_1, w_2, w_3]\;\frac{4\,H}{l}\begin{bmatrix} 2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}\begin{bmatrix} w_1\\ w_2\\ w_3\end{bmatrix} - [f_1, f_2, f_3]\begin{bmatrix} w_1\\ w_2\\ w_3\end{bmatrix}\\
&= \frac{4\,H}{l}[w_1^2 - w_1\,w_2 + w_2^2 - w_2\,w_3 + w_3^2] - f_1\,w_1 - f_2\,w_2 - f_3\,w_3\,,
\end{aligned} \tag{1.15}$$

where the matrix $\boldsymbol{K}$ and the vector $\boldsymbol{f}$ have the elements

$$k_{ij} = \int_0^l H\varphi_i'\,\varphi_j'\,dx \qquad f_i = \int_0^l p\;\varphi_i\,dx = p\,l_e = p\,\frac{l}{4}\,. \tag{1.16}$$

Finding the minimum value of Π on V_h is therefore equivalent to finding the vector $\boldsymbol{w}$—the “address” of $w_h \in V_h$—for which the *function* $\Pi(\boldsymbol{w})$ becomes a minimum. A necessary condition is, that the first derivatives of the function $\Pi(\boldsymbol{w})$ vanish at this point $\boldsymbol{w}$:

$$\frac{\partial \Pi}{\partial w_i} = \sum_{j=1}^{3} k_{ij}\,w_j - f_i = 0\,, \qquad i = 1, 2, 3\,, \tag{1.17}$$

which leads to the system of equations

$$\boldsymbol{K}\boldsymbol{w} = \boldsymbol{f} \tag{1.18}$$

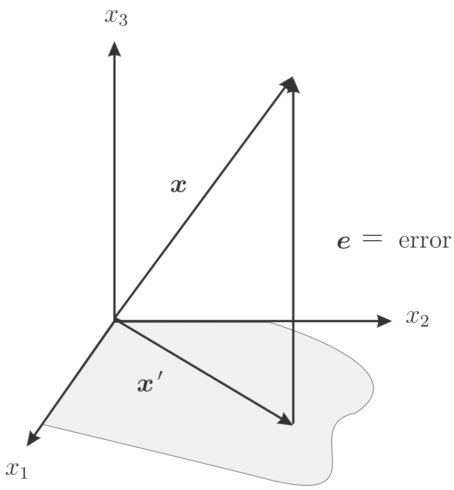


Fig. 1.6. The error $\boldsymbol{e}$ is orthogonal to the plane

or

$$\frac{4\,H}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \frac{p\,l}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{1.19}$$

which has the solution $w_1 = w_3 = 1.5\,p\,l^2/(16\,H)\,, w_2 = 2.0\,p\,l^2/(16\,H)$. Hence the deflection

$$w_h(x) = \frac{p\,l^2}{16\,H}\,[1.5 \cdot \varphi_1(x) + 2.0 \cdot \varphi_2(x) + 1.5 \cdot \varphi_3(x)] \tag{1.20}$$

is the best approximation on V_h.

1.4 Projection

Work is a scalar quantity, as are temperature and pressure. This is nearly the most important statement that can be made about work. Work is *force* × *displacement*. Work and energy are the same. The integral

$$\frac{1}{2}\int_0^l \frac{T^2}{H}\,dx\,, \qquad T = Hw'\,, \tag{1.21}$$

is the internal energy of the rope. It measures the strain energy stored in the rope.

Energy can also serve as a scale. It is the scale FE methods work with. Having a scale means having a topology, which in turn defines “far away” and “nearby”. To measure the length of a vector the *Euclidean norm* is used:

$$|\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}\,. \tag{1.22}$$

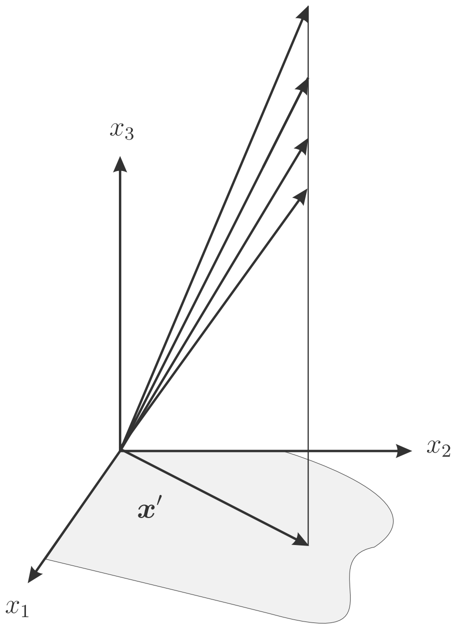


Fig. 1.7. All vectors have the same shadow $\boldsymbol{x}'$

In this *topology* two cities A and B are close neighbors if the difference between their position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (with reference to the origin of a map) is small:

$$|\boldsymbol{a} - \boldsymbol{b}| \qquad \text{“small”} \qquad \Longrightarrow \qquad \text{A and B are neighbors}. \tag{1.23}$$

Projections only make sense if distances can be measured. The shadow $\boldsymbol{x}'$ of a 3-D vector $\boldsymbol{x}$ is the vector in the plane which has the smallest distance to the tip of $\boldsymbol{x}$; see Fig. 1.6. The distance between the original vector and its shadow is the length of the vector

$$\boldsymbol{e} = \boldsymbol{x} - \boldsymbol{x}', \tag{1.24}$$

which points from the tip of the shadow to the tip of the vector $\boldsymbol{x}$. The shadow $\boldsymbol{x}'$ renders this distance a minimum

$$|\boldsymbol{e}| = \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - 0)^2} = \text{minimum}. \tag{1.25}$$

Any other vector $\tilde{\boldsymbol{x}}'$ in the plane has a greater distance from the vector $\boldsymbol{x}$

$$|\tilde{\boldsymbol{e}}| = |\boldsymbol{x} - \tilde{\boldsymbol{x}}'| > |\boldsymbol{e}| = |\boldsymbol{x} - \boldsymbol{x}'|. \tag{1.26}$$

This is the *first* feature of a projection: the shadow solves a minimum problem.

The *second* feature is that the residual vector, the error $\boldsymbol{e}$, is orthogonal to the $x_1 - x_2$-plane (assuming that the sun shines from straight above), because the scalar product between the error and the shadow is zero:

$$\boldsymbol{e}^T \boldsymbol{x}' = 0\,. \tag{1.27}$$

This is equivalent to saying that the shadow of the error $\boldsymbol{e}$ has no physical extent, but only if the line of sight coincides with the direction of the projection! Seen from any other direction the length of $\boldsymbol{e}$ is not zero. Hence a projection method is *blind with respect to errors which lie in the line of sight.* All vectors $\tilde{\boldsymbol{x}}$ that lie "above" the vector $\boldsymbol{x}$, which differ from $\boldsymbol{x}$ only by an additive term parallel to the line of sight (i.e., projection), have the same shadow; see Fig. 1.7.

The *third* feature is that the result of a projection cannot be improved. Repeating a projection changes nothing: the shadow of the shadow is the shadow. Which means that a projection method freezes after the first step, while other operations, such as squaring a number, can be repeated infinitely often.

The *fourth* feature of a projection is that the length of the shadow is shorter than the length of the original vector; see Fig. 1.7. This is not only true for vectors, but also for functions: the *Fourier series* $f_n(x)$ of a function $f(x)$ is the projection of $f(x)$ onto the trigonometric functions in the sense of the L_2-scalar product, and according to *Bessel's inequality* the length ($= L_2$-norm) of the Fourier series f_n is less than the L_2-norm of f:

$$||f_n||_0 = [\int_0^l f_n^2(x)\,dx]^{1/2} \leq [\int_0^l f^2(x)\,dx]^{1/2} = ||f||_0\,. \tag{1.28}$$

All this applies now to the FE method as well: the exact deflection curve $w \in V$ is projected onto a subspace V_h, and the shadow w_h is the FE solution.

In the case of the rope the space V_h contains all the deformations which are expansions in terms of the three unit displacements $\varphi_i(x)$,

$$w_h(x) = w_1 \cdot \varphi_1(x) + w_2 \cdot \varphi_2(x) + w_3 \cdot \varphi_3(x)\,, \tag{1.29}$$

and the FE solution is the solution of the following minimum problem:

Find the deflection

$$w_h(x) = w_1 \cdot \varphi_1(x) + w_2 \cdot \varphi_2(x) + w_3 \cdot \varphi_3(x) \tag{1.30}$$

in V_h which has the shortest distance (= strain energy) from the exact deflection w.

In FE analysis the strain energy is usually expressed

$$a(w, w) := \int_0^l H\,(w')^2\,dx = \int_0^l \frac{T^2}{H}\,dx\,. \tag{1.31}$$

If

$$e(x) = w(x) - w_h(x) \tag{1.32}$$

is the error of the FE solution, then the FE solution is that function in V_h for which the strain energy of the error $e(x)$ becomes a minimum:

$$a(e,e) = \frac{1}{2}\int_0^l \frac{(T - T_h)^2}{H}\,dx = \text{minimum}\,. \tag{1.33}$$

Any other function w_h in V_h has a larger distance—in terms of energy—than the FE solution. This property of the FE solution w_h can also be expressed as follows, see (7.413) p. 572,

$$a(e,e) \leq a(w - v_h, w - v_h) \qquad \text{for all } v_h \in V_h\,. \tag{1.34}$$

We also know that the strain energy of the FE solution is always less than the strain energy of the exact solution:

$$a(w_h, w_h) = \int_0^l \frac{T_h^2}{H}\,dx < \int_0^l \frac{T^2}{H}\,dx = a(w,w)\,, \tag{1.35}$$

i.e., the shadow w_h has a shorter length (= strain energy) than w. This inequality follows directly from

$$\begin{aligned} 0 < a(w,w) &= a(w_h + e, w_h + e) \\ &= a(w_h, w_h) + 2\,\underbrace{a(e, w_h)}_{=0} + \underbrace{a(e,e)}_{>0}\,, \end{aligned} \tag{1.36}$$

where

$$a(e, w_h) = \int_0^l \frac{(T - T_h)\,T_h}{H}\,dx = 0 \tag{1.37}$$

is a consequence of the *Galerkin orthogonality*

$$a(e, \varphi_i) = 0 \qquad i = 1,2,3 \tag{1.38}$$

i.e., the fact that the error e is orthogonal in terms of the strain energy to all unit displacements φ_i, and therefore also to $w_h = w_1 \cdot \varphi_1 + w_2 \cdot \varphi_2 + w_3 \cdot \varphi_3$.

Hence the strain energy or internal energy is the metric FE methods work with. Distance is measured in this metric and therefore also convergence.

The internal energy induces a topology on the space V which is even a *norm* on this space, because it *separates* the elements of V. Two functions w_1 and w_2 are identical if and only if their distance in terms of the strain energy is zero:

$$\frac{1}{2}\int_0^l \frac{(T_1 - T_2)^2}{H}\,dx = \frac{1}{2}\int_0^l H\,(w_1' - w_2')^2\,dx = 0 \quad \Leftrightarrow \quad w_1 = w_2 \tag{1.39}$$

that is if $w_1 - w_2$ has zero energy.

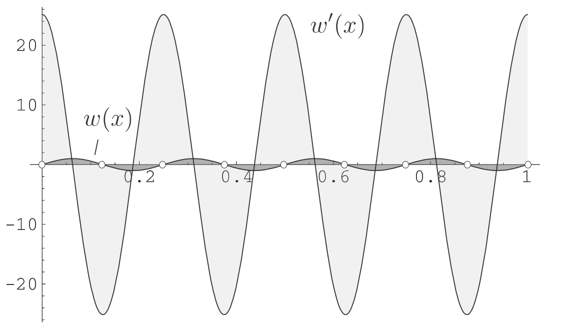


Fig. 1.8. A small deflection curve can hide a large strain energy

A function w is small in this metric if its energy (essentially the square of the first derivative) is small, and the exact deflection w and the FE solution w_h are close in this metric if the strain energy of the error

$$e(x) = w(x) - w_h(x) \qquad (e = \text{error}) \tag{1.40}$$

is small

$$\frac{1}{2}\int_0^l \frac{T_e^2}{H}\,dx = \frac{1}{2}\int_0^l H(w' - w_h')^2\,dx = \text{small} \implies e(x) = \text{small}\,. \tag{1.41}$$

This energy metric makes more sense than a naive metric that considers a function such as $w(x) = \sin(8\,\pi\,x)$ a "small" function (see Fig. 1.8), while for the FE method it is a "large" function, because the strain energy due to the rapid oscillations is large

$$\int_0^1 w(x)^2\,dx = 0.5\,, \qquad \frac{1}{2}\int_0^1 H w'(x)^2\,dx = 316 \cdot H\,. \tag{1.42}$$

Hence from an engineering standpoint it makes more sense to classify functions with regard to the strain energy than their amplitude or their L_2-norm.

A better strategy would it be to base the metric on both components, the zero-order and the first-order derivative. This leads to the so-called *Sobolev norms*, which, depending on the index n, measure the derivatives up to order n

$$||w||_n = \left[\int_0^l \left[w(x)^2 + w'(x)^2 + \ldots + w^{(n)}(x)^2\right]\,dx\right]^{1/2} \tag{1.43}$$

and classify functions according to this metric. By increasing the index n different topologies can be generated on V. In the same way the distance between two vectors does not depend on the difference of the first two components alone, $|\boldsymbol{a} - \boldsymbol{b}| = \sqrt{(a_1 - b_1)^2}$ (which would be a very crude topology) but on the difference of *all* components

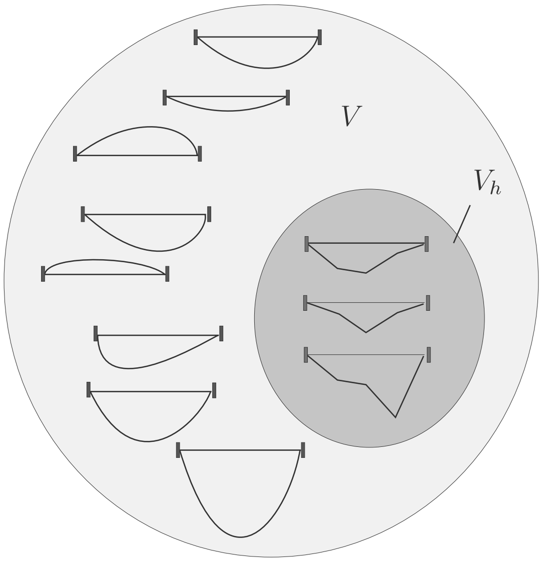


Fig. 1.9. The shapes which the FE program can represent constitute a subset V_h of V

$$|\boldsymbol{a} - \boldsymbol{b}| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \ldots + (a_n - b_n)^2}\,. \tag{1.44}$$

This metric generates the finest possible topology, just as in a lottery the prize money increases, the more figures on a ticket agree with the number drawn.

Remark 1.1. Later it will be seen that in so-called load cases δ when displacements are prescribed the projection is no longer orthogonal but "skew" this implies that the length of the shadow (the strain energy) will be *greater* than the strain energy of the exact solution; see Sect. 1.38, p. 187. This is to be expected: the stiffer a structure the greater the strain energy developed by displacing a support.

1.5 The error of an FE solution

The FE method is an approximate method, see Fig. 1.9. As such it must approximate *three* functions:

- the deflection w
- the vertical force $T = Hw'$
- the load $p = -Hw''$

i.e., the zero-order, first-order, and second-order derivative of the deflection w. All three derivatives of w are relevant to the structural analysis, and hence it is legitimate to ask which of the three errors

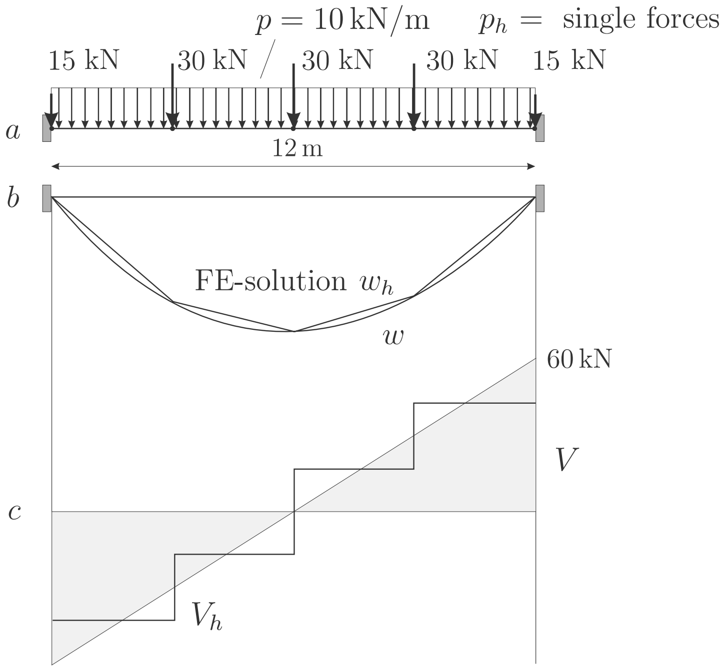


Fig. 1.10. The error in the displacement is zero at the nodes, while the error in the stresses is zero at the midpoints of the elements. This is a typical pattern in FE analysis

$w - w_h$	error in the deflection
$T - T_h$	error in the internal action
$p - p_h$	error in the load

is to be minimized? In principle we have already given the answer. The FE solution aims at minimizing the square of the error of the internal action $T - T_h$,

$$\int_0^l \frac{(T - T_h)^2}{H}\, dx = \int_0^l H(w' - w_h')^2\, dx \quad \rightarrow \quad \text{minimum}\,. \tag{1.45}$$

Hence an FE solution does not tend to win a beauty contest by imitating the original shape w as closely as possible nor does it aim to simulate the loading; rather, the solution tends to minimize the error in the strain energy (the internal energy).

The load case p_h

A closer study of the FE solution reveals that w_h is the equilibrium position of the rope if the distributed load were concentrated at the nodes, $f_i = p\, l_e$. This load case is called the FE load case p_h, (see Fig. 1.10).

Of course we would like to know what the consequences are. How far are the results of the load case p_h (= nodal forces) from p (= distributed load)? Stated otherwise: given the error in the load

$$r := p - p_h \qquad \text{(residual forces)} \tag{1.46}$$

how large is the error in the vertical force

$$T_e := T - T_h \tag{1.47}$$

and the difference in the deflection

$$e := w - w_h\,? \tag{1.48}$$

In other words what can be said about the error in the first-order, $T - T_h = H(w - w_h')$, and zero-order derivative, $w - w_h$, if the error in the second-order derivative $p - p_h$ is known?

The normal procedure is to differentiate the deflection w, yielding the vertical force T, and to differentiate T to find the load p

$$w \qquad \Rightarrow \qquad T = H\,w' \qquad \Rightarrow \qquad p = -H\,w''\,. \tag{1.49}$$

In a reverse order, the functions must be integrated

$$w = \iint -\frac{p}{H}\,dx\,dx \quad \Leftarrow \quad T = \int -p\;dx \quad \Leftarrow \quad p = -H\,w'' \tag{1.50}$$

and integration smoothes the wrinkles; see Fig. 1.10.

But is there a reliable method to make predictions about the distance in the first-order derivatives by looking at the distance in the second-order derivative? The answer is *no.* Otherwise it would suffice to calculate an approximate solution on a coarse mesh, and extrapolate from this solution to the exact solution. In general this seems not to be possible, certainly not in one step. There exist only different techniques which provide upper or lower bounds for the error. The development of such *error estimators* is the subject matter of adaptive methods.

1.6 A beautiful idea that does not work

- An FE solution cannot be improved on the same mesh.

Once it is understood that the error of an FE solution can be traced back to deviations in the load, could the situation not be improved by applying the residual forces $p - p_h$, solving this load case again with finite elements, and repeating this loop as long as the error is greater than a preset error margin ε?

This idea does not work, because the residual forces

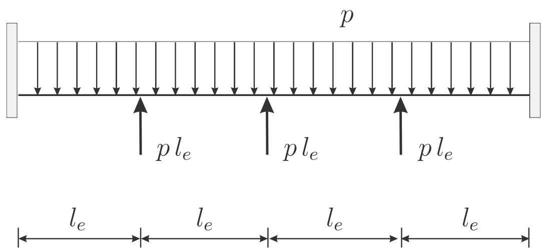


Fig. 1.11. The FE solution of this load case is zero

$$r = p - p_h \tag{1.51}$$

leave no traces on the mesh, i.e., all the equivalent nodal forces f_i^r vanish,

$$f_i^r = \int_0^l p\,\varphi_i\,dx - \sum_{j=1}^{3} f_j \cdot \varphi_i(x_j) = f_i - f_i = 0 \qquad \text{for all } \varphi_i\,, \tag{1.52}$$

so that the rope will not deflect, because zero nodal forces mean zero deflection:

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{u} = \boldsymbol{0}\,. \tag{1.53}$$

This riddle is easily solved by recalling that the exact curve w is projected onto the trial space V_h. But because the error $w - w_h$ is orthogonal (in the energy sense) to the space V_h,

$$\int_0^l H(w' - w_h')\,\varphi_i{}'\,dx = 0 \qquad \text{for all } \varphi_i\,, \tag{1.54}$$

it casts no shadow, i.e., $e = 0$.

It follows that there are load cases which cannot be solved on an FE mesh (see Fig. 1.11) namely all load cases where the load p is so arranged that it contributes no work. This is the case if all equivalent nodal forces f_i are zero:

$$f_i = \delta W_e(p, \varphi_i) = 0\,, \qquad i = 1, 2, \ldots n\,. \tag{1.55}$$

Loads that happen to be parallel to the line of sight have a “null shadow”.

1.7 Set theory

In their lowest level, many systems are at their most stable position. Many processes in physics are governed by a minimum principle. The same holds in beam analysis: the deflection curve w of a continuous beam minimizes the potential energy of the beam

$$\Pi(w) = \frac{1}{2}\int_0^l \frac{M^2}{EI}\,dx - \int_0^l p\,w\,dx \qquad \rightarrow \qquad \text{minimum} \tag{1.56}$$

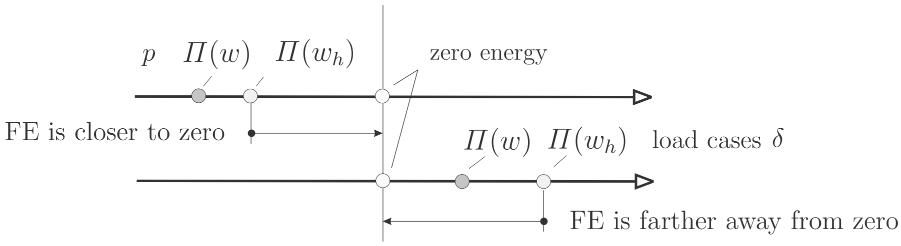


Fig. 1.12. The potential energy $\Pi(w_h)$ of the FE solution always lies to the right of the exact potential energy $\Pi(w)$

on V, which is the set of all functions w that satisfy the support conditions, i.e., that have zeros, $w = 0$, at all supports. All such functions w compete for the minimum value of $\Pi(w)$.

The winner is the deflection curve w of the continuous beam. According to Green's first identity, $G(w, w) = 0$ (see Sect. 7.2, p. 508)

$$\int_0^l \frac{M^2}{EI}\, dx = \int_0^l p\, w\, dx \qquad (2\, W_i = 2\, W_e)\,, \tag{1.57}$$

hence the minimum of the potential energy is

$$\Pi(w) = \frac{1}{2} \int_0^l \frac{M^2}{EI}\, dx - \int_0^l p\, w\, dx = -\frac{1}{2} \int_0^l p\, w\, dx\,. \tag{1.58}$$

Obviously the potential energy is *negative* in the equilibrium position, because the integral (p, w) itself is positive. It is the work done by the distributed load p inducing its own deflections, and such work (*eigenwork*) is always positive.

If no load p is applied, but instead displacements δ are prescribed at one or more supports then the potential energy is

$$\Pi(w) = \frac{1}{2} \int_0^l \frac{M^2}{EI} dx > 0\,, \tag{1.59}$$

(support displacements δ never enter into the potential energy—they only appear in the definition of the space V), i.e., the minimum value of Π must be greater than zero, because the integral of M^2 is positive. Hence the two types of load cases differ by the sign of the potential energy:

- load cases p $\quad \Pi < 0$
- load cases δ $\quad \Pi > 0\,.$

Now if a continuous beam is placed on additional supports as in Fig. 1.13, the set V "shrinks" because the candidates—the deflection curves w that compete for the minimum value of $\Pi(w)$—must have zeros, $w = 0$, at additional points. In contrast if supports are removed, then V increases, because the numbers of constraints $w = 0$ shrinks. Therefore the "size" of V is proportional to

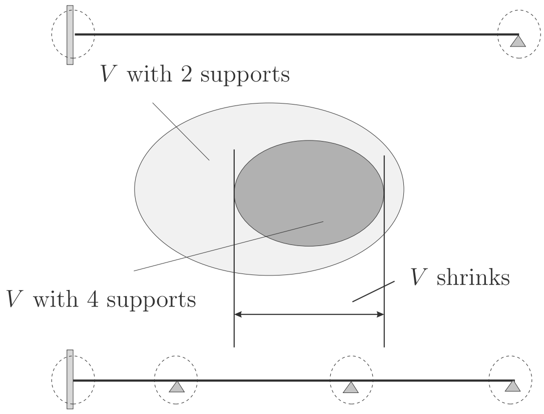


Fig. 1.13. The more supports, the smaller the space V

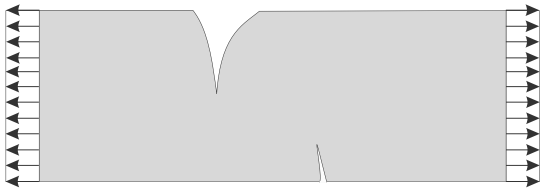


Fig. 1.14. With each crack, both the space V and $|\Pi(\boldsymbol{u})|$ increase

the number of constraints and consequently the absolute value $|\Pi(w)|$ of the potential energy must decrease (V shrinks) or increase (V grows).

Or imagine that a crack develops in a plate; see Fig. 1.14. Then the space V increases, because then also those displacement fields that are discontinuous at the faces of the crack can compete for the minimum value of $\Pi(\boldsymbol{u})$ whereupon the minimum value of $\Pi(\boldsymbol{u})$ decreases, which actually means that $|\Pi(\boldsymbol{u})|$ increases [115].

The opposite tendency is observed in FE analysis where one seeks the minimum value of $\Pi(\boldsymbol{u})$ only on a subset V_h of V. On the subset the minimum value cannot be less than the minimum on the whole space V.

A second observation can be added to this: in a load case p, the strain energy of the FE solution is always less than the strain energy of the exact solution, see (1.36),

$$\int_0^l \frac{M_h^2}{EI}\,dx \le \int_0^l \frac{M^2}{EI}\,dx \qquad \text{(load case } p) \,, \tag{1.60}$$

while in a load case δ the situation is just the opposite, because the strain energy of the FE solution exceeds the strain energy of the exact solution

$$\int_0^l \frac{M^2}{EI}\,dx \le \int_0^l \frac{M_h^2}{EI}\,dx \qquad \text{(load case } \delta) \,. \tag{1.61}$$

Both effects suggest that an FE solution tends to overestimate the stiffness of a structure.

The potential energy of the exact solution is always less than the potential energy of the FE solution:

$$\Pi(w) < \Pi(w_h) \qquad \text{because } V_h \subset V \tag{1.62}$$

or if we identify Π with numbers on the x-axis, the point $\Pi(w_h)$ will *always* lie to the right of the point $\Pi(w)$; see Fig. 1.12.

This implies that in a load case p the potential energy of the FE solution will not be as low as the potential energy of the true solution and the structure will not deflect as much—the displacements will be smaller.

The fact that $\Pi(w_h)$ lies to the right of $\Pi(w)$ means in a load case δ that more strain energy is "stored" in the FE solution than the true solution. Obviously because more energy must be supplied, to displace the support of a stiffer structure. To sum it up we have:

- in a load case p $\quad\Pi(w_h)$ is closer to zero than $\Pi(w)$
- in a load case δ $\quad\Pi(w_h)$ lies farther from zero than $\Pi(w)$

But these observations do not imply that FE displacements are smaller than the exact displacements! This certainly will be true for some nodes, but in general it cannot be guaranteed to be true for all nodes.

There is only one example where this conclusion—at least for one node—holds, namely if a single force P acts at a point $\boldsymbol{x}_P$ of a Kirchhoff plate. In the equilibrium position the potential energy is just the (negative) work done by the force P

$$-\frac{1}{2} P\, w(\boldsymbol{x}_P) = \Pi(w) < \Pi(w_h) = -\frac{1}{2} P\, w_h(\boldsymbol{x}_P) \tag{1.63}$$

and this inequality can only be true if the FE deflection at $\boldsymbol{x}_P$ is less than the exact value, $w_h < w$.

A similar result can be observed in a beam which is loaded at midspan, $x = l/2$, with a single force P, so that

$$\Pi(w) = -\frac{1}{2} P\, w(\frac{l}{2})\,. \tag{1.64}$$

What happens next is exactly what is predicted by set theory. The more supports that are added (see Fig. 1.15), the smaller the deflection $w(l/2)$ at the center of the beam. Then V decreases, as does the absolute value $|\Pi(w)|$ of the potential energy and thus the deflection $w(l/2)$.

The same effect can be observed if the beam is placed on one or two additional *elastic supports*. Springs are different, in that they do not change the size of V, because springs have no hard supports such as $w(0) = 0$.

Braces and diaphragms also enable the absolute value of the potential energy of a structure to decrease. The more plates, beams, columns and slabs a structure contains per cubic meter, the closer the absolute value of the potential energy of the structure in a load case p will be to zero, while in a

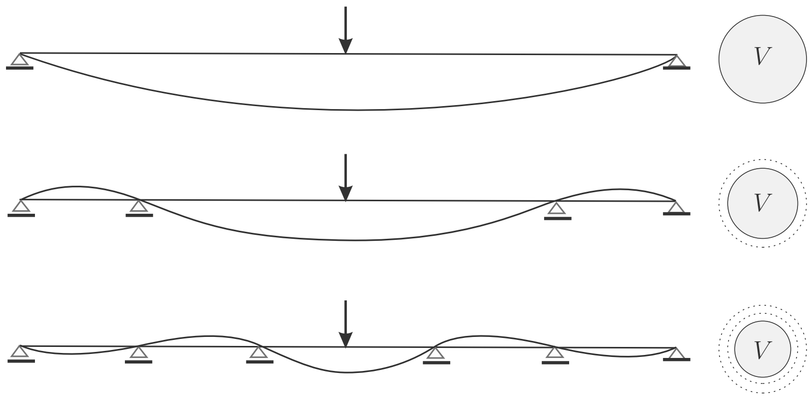


Fig. 1.15. The greater the number of supports, the smaller the value of $|\Pi|$, the smaller the deflection w, and the smaller the size of the space V

load case δ the opposite will be true. If in addition such a complex structure is modeled with just the bare minimum of elements, the structure will be very stiff.

Minimum or maximum ?

In some sense the principle of minimum potential energy could also be called a maximum principle—at least for load cases p. Calling it a minimum principle is attractive, because many processes in nature follow a principle of *least action*, but in reality the load p on a beam tends to push the beam downwards as far as possible, transforming positional energy into potential energy:

$$\Pi(w) = -\frac{1}{2} \int_0^l p\, w\, dx\,, \qquad \text{at } w = \text{equilibrium point} \tag{1.65}$$

in mathematical terms, it pushes the point $|\Pi(w)|$ as far away from zero as possible.

The movement stops at the equilibrium point. This is the point at which the external work W_e equals the internal energy W_i,

$$W_e = \frac{1}{2} \int_0^l p\, w\, dx = \frac{1}{2} \int_0^l \frac{M^2}{EI}\, dx = W_i \quad \text{at the equilibrium point } w\,. \tag{1.66}$$

The more the load presses the beam down (W_e increases), the more resistance the load feels because the beam bends; the bending moments increase, thereby increasing the internal energy W_i; see Fig. 1.16. The equilibrium point is the point at which the two trends balance.

Only in load cases δ does the minimum keep its original meaning. Then the structure tries to avoid any excess strain energy, and follows with as little