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Michael Hrušák · Ángel Tamariz-Mascarúa
Mikhail Tkachenko *Editors*

Pseudocompact Topological Spaces

A Survey of Classic and New Results with Open Problems

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Preface

In 1948, E. Hewitt introduced the concept of pseudocompactness which generalizes a property of compact subsets *X* of the real line; specifically, the range of any real-valued continuous function defined on such a compact subset is a bounded subset of the real line. That pseudocompactness is a key property of topological spaces can be appreciated on examining the volume of the mathematical literature dedicated to it.

Pseudocompact spaces constitute a natural and fundamental class of objects in General Topology, and research into their properties has important repercussions in diverse branches of mathematics such as functional analysis, dynamical systems, set theory, and topological-algebraic structures. Among the many authors who have made important contributions to the theory of pseudocompact spaces, we mention J. Colmez, W. W. Comfort, Z. Frolík, I. Glicksburg, M. Katětov, S. Mrówka, N. Noble, K. A. Ross, V. Saks, and H. Tamano.

A small number of well-known books on topology contain a considerable part of the development of the theory of pseudocompact spaces: to name but two, *Rings of Continuous Functions* by L. Gillman and M. Jerison, and *General Topology* by R. Engelking. However, in spite of the importance of this concept, there is as yet no text that systematically compiles and develops the, by now, extensive theory of pseudocompact spaces. The aim of the editors and authors contributing to this book is to correct, at least in part, this absence, to present in a book, intended for postgraduate students and researchers, many of the results of historical importance on this subject, and to develop material which has not been published in any previous work of this kind.

In addition, this book exhibits facets of the research that has been conducted over the past 50 years by what we may call "the Mexican School of General Topology." This work constitutes a testimony to, and a legacy of, the research of the editors and authors of this school. The authors involved in each chapter present studies of pseudocompact spaces from the point of view of their personal research interests.

We present various topics in this text related to pseudocompact spaces, both classic theorems and recent results on various generalizations of pseudocompactness, weakly pseudocompact spaces, bounded subsets, maximal pseudocompact spaces, pseudocompact topological groups, and pseudocompactness in the realm of topological transformation groups. Moreover, we include a detailed study of Mrówka spaces. A list of open problems is included within each chapter.

It is important to mention that when this work was initially conceived, Prof. Adalberto García-Máynez was in the process of planning a chapter concerning his research on pseudocompact spaces. Unfortunately, his untimely death in the Spring of 2016 prevents us from including what would have been a most valuable contribution. The efforts of all those involved in this collective academic project are a tribute to his memory.

Mexico City, Mexico
February 2017

Michael Hrušák
Ángel Tamariz-Mascarúa
Mikhail Tkachenko

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Symbols

$[0,1]$ or $\mathbb{I}$	the closed unit interval, 2
$A \rightrightarrows_{\varphi} B$	B dominates A, 280
$A \rightrightarrows^{*}_{\psi} B$	B almost dominates A, 280
$\mathfrak{a}$	the almost disjointness number, 256
$\alpha_0(X)$	$= X \cup (\beta X \backslash \upsilon X)$, 158
$\alpha(g,x)$	group action, 218
$\mathfrak{b}$	the boundedness number, 265
βX	the Stone–Čech compactification, 13
$\mathbb{C}$	the complex plane, 40
$\mathfrak{c}$	the power of the continuum, 2
$Cone(X)$	the cone over X, 160
$C(X)$	the set of real-valued continuous functions on X, 2
$C^*(X)$	the set of bounded real-valued continuous functions on X, 2
$CL(X)$	the set of non-empty closed subsets of $X, \neq \emptyset$, 97
$C_p(X)$	the space of real-valued continuous functions on X with the topology of pointwise convergence, 38
$C_p(X,Y)$	the space of continuous functions from X to Y with the topology of pointwise convergence, 35
$c(X)$	the cellularity of X, 70
$\chi(X)$	the character of X, 229
Δ_X	the diagonal in $X \times X$, 55
$\Delta\mathscr{F}$	the diagonal map of $\mathcal{F}$, 178
$\diamondsuit$	the Jensen's diamond principle, 215
$GL(2,\mathbb{R})$	the real general linear group of degree 2, 41
$GL(n,\mathbb{R})$	the real general linear group of degree n, 41
$G(x)$	the G-orbit of x, 213
G_x	the stabilizer of x, 228
$\mathfrak{h}$	the distributivity number, 274
$\mathcal{H}(X)$	the group of homeomorphisms of X, 220

$Home^+(\mathbb{I})$	the group of orientation-preserving homeomorphisms of the unit interval, 233
$Homeo^+(\mathbb{I})$	the subgroup of $Homeo^+(\mathbb{I})$ consisting of homeomorphisms fixing each point of the form $1/n$, 233
I_a	the inner automorphism corresponding to a, 42
$J(\kappa)$	the hedgehog with κ spines, 155
ker(f)	the kernel of f, 42
$K(X)$	the set of non-empty compact subsets of $X, \neq \emptyset$, 97
$L(A)$	the level of A, 263
$L(p, \{(S_n)_{n\in\mathbb{N}}\})$	The set of p-limit points of the sequence $(S_n)_{n\in\mathbb{N}}$, 81
$m_Q X$	the set of maximal ideals of Q, 187
μX	the Dieudonné completion of X, 109
$\mathbb{N}$	the set of natural numbers, 2
$\mathbb{N}^*$	the remainder of $\mathbb{N}$ in $\beta\mathbb{N}$, 78
ω	the first infinite ordinal, 2
$\mathfrak{p}$	the pseudo-intersection number, 275
$P_{RK}(p)$	the set of Rudin-Keisler predecessors of p, 78
$p \leq_R q$	the Rudin pre-order, 83
$p \leq_{RK} q$	the Rudin-Keisler pre-order, 78
$\varphi \approx \psi$	the isomorphism of weak selections, 280
$\pi\chi(X)$	the π-character of X, 201
$\pi\chi(x, X)$	the π-character of x in X, 201
$\psi(x, X)$	the pseudo-character of x in X, 47
$\Psi(\mathcal{A})$	the Mrówka-Isbell space corrsponding to $\mathscr{A}$, 7
$\mathbb{Q}$	the set of rational numbers, 69
$\mathbb{R}$	the real line, 2
ϱG	the Weil completion of G, 43
$S_{RK}(p)$	Rudin–Keisler successors, 78
$so(X)$	the sequetial order of X, 263
$\mathbb{T}$	the circle group, 40
$\mathfrak{t}$	the tower number, 267
T_0	the T_0-separation axiom, 39
T_1	the T_1-separation axiom, 39
T_2	the Hausdorff separation axiom, 39
T_3	regularity, 39
$T_{3.5}$	complete regularity, or the Tychonoff separation axiom, 39
$tor(G)$	the torsion part of G, 59
$T(p)$	the type of an ultrafilter p, 78
$T^*(p)$	$= T(p) \cup \mathbb{N}83$
$t(X)$	the tightness of X, 215
υX	the Hewitt-Nachbin completion, 16
V^+	$= A \in CL(X) : A \subseteq V$, 97
V^-	$= A \in CL(X) : A \cap V \neq \emptyset$, 97
$\mathcal{V}^l_G$	the left uniformity of G, 56

$\mathcal{V}_G^r$	the right uniformity of G, 56
$\mathcal{V}_G$	the two-sided uniformity of G, 56
X/G	the orbit space of the action of G on X, 222
$\mathbb{Z}$	the group of integers, 3
ζX	$= X \cup (\beta X \setminus \mathrm{i}nt_{\beta X} \upsilon X)$, 159

Chapter 1
Basic and Classic Results on Pseudocompact Spaces

J. Angoa-Amador, A. Contreras-Carreto, M. Ibarra-Contreras and Á. Tamariz-Mascarúa

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1.1 Basic Properties

E. Hewitt, in 1948, introduced the concept of pseudocompact space [15]; from then to the present multiple generalizations and variations of this concept have been defined.

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The intention of this chapter is to present the classic results on this theme in a succinct manner.[1]

Every space considered in this chapter is assumed to be nonempty completely regular and T_2 (that is, Tychonoff).

As usual, $\mathbb{R}$ is the space of the real numbers with its Euclidean Topology, and $\mathbb{N}$ and $[0, 1]$ are the subspaces of $\mathbb{R}$ formed by the natural numbers and the unit interval, respectively.

For each topological space X, we denote by $C(X)$ the set of real-valued continuous functions defined on X, and for $f, g \in C(X)$ we denote by $f \wedge g$ the function which associates to each $x \in X$ the number $\min\{f(x), g(x)\}$. As usual $C^*(X)$ stands for the subset of $C(X)$ constituted by the bounded continuous functions. With the symbols $\mathrm{cl}_X(A)$ and $\mathrm{int}_X(A)$, or simply $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ if there is no possibility of confusion, we will designate the closure and interior, respectively, of a subset A in a space X. For a point x in a space X, $\mathcal{V}(x)$ will mean the collection of neighborhoods of x in X. We will denote by ω the first infinite ordinal and by ω_1 the first non countable ordinal. We will not distinguish between ω and the first infinite cardinal $\aleph_0$, and $\mathfrak{c}$ is equal to 2^ω. For every ordinal number α, $[0, \alpha)$ (respectively, $[0, \alpha]$) will mean the set of all ordinals less than (respectively, less or equal than) α with its order topology. By $\mathcal{P}(X)$ we denote the power set of X, and $[X]^\omega$ is the collection of infinite countable subsets of X. For two spaces X and Y, the expression $X \cong Y$ will mean that they are homeomorphic. The terms used and not defined here can be consulted in [7].

For a family of spaces $\{X_s : s \in S\}$ and $T \subset S$, $\pi_T : \prod_{s\in S} X_s \to \prod_{t\in T} X_t$ will designate the natural projection. If $T = \{t\}$, we will use π_t or π_{X_t} instead of $\pi_{\{t\}}$.

Definition 1.1.1 A space X is pseudocompact if for every $f \in C(X)$, we have that $f[X]$ is bounded, i.e. $C(X) = C^*(X)$.

Since the continuous image of a compact space is compact, and the compact subsets of $\mathbb{R}$ are bounded, then any compact space is pseudocompact, and given that $id : \mathbb{R} \to \mathbb{R}$ is not bounded, the real line $\mathbb{R}$ is not pseudocompact. So, pseudocompactness is not hereditary; $[0, 1]$ is compact but its open subset $(0, 1) \cong \mathbb{R}$ is not.

Proposition 1.1.2 *The continuous image of a pseudocompact space is pseudocompact.*

Proof Let X be a pseudocompact space and let $f : X \to Y$ be continuous and onto. If Y is not pseudocompact, then there is a continuous function $g : Y \to \mathbb{R}$ such that $g[Y]$ is not bounded. Clearly, $g \circ f \in C(X)$ and $(g \circ f)[X]$ is not bounded. □

Recall that $Z \subset X$ is a *zero-set* if for a $f \in C(X)$ we have that $Z = f^{-1}[\{0\}]$. The complement of a zero-set is called *cozero*. It is possible to prove that for every $f \in C(X)$ and for every closed $F \subset \mathbb{R}$, $f^{-1}[F]$ is a zero-set. It is said that $A \subset X$ is C-*embedded* (C^*-*embedded*) if for every $f \in C(A)$ ($f \in C^*(A)$), there exists

[1] Refer to Chap. 7 for a survey on equivariant pseudocompact spaces.

$F \in C(X)$ ($F \in C^*(X)$) such that $F \restriction A = f$. Every C-embedded subset of X is C^*-embedded in X.

We shall say that a family $\mathcal{F} \subset \mathcal{P}(X)$ has the *finite intersection property* (*f.i.p.*) if for every finite $\mathcal{F}' \subset \mathcal{F}$, we have that $\bigcap \mathcal{F}' \neq \emptyset$.

A collection $\mathcal{C}$ of subsets of X is *locally finite* if for each point $x \in X$, there is an open neighborhood V of x such that $|\{C \in \mathcal{C} : C \cap V \neq \emptyset\}| < \omega$.

Theorem 1.1.3 *For a topological space X, the following are equivalent:*

(1) X is pseudocompact;
(2) [12] any locally finite family of open sets is finite;
(3) every pairwise disjoint locally finite family of open sets in X is finite;
(4) [2] for any family of open sets $\{U_n : n \in \omega\}$ with the f.i.p., $\bigcap\{\mathrm{cl}(U_n) : n \in \omega\} \neq \emptyset$;
(5) any countable open cover of X has a finite subcollection such that its union is dense in X;
(6) any countable cozero cover in X has a finite subcover;
(7) X does not have C-embedded copies of $\mathbb{N}$.

Proof (1) $\Rightarrow$ (2): Let us suppose that there exists an infinite locally finite family $\mathcal{F}$ of open subsets of X. Suppose that $\mathcal{F} = \{U_n : n < \omega\}$ and if $n \neq m$, then $U_n \neq U_m$. For every $n < \omega$ we choose $x_n \in U_n$. Also, for every $n < \omega$ there exists a continuous $f_n : X \to [0, n]$ such that $f_n(x_n) = n$ and $f_n[X \setminus U_n] \subset \{0\}$. Since the subcollection of elements in $\mathcal{F}$ which contain a given point x of X is finite, $f = \sum_{n<\omega} f_n$ is well defined.

Now, let $x \in X$, $f(x) \in (a, b)$ and $O \in \mathcal{V}(x)$ for which $C = \{n : U_n \cap O \neq \emptyset\}$ is finite. Let $C_1 = \{n < \omega : f_n(x) > 0\}$. It is clear that $C_1 \subset C$ and that $f(y) = \sum_{n \in C_1} f_n(y)$ for every $y \in O$. The function $\sum_{n \in C_1} f_n$ belongs to $C(X)$, so there is an open set W, such that $\sum_{n \in C_1} f_n[W] \subset (a, b)$ and $x \in W$. Let $V = O \cap W$. It is clear that $x \in V$ and that $f[V] \subset (a, b)$. Then, f is continuous.

Moreover, it is clear that f is unbounded.

The statement (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (2): We are going to establish that the negation of statement (2) implies the existence of an infinite pairwise disjoint locally finite collection of nonempty open subsets in X.

Indeed, assume that $\mathcal{C} = \{U_0, ..., U_n, ...\}$ is a locally finite family of nonempty open subsets in X such that $U_i \neq U_j$ if $i \neq j$. Take a point $x_0 \in U_0$. There is an open neighborhood V_0 of x_0 such that $V_0 \subset U_0$ and $|\{U \in \mathcal{C} : U \cap V_0 \neq \emptyset\}| < \aleph_0$. Assume that we have already found a sequence $0 = n_0 < n_1 < \cdots < n_k$ of natural numbers, and a sequence $V_{n_0}, \ldots, V_{n_k}$ of nonempty pairwise disjoint open sets such that (i) $V_i \subset U_{n_i}$, and (ii) $|\{U \in \mathcal{C} : U \cap V_i \neq \emptyset\}| < \aleph_0$ for each $i \in \{0, ..., k\}$. Then, since $\mathcal{C}$ is infinite, there is $n_{k+1} > n_k$ such that $V_i \cap U_{n_{k+1}} = \emptyset$ for all $i \in \{0, ..., k\}$. Let $x_{k+1} \in U_{n_{k+1}}$, and let $V_{n_{k+1}}$ be an open neighborhood of $x_{n_{k+1}}$ contained in $U_{n_{k+1}}$ such that $|\{U \in \mathcal{C} : U \cap V_{n_{k+1}} \neq \emptyset\}| < \aleph_0$.

In this manner we obtain the infinite pairwise disjoint locally finite collection of open sets $\{V_0, \ldots, V_n, \ldots\}$ in X.

(2) $\Rightarrow$ (4): Let us suppose that there exists $\{U_n : n < \omega\}$, a family of open subsets with the f.i.p. such that $\bigcap\{\mathrm{cl}(U_n) : n < \omega\} = \emptyset$.

For each $n < \omega$ we define $W_n = \bigcap_{i \leq n} \mathrm{cl}(U_i)$. For each $x \in X$, there is n_0 such that $x \notin \mathrm{cl}(U_{n_0})$. Therefore $x \notin W_{n_0}$. Since W_{n_0} is closed, there is an open set O such that $x \in O$ and $O \cap W_{n_0} = \emptyset$. So for every $n \geq n_0$ we have that $O \cap W_n = \emptyset$. If $V_n = \bigcap_{i \leq n} U_n$, we have proved that the family $\{V_n : n \in \mathbb{N}\}$ is locally finite; that is, it is finite.

Let n_1 be such that for every $n \geq n_1$ we have that $V_n = V_{n_1}$. It is clear that $V_{n_1} \subset U_n \subset \mathrm{cl}(U_n)$ for every $n \leq n_1$. Now if $n > n_1$, since $V_{n_1} = V_n \subset U_n \subset \mathrm{cl}(U_n)$, then $V_{n_1} \subset \bigcap\{\mathrm{cl}(U_n) : n < \omega\}$. Since by hypothesis $V_{n_1} \neq \emptyset$, we have that $\bigcap\{\mathrm{cl}(U_n) : n < \omega\} \neq \emptyset$; a contradiction.

(4) $\Rightarrow$ (5): Let $\mathcal{C} = \{U_n : n < \omega\}$ be an open cover of X such that for every finite set $F \subset \omega$ it happens that $X \setminus \bigcup_{n \in F} \mathrm{cl}(U_n) \neq \emptyset$.

We define for every $n < \omega$, the open set $W_n = X \setminus \mathrm{cl}(U_n)$. Since $\bigcap_{i \leq r} W_i = X \setminus (\bigcup_{i \leq r} \mathrm{cl}(U_i))$, and by hypothesis $X \setminus (\bigcup_{i \leq r} \mathrm{cl}(U_i)) \neq \emptyset$, we infer that $\{\bigcap_{i \leq r} W_i : r < \omega\}$ has the f.i.p.

By hypothesis, we have that $\bigcap_{n<\omega} \mathrm{cl}(W_n) \neq \emptyset$. On the other hand, $\bigcap_{n<\omega}(X \setminus U_n) = X \setminus \bigcup_{n<\omega} U_n = \emptyset$; but $\mathrm{cl}(X \setminus \mathrm{cl}(U_n)) \subset X \setminus U_n$ for every $n < \omega$. So $\bigcap_{n<\omega} \mathrm{cl}(W_n) = \bigcap_{n<\omega}(\mathrm{cl}(X \setminus \mathrm{cl}(U_n)) \subset \bigcap_{n<\omega}(X \setminus U_n) = \emptyset$, which is a contradiction.

(5) $\Rightarrow$ (6): Let $\{C_n : n < \omega\}$ be a cozero cover of X and let $f_n \in C(X)$ be such that $C_n = f_n^{-1}[[0, 1] \setminus \{0\}]$. For $i, n < \omega$, we define $C_{n,i} := f_n^{-1}[(1/i, 1]]$. Each $C_{n,i}$ is open in X and $C_n = \bigcup_{i<\omega} C_{n,i} = \bigcup_{i<\omega} \mathrm{cl} C_{n,i}$. In particular, $\mathcal{C} := \{C_{n,i} : n, i < \omega\}$ is an open cover of X. Since $\mathcal{C}$ is countable, there is a finite subfamily $\mathcal{F}$ of $\mathcal{C}$ satisfying that $\bigcup_{C \in \mathcal{F}} C$ is dense in X. Say that $\mathcal{F} = \{C_{n_0,i_0}, ..., C_{n_k,i_k}\}$. This implies that $X = \mathrm{cl} \bigcup_{j \leq k} C_{n_j,i_j} = \bigcup_{j \leq k} \mathrm{cl} C_{n_j,i_j}$. Observe that for $n, i < \omega$, $\mathrm{cl} C_{n,i} \subset C_{n,i+1} \subset C_n$. So, $X = \bigcup_{j \leq k} C_{n_j}$.

(6) $\Rightarrow$ (7): Assume that $D \subset X$ is an infinite countable discrete subspace of X which is C-embedded in X. Let $D = \{x_n : n < \omega\}$ with $x_n \neq x_m$ if $n \neq m$, and let $f : D \to \mathbb{R}$ be defined as $f(x_n) = n$. Suppose that $\tilde{f}$ is the continuous extension of f to all of X and define $C_n = \{x \in X : \tilde{f}(x) < n\}$.

The family $\{C_n : n < \omega\}$ is a cover of X constituted by cozero sets. For each $n < \omega$, $x_n \notin C_1 \cup \cdots \cup C_n$. So, there is no finite subcover of this cozero cover.

(7) $\Rightarrow$ (1): If $f \in C(X)$ is not bounded, we can define

$$C_1 = \{x \in X : 1 < f(x)\},$$

(we can suppose that f is not negative) and for each $n < \omega$, if we take $x_n \in C_n$ we define C_{n+1} as

$$\{x \in X : \max\{n, f(x_n)\} < f(x)\}.$$

Then, for each $n < \omega$, $C_n \neq \emptyset$. Let A be the set $\{x_n : n < \omega\}$. Let $O_1 := f^{-1}[(-\infty, f(x_2))]$ and, for $n > 1$, $O_n := f^{-1}[(f(x_{n-1}), f(x_{n+1}))]$. It is clear that $O_n \cap A = \{x_n\}$, therefore A is discrete. We will see that A is even C-embedded.

Denote by F_0 the set

$$\{x \in X : f(x) \leq 1\} \cup \{x \in X : f(x_2) \leq f(x)\},$$

and for $n \geq 1$ let

$$F_n = \{x \in X : f(x) \leq f(x_n)\} \cup \{x \in X : f(x_{n+2}) \leq f(x)\}.$$

Each F_n is closed and $x_{n+1} \notin F_n$.

Now, if $h : A \to \mathbb{R}$ is continuous, for every $n \geq 0$ there exists $f_n \in C(X)$ such that $f_n[F_n] \subset \{0\}$ and $f_n(x_{n+1}) = h(x_{n+1})$. Define $F : X \to \mathbb{R}$ as $F(x) = \sum_{i<\omega} f_i(x)$. If $x \in X$, there is $n < \omega$ such that $f(x) < n$, and so $x \in F_{n+1}$. But moreover, $x \in f^{-1}[(-\infty, n)] \subset F_m$ for all $m \geq n+1$, which implies $F(x) = \sum_{i \leq n} f_i(x)$ (of course, this is true for every $y \in f^{-1}[(-\infty, n)]$).

Since $\sum_{i \leq n} f_i \in C(X)$, it follows that for $\epsilon > 0$ there is an open subset O of X such that

$$\sum_{i \leq n} f_i[O] \subset (\sum_{i \leq n} f_i(x) - \epsilon, \sum_{i \leq n} f_i(x) + \epsilon)$$

with $x \in O$. Take $V = f^{-1}[(-\infty, n)] \cap O$ which is an open neighborhood of x in X satisfying that for every $y \in V$

$$F(y) = \sum_{i \leq n} f_i(y) \in (\sum_{i \leq n} f_i(x) - \epsilon, \sum_{i \leq n} f_i(x) + \epsilon).$$

This shows that $F \in C(X)$.

Besides, for $x_n \in A$ we know that $F(x_n) = \sum_{i<\omega} f_i(x_n)$. For $m \geq n$ we have that $x_n \in F_m$, and so $f_m(x_n) = 0$. Additionally, $f_{n-1}(x_n) = h(x_n)$. And if $k < n-1$, we have that $x_n \in F_k$; thus $f_k(x_n) = 0$. We conclude that $F(x_n) = f_{n-1}(x_n) = h(x_n)$, and that F is a continuous extension of h. □

As a consequence of Theorem 1.1.3 we obtain the following corollaries:

Corollary 1.1.4 *For a topological space X the following statements are equivalent:*

(1) X is pseudocompact;
(2) [19] *every locally finite open cover of X is finite;*
(3) [31] *every locally finite open cover of X has a finite subcover.*

Proof The implication (1) ⇒ (2) follows from Theorem 1.1.3.(2). The implication (2) ⇒ (3) is obvious.

In order to prove (3) implies (1), we consider a continuous function $f : X \to \mathbb{R}$. Since $\{(n-1, n+1) : n \in \mathbb{Z}\}$ is a locally finite open cover of $\mathbb{R}$, then $\{f^{-1}[(n-1, n+1)] : n \in \mathbb{Z}\}$ is a locally finite open cover of X (here $\mathbb{Z}$ is the set of integers). Thus there is $\{n_1, \cdots, n_r\} \subset \mathbb{Z}$ such that $\{f^{-1}[(n_i - 1, n_i + 1)] : i \in \{i, \cdots, r\}\}$ covers X. As a consequence, if $s = \max\{n_i : i \in \{i, \cdots, r\}\}$, we obtain that for every $x \in X$, $|f(x)| \leq |s|$ and therefore f is bounded. □

Corollary 1.1.5 [2] *For a topological space X the following statements are equivalent:*

(1) X es pseudocompact;
(2) if $\{W_n : n < \omega\}$ is a sequence of nonempty open subsets of X such that for all $n < \omega$, $W_{n+1} \subset W_n$, then $\bigcap_{n<\omega} \mathrm{cl}(W_n) \neq \emptyset$.

Proof Observe that the statement in (2) is equivalent to (3) in Theorem 1.1.3. □

A collection of subsets $\mathcal{C}$ of X is *discrete* in X if for each $x \in X$ there is a neighborhood V of x, such that $|\{C \in \mathcal{C} : C \cap V \neq \emptyset\}| \leq 1$. Since every discrete family is locally finite, from Theorem 1.1.3.(2), we also obtain that a space X is pseudocompact if and only if every discrete family of open subsets of X is finite.

Definition 1.1.6 Let X be a topological space and let $\{U_n : n < \omega\}$ be a sequence of nonempty open subsets of X. A point $x \in X$ is a *limit point* of $\{U_n : n < \omega\}$ if for every neighborhood V of x, $|\{n : U_n \cap V \neq \emptyset\}| = \aleph_0$.

Corollary 1.1.7 *A space X is pseudocompact if and only if for each countable family of nonempty open subsets of X there exists a limit point of such a family.*

Proof Suppose that X is pseudocompact and that there is a family $\{U_n : n < \omega\}$ of nonempty open subsets of X such that for each $x \in X$, there is a neighborhood V of x such that $|\{n : U_n \cap V \neq \emptyset\}| < \aleph_0$. Then $\{U_n : n < \omega\}$ is locally finite and, according to Theorem 1.1.3.(2), $\{U_n : n < \omega\}$ is actually finite. Therefore, there exists $m_0 < \omega$ such that for every $n \geq m_0$, we have that $U_n = U_{m_0}$. Let $x \in U_{m_0}$. It is clear that for every neighborhood V of x, $\{m : m \geq m_0\} \subset \{m : U_m \cap V \neq \emptyset\}$; which is a contradiction.

Now, because of the remark made before Definition 1.1.6, if X is not pseudocompact there exists an infinite discrete family of nonempty open subsets of X. □

Theorem 3.2.1 shows us a long list of equivalent formulations of pseudocompactness, including those we have already seen here.

Recall that a space X is *countably compact* if every infinite subset of it has an accumulation point. This is equivalent to saying that every countable open cover of X posses a finite subcover. So, as a consequence of Theorem 1.1.3.(5), we have that:

Corollary 1.1.8 [15] *Every countably compact space is pseudocompact.*

For an ordinal number α, the space $[0, \alpha)$ of ordinal numbers strictly less than α with its order topology is countably compact if α is a successor ordinal or if its cofinality is greater than ω. Then $[0, \alpha)$ is pseudocompact in any of these cases. If α is a limit ordinal, then $[0, \alpha)$ is not compact. So, in this way we obtain examples of pseudocompact spaces which are not compact.

The *Tychonoff plank* T is an example of a pseudocompact space which is not countably compact. Recall that T is the set $([0, \omega] \times [0, \omega_1]) \setminus \{(\omega, \omega_1)\}$ endowed with the topology of subspace of the product $[0, \omega] \times [0, \omega_1]$. The infinite subset $\{(n, \omega_1) : n < \omega\}$ is discrete and closed in T; so, T is not countably compact. On the other hand, T is pseudocompact. Indeed, the set of isolated points in T is a dense subset of T. Moreover, it is possible to prove that the closure of any countable subset of isolated points in T is compact. This means that if $f : T \to \mathbb{R}$ is continuous and not bounded, we can obtain a countable set $\{t_n : n < \omega\}$ of isolated points in T such that $\{f(t_n) : n < \omega\}$ is not bounded in $\mathbb{R}$; then $f(\text{cl}_T[\{t_n : n < \omega\})$ is compact and not bounded. This is a contradiction. That is, every real-valued continuous function defined in T must be bounded.

Other examples of pseudocompact spaces which are not countably compact, which we present next, are the Mrówka-Isbell spaces.

Definition 1.1.9 A family $\mathscr{A} \subset \mathcal{P}(\omega)$ is *almost disjoint* if:

(1) $|\mathscr{A}| \geq \omega$;
(2) for every $A \in \mathscr{A}$, $|A| = \omega$;
(3) for every $A, B \in \mathscr{A}$ with $A \neq B$, $|A \cap B| < \omega$.

A consequence of Zorn's Lemma is that every almost disjoint family on ω is contained in a maximal almost disjoint family.

Let $\mathscr{A}$ be a maximal almost disjoint family and let $\Psi(\mathscr{A})$ be the set $\omega \cup \mathscr{A}$ equipped with the following topology: for each $n \in \omega$, $\{n\}$ is open, and for each $A \in \mathscr{A}$, a base of neighborhoods of A is the collection of all the sets of the form $\{A\} \cup B$, where $|A \setminus B| < \omega$. The space $\Psi(\mathscr{A})$, so defined, is the so called *Mrówka-Isbell space* determined by $\mathscr{A}$. Chapter 8 by F. Hernández-Hernández and M. Hrušák in this book is devoted to analyzing these spaces. We mention here some basic facts about them.

The space $\Psi(\mathscr{A})$ is not countably compact because its subset $\mathscr{A}$ is infinite, discrete and closed in $\Psi(\mathscr{A})$. Next, we are going to see that $\Psi(\mathscr{A})$ is pseudocompact.

Proposition 1.1.10 [21] *For an almost disjoint family $\mathscr{A}$, $\Psi(\mathscr{A})$ is pseudocompact if and only if $\mathscr{A}$ is maximal.*

Proof Indeed, assume that $\Psi(\mathscr{A})$ is not pseudocompact. Then there is $f \in C(\Psi(\mathscr{A}))$ such that f is not bounded and not negative. We take $x_1 \in \omega$ and $c_1 = \max\{1, f(x_1)\}$. Then $f^{-1}[(c_1, \infty)]$ is an open and nonempty subset of $\Psi(\mathscr{A})$. We can take $x_2 \in \omega \cap f^{-1}[(c_1, \infty)]$. Since ω is dense in $\Psi(\mathscr{A})$, we can construct a sequence $C = \{x_n : n \in \mathbb{N}\} \subset \omega$ such that $f(x_{n+1}) > \max\{n, f(x_n)\}$.

By virtue of the maximality of $\mathscr{A}$, there is $A_0 \in \mathscr{A}$ satisfying $|A_0 \cap C| = \aleph_0$. Because of the continuity of f we know that there exists $B \subset \omega$ with $|A_0 \setminus B| < \aleph_0$ such that $f[\{A_0\} \cup B] \subset (f(A_0) - \epsilon, f(A_0) + \epsilon)$. Thus $|B \cap C| = \aleph_0$. So, for each $n \in \mathbb{N}$, there exists $x_{m_n} \in B \cap C$ with $m_n > n$. Moreover, there is $l \in \mathbb{N}$ such that $f(A_0) + \epsilon < l$. For this natural number we have that $f(x_{m_l}) > l > f(A_0) + \epsilon$, but this is not possible because $x_{m_l} \in B$.

If $\mathscr{A}$ is not maximal, there is an infinite $F \subset \omega$ not belonging to $\mathscr{A}$ such that $|F \cap A| < \aleph_0$ for each $A \in \mathscr{A}$. This set F is a closed discrete and locally finite subset of $\Psi(\mathscr{A})$. Therefore, if $\{a_n : n < \omega\}$ is an exact enumeration of F, the function $h : \Psi(\mathscr{A}) \to \mathbb{R}$ defined as $h(a_n) = n$ and $h(x) = 0$ for every $x \in \Psi(\mathscr{A}) \setminus F$ is continuos and unbounded. That is, $\Psi(\mathscr{A})$ is not pseudocompact. □

Next we present a result which gives us conditions that imply the countable compactness from pseudocompactness.

Theorem 1.1.11 *Let X be a pseudocompact space. X is countably compact if and only if every countable closed discrete subspace F of X can be separated with disjoint open subsets from any closed subset H which does not meet F.*

Proof Assume that X is not countably compact. Then, there is an infinite countable closed discrete subspace D of X. Let us say that $\{x_n : n < \omega\}$ is a faithful enumeration of D. Let $\mathcal{A} = \{A_n : n < \omega\}$ be a family of open subsets of X such that $A_n \cap D = \{x_n\}$ for each $n < \omega$. We can suppose that the elements of $\mathcal{A}$ are pairwise disjoint.

The set $F = \{x \in X : \forall V \in \mathcal{V}(x)(|\{n : V \cap A_n \neq \emptyset\}| = \omega)\}$ is a closed set which does not meet D; so, by hypothesis, there exist disjoint open sets V and W such that $F \subset V$ and $D \subset W$. Then, $\{A_n \cap W : n < \omega\}$ is a locally finite family of open sets; but this is impossible because X is pseudocompact.

The inverse implication can be proved because in every countably compact space each closed discrete subset is finite. □

A space X is *pseudonormal* if every pair of disjoint closed subsets F and G of X, one of which is countable, there are two disjoint open subsets A and B such that $F \subset A$ and $G \subset B$. Of course, every normal space is pseudonormal. Observe that every countably compact space is pseudonormal.

Corollary 1.1.12 [15] *Every pseudonormal pseudocompact space is countably compact.*

This fact allows us to prove that compactness and pseudocompactness are equivalent properties in the class of metrizable spaces. Indeed, every metrizable space is normal and countable compactness and compactness are equivalent in this class of spaces.

Proposition 1.1.13 *If X is a metrizable space, then X is compact if and only if X is pseudocompact.*

Corollary 1.1.14 *A space X is pseudocompact if and only if for each $f \in C(X)$, $f[X]$ is compact.*

The property of being a pseudocompact space is not necessarily inherited to all closed subsets. Indeed, if $\mathscr{A} \subset \mathcal{P}(\omega)$ is a maximal almost disjoint family on ω, then $\mathscr{A}$ is an infinite discrete and closed subset of the pseudocompact space $\Psi(\mathscr{A})$.

Recall that a *regular closed* subset of a space X is a closed subset F which satisfies $\mathrm{cl}_X(\mathrm{int}_X F) = F$.

Proposition 1.1.15 *(1) If X has a pseudocompact dense subspace, then X itself is pseudocompact.*

(2) If every closed subset of a space X is pseudocompact, then X is countably compact.

(3) [2] *Every regular closed subset of a pseudocompact space is pseudocompact.*
(4) *The free topological sum* $\oplus\{X_s : s \in S\}$ *is pseudocompact if and only if each* X_s *is pseudocompact and* $|S| < \omega$.

Proof (1) Let $D \subset X$ be dense and pseudocompact. Let $f : X \to \mathbb{R}$ be a continuous function. Then there is a bounded closed interval K such that $f[D] \subset K$. Hence, $f[X] \subset f[\mathrm{cl}D] \subset \mathrm{cl}(f[D]) \subset K$.

(2) Let $D = \{x_n : n < \omega\}$ be a closed and discrete subspace of X. Since D is discrete, the collection $\{\{x_n\} : n < \omega\}$ is a locally finite family of open subsets of D. By hypothesis D is pseudocompact, hence D has to be finite.

(3) Let X be a pseudocompact space and let A be an open subset of X. Let $\mathcal{U}$ be a pairwise disjoint locally finite family of nonempty open subsets of $\mathrm{cl}A$. Then, $\mathcal{V} = \{A \cap U : U \in \mathcal{U}\}$ is a pairwise disjoint locally finite collection of open subsets of X. Hence, $\mathcal{V}$ and $\mathcal{U}$ must be finite.

(4) If $X = \oplus\{X_s : s \in S\}$ is pseudocompact, being X_s a regular closed subset of X for every $s \in S$, then each X_s is pseudocompact. Furthermore, the family $\{X_s : s \in S\}$ is locally finite in X, so it is finite.

In order to prove the other implication, suppose that there exists $f \in C(X)$ which is not bounded. There is $\{x_n : n < \omega\} \subset X$ with $x_n \neq x_m$ if $n \neq m$ and $|f|(x_n) > n$. Then, there are $s_0 \in S$ and $A \subset \{x_n : n < \omega\}$ with A infinite and $A \subset X_{s_0}$. Therefore, $|f| \upharpoonright X_{s_0}$ is continuous and not bounded; this contradicts the pseudocompactness of X_{s_0}. □

We finish this section by proving that every Tychonoff space X can be considered as a closed subset of a pseudocompact space.

Theorem 1.1.16 [25] *Every Tychonoff space can be embedded as a closed subspace of a pseudocompact space.*

Proof Let X be a Tychonoff space, and let $Y = (\beta X \times [0, \omega_1)) \cup (X \times \{\omega_1\})$ be considered as a subspace of $\beta X \times [0, \omega_1]$. We have that X is homeomorphic to $X \times \{\omega_1\}$. Moreover, $X \times \{\omega_1\}$ is closed in Y.

To see that Y is pseudocompact, observe that $\beta X \times [0, \omega_1)$ is dense in Y, and if $\{(z_n, \alpha_n) : n < \omega\}$ is a countable subset in $X \times [0, \omega_1)$, it is contained in the compact subspace $\beta X \times [0, \eta] \subset \beta X \times [0, \omega_1)$ where $\eta = \sup\{\alpha_n : n < \omega\}$. Then, $\beta X \times [0, \omega_1)$ is countably compact, and so Y is pseudocompact (see Corollary 1.1.8 and Proposition 1.1.15.(1)). □

1.2 Baire Property and Metacompactness

Definition 1.2.1 A topological space X is a *Baire space* if for every sequence $\{G_n : n < \omega\}$ of open dense subsets of X, the intersection $\bigcap\{G_n : n < \omega\}$ is a dense subset of X.

The following theorem was proved by J. Colmez in [3].

Theorem 1.2.2 *Every pseudocompact space is a Baire space.*

Proof Let X be a pseudocompact space and let $\{G_n : n \in \omega\}$ be a sequence of dense open subsets of X, and let U be a nonempty open set of X. Then, for all $n \in \omega$, $U \cap G_n \neq \emptyset$. Since X is regular, it follows that there is an open set W_0 such that $\emptyset \neq W_0 \subset \mathrm{cl}(W_0) \subset U \cap G_0$. This implies that $\emptyset \neq W_0 \cap G_0 \cap G_1$. So, there exists an open set W_1 such that $\emptyset \neq W_1 \subset \mathrm{cl}(W_1) \subset W_0 \cap G_1 \subset U \cap G_1$.

Continuing this way, we obtain a decreasing family $\{W_n : n \in \omega\}$ of open subsets of X with the finite intersection property. Because of item (3) of Theorem 1.1.3, $\bigcap\{\mathrm{cl}(W_n) :\in \omega\} \neq \emptyset$. Since $\bigcap\{\mathrm{cl}(W_n) : n \in \omega\} \subset U \cap (\bigcap_{n\in\omega} G_n)$, then $U \cap (\bigcap_{n\in\omega} G_n) \neq \emptyset$. Therefore, $\bigcap_{n\in\omega} G_n$ is dense in X. □

A family $\{A_s : s \in S\}$ of subsets of X is *point-finite* if for every $x \in X$ the set $\{s \in S : x \in A_s\}$ is finite.

Now we are going to see that in a Baire space, every point-finite collection is locally finite at the points of a dense subset.

Theorem 1.2.3 [9, 10] *Let $\mathcal{U}$ be a point-finite cover of a topological space X. Then, the set A of all the points in X which have a neighborhood which intersects only a finite subcollection of elements in $\mathcal{U}$, contains the intersection of a countable family of dense open subsets of X. In particular, if X is a Baire space, A is dense in X.*

Proof For each $x \in X$, let

$$n(x) = |\{U \in \mathcal{U} : x \in U\}|.$$

Now, for each $n < \omega$, let

$$G_n = \{x \in X : n(x) > n\} \cup \{x \in X : \exists V \in \mathcal{V}(x)(\forall y \in V(n(y) \leq n))\}.$$

We will prove that for every $n < \omega$, G_n is open. Let $n < \omega$ and $z \in G_n$. Assume first that $n(z) = j > n$. Let $U_1, \cdots, U_j$ be the elements of $\mathcal{U}$ which contain z. Thus, $B = \bigcup_{k\leq j} U_k$ is an open set which contains z. Moreover, $B \subset G_n$ because if $y \in B$, then for each $k \in \{1, \cdots j\}$, $y \in U_k$. This implies that $n(y) \geq j > n$ and so $y \in G_n$.

Now assume that $n(z) \leq n$. Then there is a neighborhood V of z such that, for all $y \in V$, $n(y) \leq n$. We claim that $V \subset G_n$ because if $y \in V$, V is a neighborhood of y and $u \in V$, we have $n(u) \leq n$. This implies that $y \in G_n$.

Next, we shall see that for each $n < \omega$, G_n is dense in X. Let $n < \omega$ and V a nonempty open subset of X. Let z be an element in V. If $n(z) > n$, then $z \in G_n$, and so $G_n \cap V \neq \emptyset$. If for all $y \in V$ $n(y) \leq n$, then $z \in G_n$ and, again, G_n meets V. Finally, if there is $y \in V$ such that $n(y) > n$, then $y \in G_n$ and so $G_n \cap V \neq \emptyset$. In any case, G_n meets V. Since V is an arbitrary nonempty open subset of X, G_n is dense in X.

It is our turn to prove that $\bigcap_{n<\omega} G_n \subset A$. Let $x \in \bigcap_{n<\omega} G_n$. Thus, since $n(x) < \omega$ (because $\mathcal{U}$ is point-finite), $x \in G_{n(x)}$. Of course $n(x)$ is not greater than itself, hence there is a neighborhood V of x such that, for each $y \in V$, $n(y) \leq n(x)$. For each $y \in V$ we define $\mathcal{U}_y = \{U \in \mathcal{U} : y \in U\}$. Observe that $\mathcal{U}_y$ contains $n(y)$ elements and that, since $\mathcal{U}$ covers X, $n(y) \geq 1$. In particular we can write $\mathcal{U}_x = \{U_1, U_2, \cdots, U_{n(x)}\}$.

Let $W := \left(\bigcap_{j \leq n(x)} U_j\right) \cap V$. W is a neighborhood of x. We are going to see that the elements of $\mathcal{U}_x$ are the only elements of $\mathcal{U}$ that intersect W. This will prove that $x \in A$. Let $U \in \mathcal{U}$ be such that $U \cap W \neq \emptyset$. Then, there is $y \in U \cap W$, and so $U \in \mathcal{U}_y$. Since $y \in \bigcap_{j \leq n(x)} U_j$, then $\mathcal{U}_x \subset \mathcal{U}_y$. But since $y \in V$, we have that $n(y) \leq n(x)$ and therefore $\mathcal{U}_x = \mathcal{U}_y$. So $U \in \mathcal{U}_x$.

Finally, if X is a Baire space, $\bigcap_{n \in \omega} G_n$ is dense in X and so A is also dense. □

As usual, if $\mathcal{U}$ and $\mathcal{V}$ are covers of a space X, we will say that $\mathcal{U}$ *refines* (or it is *a refinement* of) $\mathcal{V}$ if each $U \in \mathcal{U}$ is contained in some $V \in \mathcal{V}$. And recall that a topological space is *metacompact* if every open cover of X has a point-finite open refinement. Finally, X is *paracompact* if every open cover of X has a locally finite open refinement.

So every paracompact space is metacompact and there are metacompact spaces which are not paracompact. Moreover, every Lindelöf space and every metrizable space are paracompact, and paracompact spaces are normal (see for instance, [7, Sect. 5.1]).

Using part (2) of Theorem 1.1.3, the following equivalence is clear:

Theorem 1.2.4 *A space is compact if and only if it is pseudocompact and paracompact.*

Then every pseudocompact Lindelöf space is compact. Also, we infer that the space of ordinals $[0, \omega_1)$ is not paracompact even in spite of being collectionwise normal. This is also the case of the Σ-product $\Sigma[0, 1]^{\omega_1}$.

An improvement of Theorem 1.2.4 is the following:

Theorem 1.2.5 [30, 38] *A space X is compact if and only if it is pseudocompact and metacompact.*

Proof Of course, we only have to prove the sufficiency, so we assume that X is pseudocompact and metacompact and let $\mathcal{L}$ be an open cover of X. For each $x \in X$, there exists $L_x \in \mathcal{L}$ and an open neighborhood N_x of x, such that $x \in N_x \subset \text{cl}(N_x) \subset L_x$. Then, the family $\mathcal{N} := \{N_x : x \in X\}$ is an open cover of X and $\mathcal{N}$ is a refinement of $\mathcal{L}$. Let $\mathcal{U}$ be a point-finite refinement of $\mathcal{N}$. We want to find a maximal family $\mathcal{V}$ of open subsets of X possessing the following properties:

(1) $|\{V \in \mathcal{V} : U \cap V \neq \emptyset\}| \leq 1$ for every $U \in \mathcal{U}$;
(2) $|\{U \in \mathcal{U} : U \cap V \neq \emptyset\}| < \aleph_0$ for every $V \in \mathcal{V}$.

Since X is a Baire space, Theorem 1.2.3 implies that the set A of all the points of X which have a neighborhood that intersects only the elements of a finite subset of $\mathcal{U}$, is dense in X. In particular $A \neq \emptyset$, so we can take $y \in A$. Let V_y be a neighborhood of y which only intersects the elements of a finite subcollection of $\mathcal{U}$, and let $\mathcal{V} := \{V_y\}$.

It happens that property (1) holds because $|\mathcal{V}| = 1$. Since V_y intersects only a finite collection of elements of $\mathcal{U}$, property (2) holds as well.

Let $\mathcal{O}$ be the set of families $\mathcal{V}$ of open subsets of X which satisfy properties (1) and (2). As we have already seen, $\mathcal{O} \neq \emptyset$. Consider $\mathcal{O}$ ordered by the relation $\subset$. Let $\mathcal{C}$ be a chain in $\mathcal{O}$ and define $\mathcal{A} = \bigcup_{\mathcal{V}\in\mathcal{C}} \mathcal{V}$. Now, we verify that $\mathcal{A}$ is an upper bound of $\mathcal{C}$ in $\mathcal{O}$. Indeed, $\mathcal{A}$ is a family of open subsets of X. Let us see that $\mathcal{A}$ has properties (1) and (2).

Assume that $\mathcal{A}$ does not have property (1). Then, there is $U \in \mathcal{U}$ such that $|\{V \in \mathcal{A} : U \cap V \neq \emptyset\}| \geq 2$. Thus, let V_1 and V_2 be two different elements in $\mathcal{A}$. According to the definition of $\mathcal{A}$, there are $\mathcal{V}_1$ and $\mathcal{V}_2$ in $\mathcal{C}$ such that $V_1 \in \mathcal{V}_1$ and $V_2 \in \mathcal{V}_2$. Since $\mathcal{C}$ is a chain, we can suppose that $\mathcal{V}_1 \subset \mathcal{V}_2$. Therefore, we have that V_1 and V_2 are elements of $\mathcal{V}_2$, and so $|\{V \in \mathcal{V}_2 : U \cap V \neq \emptyset\}| > 1$. This contradicts the fact that $\mathcal{V}_2$ satisfies condition (1).

Now we shall see that $\mathcal{A}$ satisfies condition (2): Let $V \in \mathcal{A}$. Hence, there is $\mathcal{V} \in \mathcal{C}$ such that $V \in \mathcal{V}$. Since $\mathcal{V} \in \mathcal{O}$, $\mathcal{V}$ satisfies condition (2). This implies that $|\{U \in \mathcal{U} : U \cap V \neq \emptyset\}| < \aleph_0$. Therefore, $\mathcal{A}$ satisfies this condition.

Given that $\mathcal{A}$ possesses properties (1) and (2), then $\mathcal{A} \in \mathcal{O}$ and of course it is an upper bound of $\mathcal{C}$ in $\mathcal{O}$. By Zorn's Lemma, there exists a maximal family $\mathcal{V}_0 \in \mathcal{O}$.

Let $Y := \bigcup\{U \in \mathcal{U} : U \cap \bigcup \mathcal{V}_0 \neq \emptyset\}$. We are going to see that Y is a dense subspace of X. Assume that Y is not dense in X. Then there is a nonempty open subset A of X such that $A \subset X \setminus Y$. Let $x \in A$. Again, by using Theorem 1.2.3, there is a point $y \in A$ which has an open neighborhood W_y which only intersects a finite quantity of elements of $\mathcal{U}$. Thus $A \cap W_y$ is a nonempty open set contained in $X \setminus Y$. In order to arrive to a contradiction we are going to prove that the family $\mathcal{V}' = \mathcal{V}_0 \cup \{A \cap W_y\}$ satisfies conditions (1) and (2).

If $U \in \mathcal{U}$ intersects an element of $\mathcal{V}_0$, then $U \subset Y$ and so $U \cap (A \cap W_y) = \emptyset$. Since $\mathcal{V}_0$ satisfies (1), U only intersects an element of $\mathcal{V}_0$ and, therefore, intersects only an element of $\mathcal{V}'$. If U does not intersect any of the elements of $\mathcal{V}_0$, then U intersects at most one element of $\mathcal{V}'$. In any case, $\mathcal{V}'$ satisfies property (1).

Now, let $V' \in \mathcal{V}'$. If $V' \in \mathcal{V}_0$, then V' intersects at most a finite quantity of elements of $\mathcal{U}$. If $V' = A \cap W_y$, then, since $V' \subset W_y$, V' intersects at most the elements of a finite subfamily of $\mathcal{U}$. So, $\mathcal{V}'$ satisfies condition (2).

On the other hand, $\mathcal{V}_0 \subset \mathcal{V}'$, but they are not equal because $A \cap W_y$ is not an element of $\mathcal{V}_0$. This is a contradiction because $\mathcal{V}_0$ is a maximal family with respect to properties (1) and (2). Since this contradiction has its origins in our hypothesis *Y is not dense in X*, we conclude that Y is, indeed, dense in X.

The family $\mathcal{V}_0$ is locally finite in X because if $x \in X$ there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Since $\mathcal{V}_0$ satisfies property (1), it follows that U_x intersects at most one element of $\mathcal{V}_0$. Since X is pseudocompact, it turns out that $|\mathcal{V}_0| < \aleph_0$. Since $\mathcal{V}_0$ has condition (2), we have that:

$$|\{U \in \mathcal{U} : U \cap \bigcup \mathcal{V}_0 \neq \emptyset\}| \leq \sum_{V\in\mathcal{V}_0} |\{U \in \mathcal{U} : U \cap V \neq \emptyset\}| < \aleph_0.$$

Put $\{U \in \mathcal{U} : U \cap \bigcup \mathcal{V}_0 \neq \emptyset\} = \{U_i : i \in \{1, 2 \ldots, n\}\}$. Because $\mathcal{U}$ is a refinement of $\mathcal{N}$, for each $i \in \{1, 2 \ldots, n\}$ there exists $N_{x_i} \in \mathcal{N}$ such that $U_i \subset N_{x_i}$. Finally, we obtain

$$X = \mathrm{cl}(Y) = \mathrm{cl}(\bigcup\{U \in \mathcal{U} : U \cap \bigcup \mathcal{V}_0 \neq \emptyset\}) = \bigcup_{i=1}^{n} \mathrm{cl}(U_i) \subset \bigcup_{i=1}^{n} \mathrm{cl}(N_{x_i}) \subset \bigcup_{i=1}^{n} L_{x_i}.$$

Therefore, X is a compact space. □

1.3 Pseudocompactness and the Stone-Čech Compactification

We present the following result without proof (see e.g. [11]).

Proposition 1.3.1 *The following conditions are equivalent for a dense subset D of a space X.*

(1) D is C^*-embedded in X.
(2) If Z_1 and Z_2 are two disjoint zero-sets of D, then $\mathrm{cl}_{\beta X} Z_1 \cap \mathrm{cl}_{\beta X} Z_2 = \emptyset$.
(3) If Z_1 and Z_2 are two zero-sets in D, then $\mathrm{cl}_X Z_1 \cap \mathrm{cl}_X Z_2 = \mathrm{cl}_X (Z_1 \cap Z_2)$.

Let $\mathcal{Z}(X)$ be the set of all zero-sets of X; that is, $Z \in \mathcal{Z}(X)$ if there is $f \in C(X)$ such that $Z = f^{-1}[\{0\}]$; in such a case, we will write $Z = Z(f)$. A nonempty collection $\mathcal{F} \subset \mathcal{Z}(X)$ is a *z-filter* on X if $\emptyset \notin \mathcal{F}$, $\mathcal{F}$ is closed under finite intersections, and if $F \subset G$, $F \in \mathcal{F}$ and $G \in \mathcal{Z}(X)$, then $G \in \mathcal{F}$. A *z-ultrafilter* on X is a z-filter which is not contained in a different proper z-filter on X. Also we will say that a z-filter $\mathcal{F}$ is *fixed* if $\bigcap \mathcal{F} \neq \emptyset$, and in the contrary case we will say that it is *free*.

Denote by βX the set of z-ultrafilters on X. When $Z \in \mathcal{Z}(X)$, we define $\mathrm{cl}(Z) := \{p \in \beta X : Z \in p\}$. Note that $\mathrm{cl}(X) = \beta X$, $\mathrm{cl}(Z_1 \cup Z_2) = \mathrm{cl}(Z_1) \cup \mathrm{cl}(Z_2)$, and $\mathrm{cl}(\emptyset) = \emptyset$. The collection $\{\mathrm{cl}(Z) : Z \in Z(X)\}$ is a base for the closed subsets of a topology τ in βX. The space $(\beta X, \tau)$ is the *Stone-Čech compactification* of X, where X is identified with the subspace of βX formed by the fixed z-ultrafilters. Consequently $\beta X \setminus X$ denotes the set of all free z-ultrafilters on X.

The result that we are now going to present contains some of the fundamental properties of βX (see for instance [11]), and part of it is a consequence of Proposition 1.3.1.

Theorem 1.3.2 *For a space X, the compactification βX of X satisfies:*

(1) every continuous function g defined on X and with values in a compact space K has a continuous extension $\overline{g}$ from βX to K;
(2) every $f \in C^*(X)$ has an extension $f^\beta \in C(\beta X)$;
(3) every two disjoint zero-sets in X have disjoint closures in βX;
(4) for every two disjoint zero-sets in X, Z_1 and Z_2, we have: $\mathrm{cl}_{\beta X}(Z_1 \cap Z_2) = \mathrm{cl}_{\beta X} Z_1 \cap \mathrm{cl}_{\beta X} Z_2$;
(5) different z-ultrafilters on X have different limits in βX.

Moreover, βX is the only space satisfying (1)–(5) in the sense that if K is a compactification of X which satisfies one of the conditions listed above, then there is a homeomorphism $h : \beta X \to K$ such that $h(x) = x$ for every $x \in X$. We denote this fact by $K \equiv_X \beta X$.

We can obtain new characterizations of the pseudocompact property in which βX and the G_δ-sets play a central role. We say that a subset G of the space Y is *G_δ-dense in Y* if every nonempty G_δ-subset of Y meets G.

Theorem 1.3.3 *Let X be a topological space. The following conditions are equivalent.*

(1) X is pseudocompact.
(2) X is G_δ-dense in every compactification of X.
(3) [15] *X is G_δ-dense in βX.*
(4) Every nonempty zero-set in βX has a nonempty intersection with X.

Proof (1)$\Rightarrow$ (2): Let bX be a compactification of X. Let $G = \bigcap\{G_n : n < \omega\}$ be a nonempty G_δ-subset of bX, where G_n is open in bX for all $n < \omega$. Assume that $G \cap X = \emptyset$ and take $z \in G$. Then, since bX is Tychonoff, for each $n < \omega$, there is a zero-set Z_n of bX such that $z \in Z_n \subset G_n$.

The set $Z = \bigcap\{Z_n : n < \omega\}$ is a zero-set of bX and $z \in Z \subset G$. Let $f \in C(bX)$ such that $Z = Z(f)$. Without loss of generality, we can suppose that $f(x) \geq 0$ for every $x \in bX$. So $f(x) > 0$ for every $x \in X$. Since $z \in f^{-1}[[0, \frac{1}{n+1})]$ for all $n < \omega$ and X is dense in bX, then for each $n < \omega$, there exists $x_n \in X$ such that $f(x_n) < 1/(n+1)$; hence $(1/f) \restriction X \in C(X)$ and it is not bounded, which contradicts the assumption that X is pseudocompact.

(2)$\Rightarrow$ (3): The proof of this implication is clear.

(3)$\Rightarrow$ (4): It is enough to observe that if $Z = Z(f)$, then $Z = \bigcap_{n\in\mathbb{N}} f^{-1}[(-\frac{1}{n}), \frac{1}{n}]$.

(4)$\Rightarrow$ (1): Assume that there exists $f \in C(X)$, positive and not bounded. For $n < \omega$, we define $f_n = f \wedge (\mathbf{n+1})$ (here $\mathbf{n+1}$ is the constant function equal to the natural number $n+1$). Since $f_n \in C^*(X)$, then there exists $f_n^\beta \in C(\beta X)$ which is a continuous extension of f_n. Let $F_n = \left(f_n^\beta\right)^{-1}[[n+1, \infty)]$; this set is a zero-set of βX.

Let us confirm that F_n is not empty. For every $n < \omega$, there is $x_n \in X$ such that $f(x_n) > n+1$, so $f_n(x_n) = n+1$, then $x_n \in F_n$. Observe that $f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and for all $x \in X$. Therefore we obtain that $f_{n+1}^\beta(z) \leq f_n^\beta(z)$ for all $z \in \beta X$.

In short, we have that $\{F_n : n \in \mathbb{N}\}$ is a decreasing family of nonempty zero-sets of βX; so, it has the $f.i.p$. Since βX is compact, we conclude that $F = \bigcap\{F_n : n < \omega\}$ is not empty. Furthermore, since every F_n is a zero-set, it follows that F is a zero-set and therefore a G_δ-set in βX. If $z \in X \cap F$, then $n \leq f_n(z)$ for all $n < \omega$. On the other hand, there exists $n_0 \in \mathbb{N}$ such that $f(z) \leq n_0$, hence for every $m > n_0$ we have that $f_m(z) = f(z) < m$. However, $f_m(z) \geq m$. This contradiction implies that the set $F \cap X$ must be empty, and we finish the proof. □

Now we are able to present another classic example of a pseudocompact non countably compact space.

Example 1.3.4 [18] Let us consider the space $\Lambda = \beta\mathbb{R} \setminus (\beta\mathbb{N} \setminus \mathbb{N})$; clearly Λ contains the discrete space $\mathbb{N}$ as a closed subset and so it is not countably compact. We shall prove that Λ is pseudocompact. Assume that there exists a continuous function $f : \Lambda \to \mathbb{R}$ which is not bounded. The set $\mathbb{R} \setminus \mathbb{N}$, being dense in Λ, contains a sequence $x_0, x_1, \ldots, x_n, \ldots$ of distinct points such that $|f(x_n)| > n$ for $n < \omega$. Since f is continuous, the set $F = \{x_n : n < \omega\}$ has no accumulation point in Λ, which implies, in particular, that F is a closed subset of $\mathbb{R}$. So F and $\mathbb{N}$ are two disjoint closed subsets of $\mathbb{R}$. Since every closed set in a metrizable space is a zero-set, and using Theorem 1.3.2, we obtain that $\mathrm{cl}_{\beta\mathbb{R}} F \cap \mathrm{cl}_{\beta\mathbb{R}} \mathbb{N} = \mathrm{cl}_{\beta\mathbb{R}} F \cap \beta\mathbb{N} = \emptyset$. That is, $\mathrm{cl}_{\beta\mathbb{R}} F \subset \Lambda$; thus F is an infinite subset of $\beta\mathbb{R}$ with no accumulation point. This contradiction shows that Λ is pseudocompact.

Next we are going to see the relations between realcompactness and pseudocompactness. Recall the following definitions that we need for what follows.

Definition 1.3.5 Let X be a Tychonoff space.

(1) A z-ultrafilter p on X is *real* if it has the countable intersection property.
(2) A topological space X is *realcompact* if it satisfies any of the following equivalent properties:

  (a) X is not C-embedded in any proper extension $\tilde{X}$ of X;
  (b) each real z-ultrafilter on X is fixed;
  (c) X is homeomorphic to a closed subspace of a power of $\mathbb{R}$.

Proposition 1.3.6 *(1) A space X is pseudocompact if and only if each z-ultrafilter on X is real.*
(2) [15] *A space X is compact if and only if X is pseudocompact and realcompact.*
(3) A space X is pseudocompact if and only if each countable C-embedded subset of X is compact.
(4) If X is pseudocompact, then every countable zero-set of X is compact.

Proof (1) Assume that X is pseudocompact. Let p be a z-ultrafilter on X, and let $\{U_n : n < \omega\}$ be a sequence of elements of p. Let $Z(f_n) = U_n$ and take $Z_n = Z(f_n^\beta)$. Thus Z_n is a zero-set in βX such that $Z_n \cap X = U_n$. So, the collection $\{U_n : n < \omega\}$ has the finite intersection property (p is a z-ultrafilter), then $\{Z_n : n < \omega\}$ is a collection of closed subsets of βX having the finite intersection property. Therefore, $\bigcap_{n<\omega} Z_n = Z$ is a nonempty zero-set of βX. Since X is pseudocompact, we have that $Z \cap X \neq \emptyset$ (Theorem 1.3.3), but observe that $Z \cap X = \bigcap_{n<\omega} U_n$.

Suppose now that every z-ultrafilter on X has the countable intersection property, and let $f : X \to \mathbb{R}$ be a non negative non bounded continuous function. The collection $\{f^{-1}[[n, \infty)] : n < \omega\}$ is a family of zero-sets with the finite intersection property. Then, it is contained in a z-ultrafilter p on X. By hypothesis the set $\bigcap_{n<\omega} f^{-1}[[n, \infty)]$ is not empty, which is not possible.

(2) Let X be pseudocompact and realcompact. Let $p \in \beta X$. Then p is real and fixed. Therefore, $p \in X$; so $X = \beta X$ and therefore X is compact.

(3) Assume that X is pseudocompact and that $A \subset X$ is countable and C-embedded. Thus, for all $f \in C(A)$ its extension $\widehat{f}$ belongs to $C^*(X)$. Hence A is pseudocompact. Furthermore, as A is trivially Lindelöf and every Lindelöf is realcompact, we infer that A is compact because of Claim (2).

Next we are going to prove the converse. Let N be a C-embedded subset of X. If N is not compact, N is not pseudocompact because it is Lindelöf. Then, there is a non bounded $f \in C(N)$. Since N is C-embedded in X, there exists $\widetilde{f} \in C(X)$ which extends f. This means that $\widetilde{f}$ is not bounded and so X is not pseudocompact.

(4) We assume that $Z \subset X$ is a countable zero-set which is not compact and we will derive a contradiction. We have that there exists a free z-filter $\mathcal{F}$ such that $Z \in \mathcal{F}$. Take a z-ultrafilter $\mathcal{U}$ such that $\mathcal{F} \subset \mathcal{U}$. In particular, $\mathcal{U}$ is free. For every $x \in Z$, we take $U_x \in \mathcal{U}$ such that $x \notin U_x$. So, $\{U_x : x \in Z\}$ is a countable subfamily of $\mathcal{U}$. Since X is pseudocompact, $\mathcal{U}$ is real. Hence, $\bigcap\{U_x : x \in Z\}$ is an element in $\mathcal{U}$ which does not intersect Z, and this is a contradiction. □

The subspace of βX constituted by the real z-ultrafilters on X is usually denoted as υX. The space υX is always a realcompact space, and for every space X, X is C-embedded in υX (see [11], p. 118). The following result is a consequence of part (1) of Proposition 1.3.6 and of part (2) of Theorem 1.3.2.

Theorem 1.3.7 *A space X is pseudocompact if and only if $\upsilon X = \beta X$.*

Furthermore we have:

Theorem 1.3.8 *If $|\beta X \setminus X| < 2^{\mathfrak{c}}$, then X is pseudocompact.*

Proof Indeed, if X is not pseudocompact, then it contains a countable closed and discrete C-embedded subspace N. But then $\mathrm{cl}_{\beta X} N \equiv \beta\mathbb{N}$ because $\mathrm{cl}_{\beta X} N$ is a compactification of N in which N is C^*-embedded. But $2^{\mathfrak{c}} = |\mathrm{cl}_{\beta X} N \setminus N| \leq |\beta X \setminus X|$ (see [11], p. 131) because N is closed in X and then $\mathrm{cl}_{\beta X} N \setminus N \subset \beta X \setminus X$. □

Recall that two subsets A and B of a topological space are *completely separated* if there is $f \in C^*(X)$ such that $f[X] \subset [0, 1]$, $f[A] \subset \{0\}$ and $f[B] \subset \{1\}$.

Proposition 1.3.9 [14] *The following statements are equivalent for a space X.*

(1) X has only one compactification.
(2) $|\beta X \setminus X| \leq 1$.
(3) If two closed subsets of X are completely separated then at least one of them is compact.

A space X satisfying one of the equivalent conditions listed in the previous proposition is called *almost compact*.

Proposition 1.3.10 *If X is almost compact, then it is pseudocompact and locally compact.*

Proof According to Proposition 1.3.8, X is pseudocompact. To finish suppose that $X \neq \beta X$; that is, $\beta X = \{p\} \cup X$ with $p \notin X$. So X is an open subset of βX, and then X is locally compact. □

A classic example of an almost compact space which is not compact is the space of ordinals $[0, \omega_1)$. It is not difficult to give examples of spaces which are not almost compact. In general, given a Tychonoff space X, the difference $\beta X \setminus X$ is enormous. However, in the case of Mrówka-Isbell spaces $\Psi(\mathscr{A})$ that we presented after Definition 1.1.9, Mrówka himself proved that there exists a maximal almost disjoint family $\mathscr{A}$ for which $\Psi(\mathscr{A})$ is almost compact. Moreover, he explained how to construct a maximal almost disjoint family $\mathscr{B}$ for which $\beta\Psi(\mathscr{B})\setminus \Psi(\mathscr{B})$ has cardinality 2^ω. (See Sect. 8.6.)

1.4 Products of Pseudocompact Spaces

Pseudocompactness is an unstable property with respect to the products, as we will see now.

Example 1.4.1 [36] There exists a pseudocompact space X such that X^2 is not pseudocompact.

Proof The space X will be constructed inside the Stone-Čech compactification $\beta\mathbb{N}$ of the natural numbers.

Let N_1 be the set of odd numbers, and N_2 be the set of even numbers. Each one is trivially C^*-embedded in $\mathbb{N}$. Therefore,

$$\mathrm{cl}_{\beta\mathbb{N}} N_i = \beta N_i \equiv_{\mathbb{N}} \beta\mathbb{N}.$$

Moreover, $\beta N_1 \cap \beta N_2 = \emptyset$ (Theorem 1.3.2) and $\beta N_1 \cup \beta N_2 = \beta\mathbb{N}$. Let $h : \beta N_1 \to \beta N_2$ be the homeomorphism such that $h(2n-1) = 2n$ for all $n \in \mathbb{N}$, and let $\tau : \beta N_1 \cup \beta N_2 \to \beta\mathbb{N}$ be defined as follows:

$$\tau(x) = h(x) \text{ if } x \in \beta N_1 \text{ and } \tau(x) = h^{-1}(x) \text{ if } x \in \beta N_2.$$

It happens that τ does not have fixed points, and $\tau \circ \tau$ is the identity function on $\beta\mathbb{N}$.

We are going to define the space X recursively. Let $\mathcal{S}$ be the collection of all the infinite countable subsets of $\beta\mathbb{N}$. As $|\beta\mathbb{N}| = 2^{\mathfrak{c}}$, we have that $|\mathcal{S}| = (2^{\mathfrak{c}})^{\aleph_0} = 2^{\mathfrak{c}}$. Enumerate $\mathcal{S}$: $\{S_\lambda : \lambda \in 2^{\mathfrak{c}}\}$. Let $p_0 \in \mathrm{cl}_{\beta\mathbb{N}} S_0$. Assume that for all $\xi < \gamma < 2^{\mathfrak{c}}$ we have chosen a point $p_\xi \in \mathrm{cl}_{\beta\mathbb{N}} S_\xi \setminus \{\tau(p_\chi) : \chi < \xi\}$.

We have that $\left|\mathrm{cl}_{\beta\mathbb{N}} S_\gamma\right| = 2^{\mathfrak{c}}$. We can choose

$$p_\gamma \in \mathrm{cl}_{\beta\mathbb{N}} S_\gamma \setminus \{\tau(p_\xi) : \xi < \gamma\}.$$

Now we define X as the subspace $\mathbb{N} \cup \{p_\gamma : \gamma < 2^{\mathfrak{c}}\}$ of $\beta\mathbb{N}$.