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Karl-Hermann Neeb · Gestur Ólafsson

Reflection Positivity

A Representation Theoretic Perspective

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Reflection Positivity

A Representation Theoretic Perspective

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Karl-Hermann Neeb
Department Mathematik
Universität Erlangen-Nürnberg
Erlangen, Germany

Gestur Ólafsson
Department of Mathematics
Louisiana State University
Baton Rouge, LA, USA

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Preface

The concept of reflection positivity (RP) occurs as an important theme in various areas of mathematics and physics:

- In the representation theory of Lie groups, it establishes a passage of a unitary representation of a symmetric Lie group (such as the euclidean motion group) to a unitary representation of the Cartan dual group (such as the Poincaré group) [FOS83, JOl00, JOl98, NÓ14, JPT15].
- In constructive Quantum Field Theorem (QFT), it arises as the condition of Osterwalder–Schrader (OS) positivity for a euclidean field theory to correspond to a relativistic one [GJ81, Ja08, Ja18, Os95a, Os95b, OS73, OS75].
- For stochastic processes, it is weaker than the Markov property and specifies processes arising in lattice gauge theory. It plays a central role in the mathematical study of phase transitions and symmetry breaking [FILS78, JJ16, JJ17, Nel73].
- In analysis, it is a crucial condition that leads to inequalities such as the Hardy–Littlewood–Sobolev inequality [FL10].

Only recently, it became apparent that there are many hidden and still not sufficiently well-understood structures underlying the duality between unitary representations of a symmetric Lie group and its dual. Establishing reflection positivity in this context requires new analytic methods and new geometric insight into constructions and realizations of representations in analytic contexts. New developments concern analytic issues such as criteria for integrating Lie algebra representations to Lie group representations, reflection positive functions, distributions and kernels, new dilation techniques for representations and unexpected connections between Kubo–Martin–Schwinger (KMS) states of C^*-algebras and reflection positive unitary representations.

This was our motivation to write this "light" introduction to the representation theoretic aspects of reflection positivity to present this perspective on a level suitable for doctoral students.

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Erlangen, Germany Karl-Hermann Neeb
Baton Rouge, USA Gestur Ólafsson

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Contents

Chapter 1
Introduction

In the context of quantum physics reflection positivity is often related to Wick rotation, which roughly means multiplying the time coordinate by $i = \sqrt{-1}$. This can be made precise and used in analytic constructions if this process is related to analytic continuation of the time variable to a domain in the complex plane which provides a means to go back and forth between real and imaginary time.

A duality of a similar flavor also exists in the context of Lie groups, where it arose almost a century ago in the work of É. Cartan on the classification of symmetric spaces. Here one considers a *symmetric Lie group* (G, τ), i.e., a Lie group G, endowed with an involutive automorphism τ. Then the Lie algebra $\mathfrak{g}$ of G decomposes into τ-eigenspaces

$$\mathfrak{h} = \{x \in \mathfrak{g} \colon \tau x = x\} \quad \text{and} \quad \mathfrak{q} = \{x \in \mathfrak{g} \colon \tau x = -x\}.$$

From the bracket relations $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{q}] \subseteq \mathfrak{q}$, and $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{h}$ it then follows that the *Cartan dual*

$$\mathfrak{g}^c := \mathfrak{h} + i\mathfrak{q}$$

also is a Lie subalgebra of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$. We thus obtain a duality relation between symmetric Lie groups (G, τ) and (G^c, τ^c), where G^c denotes a Lie group with Lie algebra G^c and τ^c an involutive automorphism acting by $x + iy \mapsto x - iy$ on the Lie algebra $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$. The classical examples from quantum physics are the euclidean motion group $G = E(d) \cong \mathbb{R}^d \rtimes \mathrm{O}_d(\mathbb{R})$ and the automorphism τ of $E(d)$ induces by time reflection. This establishes a duality with the Poincaré group $G^c = P(d) \cong \mathbb{R}^{1,d-1} \rtimes \mathrm{O}_{1,d-1}(\mathbb{R})$.

In many cases both groups G and G^c are contained in one complex Lie group $G_{\mathbb{C}}$ and $H = G \cap G^c$ is a Lie subgroup with Lie algebra $\mathfrak{h}$, contained in both. Therefore any passage from G to G^c should be related to analytic continuation to domains in $G_{\mathbb{C}}$ whose closure intersects both groups G and G^c. On the Lie algebra level the passage from $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ to $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$ very much resembles Wick rotation because the elements of $\mathfrak{q}$ are multiplied by i (cf. [HH17], where this context is discussed for pseudo-Riemannian manifolds). Simple examples are

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- the circle group $G = \mathbb{T} \subseteq G_{\mathbb{C}} = \mathbb{C}^\times$ with the dual group $G^c = \mathbb{R}^\times$ and $\tau(z) = z^{-1}$.
- the additive group $G = \mathbb{R} \subseteq G_{\mathbb{C}} = \mathbb{C}$ with $\tau(x) = -x$ and $G^c = i\mathbb{R}$.
- the group $G = \mathrm{GL}_n(\mathbb{R}) \subseteq G_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C})$ with $\tau(g) = g^{-\top}$ and $G^c = \mathrm{U}_n(\mathbb{C})$.
- the group $G = \mathrm{O}_n(\mathbb{R}) \subseteq G_{\mathbb{C}} = \mathrm{O}_n(\mathbb{C})$ with $\tau(g) = rgr$, where r is an orthogonal reflection in a hyperplane and $G^c = \mathrm{O}_{1,n-1}(\mathbb{R})$.
- the group $G = \mathrm{O}_{1,n}(\mathbb{R}) \subseteq G_{\mathbb{C}} = \mathrm{O}_{n+1}(\mathbb{C})$ with $\tau(g) = \theta g \theta$, where θ is an orthogonal reflection in a Minkowski hyperplane and $G^c = \mathrm{O}_{2,n-1}(\mathbb{R})$.

If $U : G \to \mathrm{U}(\mathscr{E})$ is a unitary representation of G, then, for $x \in \mathfrak{g}$, the infinitesimal generator $\mathrm{d}U(x)$ of the unitary one-parameter $t \mapsto U(\exp tx)$ is a skew-adjoint operator, and multiplication by i leads to a selfadjoint operator. Therefore we cannot expect unitary representations of G and G^c to live on the same Hilbert space. What we need instead is some extra structure on $\mathscr{E}$ that permits us to construct another Hilbert space on which a unitary representation of G^c may be implemented. This is where reflection positivity comes in as a framework establishing a bridge between unitary representations of G and G^c. This perspective isolates many of the key features of reflection positivity and subsumes not only the representation theoretic aspects of classical applications along the lines of Osterwalder and Schrader [OS73, OS75], but also quite recent developments in Algebraic Quantum Field Theory (AQFT), where Haag–Kastler nets of operator algebras are constructed on space times by methods relying very much on the unitary or anti-unitary representations of the groups involved [Bo92, BJM16, NÓ17, Nel69]. Another recent branch of applications of reflection positivity for the euclidean conformal group along these lines concerns Hardy–Littlewood–Sobolev inequalities in analysis (see [FL10, FL11, NÓ14]).

The extra structure required on the Hilbert space $\mathscr{E}$ can be specified axiomatically as follows. A *reflection positive Hilbert space* is a triple $(\mathscr{E}, \mathscr{E}_+, \theta)$, consisting of a Hilbert space $\mathscr{E}$ with a unitary involution θ and a closed subspace $\mathscr{E}_+$ satisfying

$$\langle \xi, \xi \rangle_\theta := \langle \xi, \theta \xi \rangle \geq 0 \quad \text{for} \quad \xi \in \mathscr{E}_+.$$

This structure immediately leads to a new Hilbert space $\widehat{\mathscr{E}}$ that we obtain from the positive semidefinite form $\langle \cdot, \cdot \rangle_\theta$ on $\mathscr{E}_+$. We write $q : \mathscr{E}_+ \to \widehat{\mathscr{E}}, \xi \mapsto \widehat{\xi}$ for the natural map. Bounded or unbounded operators S on $\mathscr{E}_+$ preserving the kernel of q induce an operator $\widehat{S}$ on $\widehat{\mathscr{E}}$ via $\widehat{S}\widehat{\xi} := \widehat{S\xi}$. The passage $S \mapsto \widehat{S}$ is called the *Osterwalder–Schrader (OS) transform.*

On the level of $\mathscr{E}$ (the euclidean side), we consider a unitary representation U of a symmetric Lie group (G, τ) on a reflection positive Hilbert space $(\mathscr{E}, \mathscr{E}_+, \theta)$. There are several ways to express the compatibility of the representation U with $\mathscr{E}_+$ and θ. One is the compatibility relation

$$\theta U_g \theta = U_{\tau(g)} \quad \text{for} \quad g \in G$$

between τ and the unitary involution θ and another is the invariance of $\mathscr{E}_+$ under the operators U_h, where h belongs to the identity component $H := G_0^\tau$ of the group

G^τ of τ-fixed points in G. These two already ensure that the OS transform yields a unitary representation $(\widehat{U}_h)_{h\in H}$ on $\widehat{\mathscr{E}}$. We are aiming at a unitary representation of the Cartan dual group G^c on $\widehat{\mathscr{E}}$ and this can only be achieved by additional requirements. On the algebraic level, if we only consider the representation $\mathrm{d}U$ of the Lie algebra $\mathfrak{g}$ on the subspace $\mathscr{E}^\infty$ of smooth vectors for $(U, \mathscr{E})$, it suffices to have a subspace $\mathscr{D} \subseteq \mathscr{E}_+ \cap \mathscr{E}^\infty$ which is $\mathfrak{g}$-invariant. Then OS transformation immediately leads to a representation of $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$ by skew-symmetric operators on the subspace $\widehat{\mathscr{D}} \subseteq \widehat{\mathscr{E}}$ by

$$x + iy \mapsto \big(\mathrm{d}U(x)|_{\mathscr{D}} + i\mathrm{d}U(y)|_{\mathscr{D}}\big)\widehat{}.$$

This simple passage already shows how the θ-twisting of the scalar product on $\mathscr{E}_+$ turns the symmetric operators $i\mathrm{d}U(y)$, $y \in \mathfrak{q}$, on $\mathscr{D}$ into skew-symmetric operators on $\widehat{\mathscr{D}}$, but it completely ignores all issues related to essential selfadjointness and, accordingly, integrability to group representations. We therefore need more global ways to express the reflection positivity requirements.

One way to express such requirements refers to subsemigroups $S \subseteq G$ which are *symmetric* in the sense that they are invariant under the involution $s \mapsto s^\sharp := \tau(s)^{-1}$. Then we call $(U, \mathscr{E})$ *reflection positive with respect to* S if $U_s\mathscr{E}_+ \subseteq \mathscr{E}_+$ holds for $s \in S$. Then OS transformation yields a $*$-representation $(\widehat{U}_s)_{s\in S}$ of the involutive semigroup $(S, \sharp)$, and if S has interior points one can expect this representation to “extend analytically” to a unitary representation of a dual group G^c. A typical situation arises for $(G, S, \tau) = (\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_{\mathbb{R}})$ in euclidean field theory from the one-parameter group of time translations. Then $(\widehat{U}_t)_{t\geq 0}$ is a one-parameter group of hermitian contractions on $\widehat{\mathscr{E}}$, hence of the form $\widehat{U}_t = e^{-tH}$ for a positive selfadjoint operator H, and $U_t^c := e^{itH}$ defines a unitary representation of the dual group $G^c = i\mathbb{R}$ with positive spectrum (in QFT H corresponds to the Hamiltonian, the energy observable, which should be positive).

There are, however, many situations where there are no natural symmetric subsemigroups $S \subseteq G$, or some which do not have interior points, such as the subsemigroup S of the euclidean motion group mapping a closed half space into itself. In this case the reflection positivity requirements on $(U, \mathscr{E}, \mathscr{E}_+, \theta)$ have to be formulated differently. Instead of a subsemigroup, we consider a domain $G_+ \subseteq G$ (mostly open or with dense interior) and the reflection positivity condition is inspired by situations in QFT, where Hilbert spaces are generated by field operators: Instead of fixing $\mathscr{E}_+$ a priori, we consider a real linear space V and a linear map $j\colon V \to \mathscr{H}$ whose range generates $\mathscr{E}$ under U_G and θ and call $(U, \mathscr{E}, j, V)$ *reflection positive with respect to* G_+ if the subspace $\mathscr{E}_+ := [\![U_{G_+}^{-1} j(V)]\!]$ is θ-positive. The prototypical examples arise for circle groups $G = \mathbb{R}/\beta\mathbb{Z}$ with $\tau(g) = g^{-1}$, where $G_+ = [0, \pi] + \beta\mathbb{Z}$ is a half circle. In physics they occur in the context of quantum statistical mechanics, where β plays the role of an inverse temperature [Fro11]. Both approaches, the one based on semigroups S and on domains G_+ lead to situations in which we can use suitable integrability results (cf. Chap. 6) to obtain unitary representations of the 1-connected Lie group G^c with Lie algebra $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$ on $\widehat{\mathscr{E}}$.

We now turn to the contents of this book. We shall not turn to finer aspects of unitary representations of higher-dimensional Lie groups before Chap. 6. In the first half, Chaps. 2–5, we deal with rather concrete contexts and how they relate to reflection positivity. Various aspects concerning general Lie groups are postponed to Chaps. 6–9.

Chapter 2 develops the notion of a reflection positive Hilbert space $(\mathcal{E}, \mathcal{E}_+, \theta)$ from various perspectives. For instance θ-positive subspaces $\mathcal{E}_+$ can be constructed as graphs of contractions from the 1 to the -1-eigenspace of θ (Sect. 2.2). In physics reflection positive Hilbert spaces often arise from distributions. Here $\mathcal{E}$ is a Hilbert space arising by completing the space $C_c^\infty(M)$ of smooth test functions on a manifold with respect to a singular scalar product

$$\langle \xi, \eta \rangle = \int_{M\times M} \overline{\xi(x)}\eta(y)\, dD(x, y), \tag{1.1}$$

where D is a positive definite distribution on $M \times M$. Then θ is supposed to come from a diffeomorphism of M and $\mathcal{E}_+$ from an open subset $M_+ \subseteq M$, which leads to the reflection positivity condition

$$\int_{M_+\times M_+} \overline{\xi(x)}\xi(y)\, dD(\theta(x), y) \geq 0 \quad \text{for} \quad \xi \in C_c^\infty(M_+) \tag{1.2}$$

(Section 2.4). Typical concrete examples arise from reflections of complete Riemannian manifolds and resolvents $(\lambda \mathbf{1} - \Delta)^{-1}$ of the Laplacian (Sect. 2.5). Motivated by these examples, we briefly describe an abstract operator theoretic context for reflection positivity that we feel should be developed further (Sect. 2.6; [JR07, An13, AFG86, Di04]). In probabilistic contexts, one encounters situations satisfying the Markov condition, i.e., there exists a subspace $\mathcal{E}_0 \subseteq \mathcal{E}_+$ mapped isometrically onto $\widehat{\mathcal{E}}$ (Sect. 2.3).

The connection between reflection positive Hilbert spaces and representation theory is introduced in Chap. 4. After discussing some general properties of the OS transform, we introduce symmetric Lie groups (G, τ), symmetric subsemigroups $S \subseteq G$ and various kinds of reflection positivity conditions for unitary representations. As in the representation theory of operator algebras, where cyclic representations are generated from states, it is an extremely fruitful approach to generate representations of groups and semigroups by positive definite functions via the Gelfand–Naimark–Segal (GNS) construction. Here reflection positivity requirements lead to the concept of a reflection positive function whose values may also comprise bounded operators or bilinear forms (Sect. 3.4).

After these generalities, we turn in Chap. 4 to the most elementary concrete symmetric Lie group $(G, \tau) = (\mathbb{R}, -\mathrm{id}_{\mathbb{R}})$, where the RP condition is based on the subsemigroup $\mathbb{R}_+$. Although this Lie group is quite trivial, reflection positivity on the real line has many interesting facets and is therefore quite rich. As reflection positive functions play a crucial role, we start Chap. 4 with functions on intervals $(-a, a) \subseteq \mathbb{R}$ which are reflection positive in the sense that both kernels

$$(\varphi(x-y))_{-a/2<x,y<a/2} \quad \text{and} \quad (\varphi(x+y))_{0<x,y<a/2}$$

are positive definite. For $a = \infty$, this combines the positive definiteness of the group $(\mathbb{R}, +)$ with the positive definiteness on the involutive semigroup $(\mathbb{R}_+, \mathrm{id}_\mathbb{R})$. Accordingly, these two conditions ask for techniques related to Fourier- and Laplace transforms.

Reflection positive representations of $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_\mathbb{R})$ are unitary one-parameter groups $(U_t)_{t\in\mathbb{R}}$ on a reflection positive Hilbert space $(\mathscr{E}, \mathscr{E}_+, \theta)$ satisfying $U_t\mathscr{E}_+ \subseteq \mathscr{E}_+$ for $t > 0$ and $\theta U_t \theta = U_{-t}$ for $t \in \mathbb{R}$. On $\widehat{\mathscr{E}}$ this leads to a semigroup $(\widehat{U}_t)_{t\geq 0}$ of hermitian contractions and we show in particular that, under the OS transform, fixed points of U on $\mathscr{E}$ correspond to fixed points of $\widehat{U}$ in $\widehat{\mathscr{E}}$ (Proposition 4.2.6).

For $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_\mathbb{R})$, we obtain a complete classification of reflection positive representations in terms of integral formulas, resp., spectral theorems. From these results one obtains an interesting converse of the OS transform in this context. Any hermitian contraction semigroup $(C_t)_{t\geq 0}$ on a Hilbert space $\mathscr{H}$ has a so-called minimal dilation represented by the reflection positive function $\psi(t) := C_{|t|}$ on $\mathbb{R}$.

We conclude Chap. 4 by showing that, for any reflection positive one-parameter group for which $\mathscr{E}_+$ is cyclic and fixed points are trivial, the space $\mathscr{E}_+$ is outgoing in the sense of Lax–Phillips scattering theory (Proposition 4.4.2). This establishes a remarkable connection between reflection positivity and scattering theory that leads to a normal form of reflection positive one-parameter groups by translations on spaces of the form $\mathscr{E} = L^2(\mathbb{R}, \mathscr{H})$ with $\mathscr{E}_+ = L^2(\mathbb{R}_+, \mathscr{H})$. Applying the Fourier transform to our concrete dilation model leads precisely to this normal form.

In Chap. 5 we still work with the same symmetric group $(\mathbb{R}, -\mathrm{id}_\mathbb{R})$ or rather its quotient circle group $\mathbb{R}/2\beta\mathbb{Z} \cong \mathbb{T}$, but now reflection positivity is based on the interval $[0, \beta]$, where $\beta > 0$ is interpreted as an inverse temperature in physical models [NÓ15b, KL81, KL81b, Fro11, NÓ16]. In this context reflection positivity is closely connected with the Kubo–Martin–Schwinger (KMS) condition for states of C^*-dynamical systems [Fro11, BR02]. This connection is established by a purely representation theoretic perspective on the KMS condition formulated as a property of form-valued positive definite functions on $\mathbb{R}$: Let V be a real vector space and $\mathrm{Bil}(V)$ be the space of real bilinear maps $V \times V \to \mathbb{C}$. For $\beta > 0$, we consider the open strip

$$\mathscr{S}_\beta := \{z \in \mathbb{C} \colon 0 < \mathrm{Im} z < \beta\}.$$

We say that a positive definite function $\psi \colon \mathbb{R} \to \mathrm{Bil}(V)$ (Definition A.1.5) satisfies the *β-KMS condition* if ψ extends to a pointwise continuous function ψ on $\overline{\mathscr{S}_\beta}$ which is pointwise holomorphic on the interior $\mathscr{S}_\beta$ and satisfies

$$\psi(i\beta + t) = \overline{\psi(t)} \quad \text{for} \quad t \in \mathbb{R}.$$

The classification of such functions in terms of an integral representation is based on relating them to standard (real) subspaces of a complex Hilbert space which occur naturally in the modular theory of operator algebras [Lo08]. These are closed real subspaces $V \subseteq \mathscr{H}$ for which $V \cap iV = \{0\}$ and $V + iV$ is dense. Any standard subspace determines a pair (Δ, J) of *modular objects*, where Δ is a positive selfadjoint operator and J is an anti-linear involution (a *conjugation*) satisfying $J\Delta J = \Delta^{-1}$. The connection is established by

$$V = \mathrm{Fix}(J\Delta^{1/2}) = \{\xi \in \mathscr{D}(\Delta^{1/2})\colon J\Delta^{1/2}\xi = \xi\}.$$

This connects reflection positivity very naturally to the aforementioned recent developments in AQFT initiated by the work of Borchers [Bo92] and now exploited systematically in the constructions of QFTs (see [BJM16, BLS11, LL14, LW11] for typical applications). Here standard subspaces can be considered as "one-particle space analogs" of the modular data (Δ, J) arising in Tomita–Takesaki theory in the context of von Neumann algebras [Lo08, LL14].

After the discussion of the concrete examples, the reader should be prepared to appreciate the Lie theoretic aspects of the theory, which start in Chap. 6 with the development of the integration techniques that are used to obtain a unitary representation of the simply connected Lie group G^c on $\widehat{\mathscr{E}}$ from a reflection positive representation of (G, τ) on $(\mathscr{E}, \mathscr{E}_+, \theta)$. Our techniques are based on the fact that the Hilbert spaces are mostly constructed from G-invariant positive definite kernels or positive definite G-invariant distributions. We have already seen that any reflection positive representation of (G, τ) immediately yields a unitary representation U^c of $H = G^\tau_0$ on $\widehat{\mathscr{E}}$, so that it remains to find a unitary representation of the one-parameter groups $\exp_{G^c}(\mathbb{R}ix)$ for $x \in \mathfrak{q}$. By Stone's Theorem, the main point is to show that, for $y \in \mathfrak{q}$, the symmetric operator $\widehat{\mathrm{d}U}(y)$ defined on a dense subspace of $\widehat{\mathscr{E}}$ is essentially selfadjoint. In our geometric setting, this can be derived from Fröhlich's Theorem [Fro80] which provides a criterion for the essential selfadjointness of a symmetric operator in terms of the existence of enough local solutions of the corresponding linear ODE. The natural setting for the corresponding integrability results are pairs (β, σ) of a homomorphism $\beta\colon \mathfrak{g} \to \mathscr{V}(M)$ to the Lie algebra $\mathscr{V}(M)$ of smooth vector fields on a manifold M which is compatible with a smooth H-action σ. Then, for any smooth kernel K on M satisfying a suitable invariance condition with respect to (β, σ), a unitary representation of G^c on $\mathscr{H}_K$ exists (Theorem 6.2.3). We also show that a similar result holds if we replace the kernel K by a positive definite distribution $K \in C^{-\infty}(M \times M)$ compatible with (β, σ) (Theorem 6.3.6). From this we easily derive the existence of a unitary representation of the simply connected group G^c on $\widehat{\mathscr{E}}$ for a reflection positive representation $(U, \mathscr{E})$ of (G, τ). Our exposition is based on new aspects developed in [MNO15] which complements the classical approach from [FOS83].

The most effective tool to deal with reflection positive representations of symmetric Lie groups (G, τ) are reflection positive distributions on G and their relation with reflection positive distribution vectors of unitary representations. A key advantage of this method, outlined in Chap. 7, is that is leads naturally to reflection positive representations in Hilbert spaces of distributions on homogeneous spaces G/H, where H may be non-compact. To illustrate this technique, we apply it to spherical representations of the Lorentz group $G = \mathrm{O}_{1,n}(\mathbb{R})$. These representations consist of two series,