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V Lakshmibai
Justin Brown

Flag Varieties

An Interplay of Geometry, Combinatorics, and Representation Theory

Second Edition

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Preface

This book provides an introduction to flag varieties and their Schubert subvarieties. The book portrays flag varieties as an interplay of algebraic geometry, algebraic groups, combinatorics, and representation theory.

After discussing the representation theory of finite groups, the polynomial representations (in characteristic zero) of the general linear group are obtained by relating the representation theory of the general linear group to that of the symmetric group (using Schur-Weyl duality). Since the Lie algebra of a semisimple algebraic group plays a crucial role in the structure theory of semisimple algebraic groups, the book discusses the structure theory and the representation theory of complex semisimple Lie algebras. Since Bruhat decomposition is at the heart of the study of the flag variety, the book gives a quick treatment of the generalities on algebraic groups leading to the root system and Bruhat decomposition in reductive algebraic groups.

The nucleus of this book is the geometry of the Grassmannian and flag varieties. A knowledge of Grassmannian and flag varieties is indispensable for any prospective graduate student working in the area of algebraic geometry. We hope that this book will serve as a reference for basic results on Grassmannian, flag, and Schubert varieties as well as the relationship between the geometric aspects of these varieties and the representation theory of semisimple algebraic groups.

The prerequisite for this book is some familiarity with commutative algebra, algebraic geometry, and algebraic groups. A basic reference to commutative algebra is [17], algebraic geometry [28], and algebraic groups [5, 36]. The basic results from commutative algebra and algebraic geometry are summarized in Chapter 1. We have mostly used standard notation and terminology and have tried to keep notation to a minimum. Throughout the book, we have numbered theorems, lemmas, propositions etc., in order according to their chapter and section; for example, 3.2.4 refers to the fourth item of the second section in the third chapter.

This book can be used for an introductory course on flag varieties. The material covered in this book should provide adequate preparation for graduate students and researchers in the area of algebraic geometry and algebraic groups. For the interested reader, we have included several exercises at the end of almost every chapter; most of these exercises can also be found in the standard texts on their respective subjects.

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J. Brown thanks all of those who participated in the Flag Varieties course at Northeastern University, specifically K. Webster and M. Fries for their contributions and suggestions. He also thanks his wife, Jody, for her love and support.

September 1, 2007 V. Lakshmibai
Boston, MA, USA J. Brown

Preface to the Second Edition

In this second edition, we have added two recent results (from [42] and [32], respectively) on Schubert varieties in the Grassmannian. The first result, which has been added as Chapter 15, gives a free resolution of certain Schubert singularities. The second result, which has been added as Chapter 16, is about certain Levi subgroup actions on Schubert varieties in the Grassmannian and derives some interesting geometric and representation-theoretic consequences.

Contents

About the Authors

V. Lakshmibai is professor at Northeastern University, Boston, USA and a fellow of the American Mathematical Society since January 2013. She earned her Ph.D. from the Tata Institute of Fundamental Research, Mumbai, India. She, together with C. S. Seshadri and C. Musili, founded the standard monomial theory for generalized flag varieties (as well as their Schubert subvarieties), as a generalization of the classical Hodge-Young theory (for the Grassmannians as well as their Schubert subvarieties). Her areas of research are algebraic geometry, representation theory, and combinatorics, particularly in flag varieties, Grassmannian varieties as well as their Schubert subvarieties. She has authored five books including Singular Loci of Schubert Varieties with co-author Sara Billey (Birkhauser), Standard Monomial Theory with co-author K. N. Raghavan (Springer) and The Grassmannian Variety with co-author Justin Brown (Springer).

Justin Brown is associate professor of mathematics at Olivet Nazarene University in Bourbonnais, Illinois, USA. He earned his Ph.D. on the topic "Some geometric properties of certain toric varieties and Schubert varieties" at Northeastern University, under the guidance of Prof. V. Lakshmibai. His areas of research include algebra, algebraic geometry, topology, and combinatorics. He has co-authored two books with Prof. V. Lakshmibai including The Grassmannian Variety (Springer).

Introduction

This book is an expanded version of the lectures given by V. Lakshmibai on flag varieties at the Institute for Advanced Study Program for Women and Mathematics on "Algebraic Geometry and Group Actions, 2007."

An algebraic variety X with an action by an algebraic group G comes equipped with additional structures; for instance, the action of G on X induces an action of G on R, the coordinate ring of X, thus admitting representation-theoretic techniques for the study of X. One important example is the *flag variety* $\mathcal{F}$, consisting of flags in K^n, K being a field (a flag being a sequence $(0) = V_0 \subset V_1 \subset \ldots \subset V_n = K^n$, V_i being a K-vector subspace of K^n of dimension i); and the group $GL_n(K)$, of invertible $n \times n$ matrices with entries in K, acts on $\mathcal{F}$ in a natural way.

Flag varieties form an important class of geometric objects in algebraic geometry. They have close links with other areas in mathematics — commutative algebra, representation theory and combinatorics. Their richness in geometry and combinatorics makes the study of flag varieties very interesting; further, the *Schubert subvarieties* in the flag varieties provide a powerful inductive machinery for the study of flag varieties.

One may describe the flag variety as an interplay of geometry, combinatorics and representation theory in mathematics. It is this interplay on which we want to focus in this book. Fixing a bunch of r distinct integers $\underline{d} := 1 \leq d_1 < d_2 < \ldots < d_r \leq n-1$, we may talk about the *partial flag variety* $\mathcal{F}l_{\underline{d}}$, the set of partial flags of type $\underline{d}$, namely, sequences $V_{d_1} \subset V_{d_2} \subset \ldots \subset V_{d_r}$, $\dim V_i = i$. The extreme case with $r = 1$, corresponds to the celebrated *Grassmannian variety* $G_{d,n}$ consisting of d-dimensional subspaces of K^n. If $d = 1$, then $G_{1,n}$ is just the $(n-1)$-dimensional projective space $\mathbb{P}_K^{n-1}$ (consisting of lines through the origin in K^n).

It might be said that after the projective and affine spaces, Grassmannian varieties form the next important class of algebraic varieties. In 1934, *Ehresmann* (cf. [16]) showed that the classes of Schubert subvarieties in the Grassmannian give a $\mathbb{Z}$-basis for the cohomology ring of the Grassmannian, and thus established a key relationship between the geometry of the Grassmannian varieties and the theory of characteristic classes. In 1956, *Chevalley* (cf. [13]) further enhanced this relationship by showing that the classes of the Schubert varieties (in the *generalized flag variety* G/B, G a semisimple algebraic group and B a Borel subgroup) form a $\mathbb{Z}$-basis for the Chow ring of the generalized flag variety. Around the same time, *Hodge* (cf. [30, 31]) developed the *Standard Monomial Theory* for Schubert varieties in the Grassmannian. This theory describes explicit bases for the homogeneous coordinate ring of the Grassmannian and its Schubert varieties (for the Plücker embedding) in terms of certain monomials (called standard monomials). Hodge's theory was generalized to G/B, for G classical by *Lakshmibai, Musili, and Seshadri* in the series G/P I-V (cf. [48, 49, 52, 54, 74]) during 1975–1986; conjectures were then formulated (cf. [55]) by Lakshmibai and Seshadri in 1991 toward the generalization of Hodge's theory to exceptional groups. These conjectures were proved by *Littelmann* (cf. [61, 62, 63]) in 1994–1998, thus completing the standard monomial theory for semisimple algebraic groups. This theory has led to many interesting and important geometric and representation theoretic consequences (see [43, 46, 51, 53, 56, 61, 62, 63, 45]).

In this book, we confine ourselves to $GL_n(K)$ (and $SL_n(K)$) since our goal is to introduce the readers to flag varieties fairly quickly, minimizing the technicalities along the way. We have attempted to give a complete and comprehensive introduction to the flag variety — its geometric and representation-theoretic aspects.

Our discussion of the polynomial representations of $GL_n(\mathbb{C})$ is carried out using Schur-Weyl duality (between the general linear group and the symmetric group in characteristic zero). To make it more precise, we first discuss the representation theory of the symmetric group S_d, then deduce the polynomial representations of $GL_n(\mathbb{C})$ using Schur-Weyl duality. We have carried out the more general discussion of the representation theory of finite groups; this discussion is preceded by the discussion on the structure theory of semisimple algebras. Thus the discussion leading to the representation theory of $GL_n(\mathbb{C})$ occupies Chapters 2 through 6.

The second half of the book is devoted to the discussion on Grassmannian and flag varieties (and their Schubert varieties). In this part again, as a prelude to the main discussion, we have included a quick introduction to algebraic groups, followed by the structure theory (as well as the representation theory) of reductive algebraic groups — mainly root systems and Bruhat decomposition in reductive algebraic groups. Since this structure theory is built using the Lie algebra of the reductive algebraic group, we have included the discussion on structure theory and representation theory of complex semisimple Lie algebras. We have included a detailed discussion of the geometry of the Grassmannian variety and its Schubert varieties via standard monomial theory. Similar results are then described for the flag variety and its Schubert varieties. Thus the discussion in the second half of the book leading to the main theme on Grassmannian and flag varieties occupies Chapters 7 through 12.

One of the most important geometric consequences of standard monomial theory is the determination of the singular locus of a Schubert variety and we have devoted Chapter 13 for this discussion. Here again, we have confined ourselves to the special linear group $SL_n(K)$. For the discussion on the singular locus of Schubert varieties for other semisimple algebraic groups, we refer the reader to [2].

By way of applications, we have included two instances: the first one is about the connection between Schubert varieties and classical invariant theory (cf. [82]). For the diagonal action of $GL_n(K)$ on $V^{\oplus m} \oplus (V^*)^{\oplus q}$, $V = K^n$, the corresponding categorical quotient is a determinantal variety in $M_{m,q}(K)$ (the space of $m \times q$ matrices with entries in K). A determinantal variety can be identified with a certain open subset of a Schubert variety in the Grassmannian variety $G_{q,m+q}$, and hence one obtains a standard monomial basis for the corresponding ring of invariants. This discussion is carried out in Chapter 14.

The second instance describes the connection between Schubert and toric varieties; to be more precise, we present results on toric degenerations of Schubert varieties, i.e., given a Schubert variety X in the Grassmannian, there exists a flat family over $\mathbb{A}^1$ with $\hat{X}$ (the cone over X) as the generic fiber and the special fiber being a toric variety. It should be mentioned that this result generalizes to Schubert varieties in G/B, G any semisimple algebraic group, thanks to the works of *Caldero* (cf. [10]) and *Chirivi* (cf. [14]).

In this second edition, we have included chapters 15 and 16 on recent results concerning Schubert varieties of the Grassmannian. There is also an appendix giving a brief account of Chevalley groups.

The book is organized as follows: Chapter 1 is a brief review on the basics of algebraic varieties. Chapter 2 is on the structure theory of semisimple rings. Chapter 3 deals with the representation theory of finite groups and leads to the representation theory of the symmetric group in Chapter 4. Chapter 5 is a brief account of symmetric polynomials. In Chapter 6, we present the representation theory of $GL_n(\mathbb{C})$. Chapters 7 and 8 deal respectively with the structure theory and representation theory of complex semisimple Lie algebras. In Chapter 9, we discuss first the generalities on algebraic groups, and then introduce the variety of Borel subgroups. In Chapter 10, we discuss the structure theory of reductive algebraic groups. Chapter 11 is on the representation theory of semisimple algebraic groups. Chapter 12 is the "nucleus" of the book - Grassmannian, flag varieties and their Schubert varieties. Chapter 13 is the discussion on the singular locus of Schubert varieties in the flag variety. Chapter 14 discusses the connection between standard monomial theory and classical invariant theory as well as that between Schubert varieties and toric varieties. Chapter 15 gives free resolutions of a certain class of Schubert singularities. Chapter 16 is on Levi subgroup actions on Schubert varieties.

Chapter 1

Preliminaries

This chapter is a brief review of commutative algebra and algebraic geometry. We have included basic definitions and properties. For details in commutative algebra, we refer the reader to [17] and in algebraic geometry to [28, 67].

1.1 Commutative Algebra

Throughout this section, A shall denote a commutative ring with 1.

Noetherian rings

A ring A is said to be *Noetherian* if every ideal of A is finitely generated, or, equivalently, A satisfies a.c.c. (*ascending chain condition*), namely every increasing chain of ideals terminates or, equivalently, every nonempty collection of ideals has a maximal element relative to inclusion.

Theorem 1.1.1 (Hilbert Basis Theorem). *If A is Noetherian, then so is the polynomial ring $A[x]$. In particular, if K is a field, then $K[x_1, \ldots, x_n]$ is Noetherian.*

Localization

Let S be a *multiplicative set* in A, i.e. $0 \notin S$, $1 \in S$, and $a, b \in S \Rightarrow ab \in S$. The *ring of quotients* (also known as the *ring of fractions*) $S^{-1}A$ is constructed using equivalence classes of pairs $(a, s) \in A \times S$, where $(a_1, s_1) \sim (a_2, s_2)$ if there exists $s \in S$ such that $s(s_2a_1 - s_1a_2) = 0$. Denoting the equivalence class of (a, s) by $\frac{a}{s}$, the multiplication and addition in $S^{-1}A$ are defined by

$$\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1a_2}{s_1s_2}, \ \frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{s_2a_1 + s_1a_2}{s_1s_2}.$$

We have the natural map $A \to S^{-1}A$, $a \mapsto \frac{a}{1}$, which is universal for homomorphisms from A rendering the elements of S invertible. The (prime) ideals of $S^{-1}A$ correspond bijectively to the (prime) ideals of A not meeting S.

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If S is the set of all nonzero divisors in A, then $S^{-1}A$ is called the *full ring of fractions of* A; note that in this case $A \to S^{-1}A$ is injective. If A is an integral domain, then the full ring of fractions of A consists of fractions of the form $\frac{a}{s}$, with $a \in A$, $s \in A \setminus \{0\}$, and is in fact a field, called the *field of fractions* (or also the *quotient field*) of A.

If $S = A \backslash \mathcal{P}$, where $\mathcal{P}$ is a prime ideal of A, then $S^{-1}A$ is denoted $A_{\mathcal{P}}$. It is a *local ring* (i.e. has a unique maximal ideal $\mathcal{P}A_{\mathcal{P}}$, consisting of the non-units in $A_{\mathcal{P}}$, a *unit* is any element with a multiplicative inverse). The prime ideals of $A_{\mathcal{P}}$ correspond to the prime ideals of A contained in $\mathcal{P}$. If A is Noetherian, then so is $A_{\mathcal{P}}$. If $\mathfrak{m}$ is a maximal ideal, then the fields $A/\mathfrak{m}$ and $A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}$ are naturally isomorphic. The canonical map $A \to A_{\mathfrak{m}}$ induces a vector space isomorphism $\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}A_{\mathfrak{m}}/\left(\mathfrak{m}A_{\mathfrak{m}}\right)^2$.

For an A-module M, $S^{-1}M$ is defined by using classes of pairs $(m, s) \in M \times S$, where $(m_1, s_1) \sim (m_2, s_2)$ if there exists $s \in S$ such that $s(s_1 m_2 - s_2 m_1) = 0$. Then $S^{-1}M$ is an $S^{-1}A$-module (in a natural way), and is naturally isomorphic to $S^{-1}A \otimes M$. The functor $M \to S^{-1}M$ from A-modules to $S^{-1}A$-modules is exact, i.e. it takes exact sequences to exact sequences. Further, it preserves tensor and Hom in the following sense: if M and N are A-modules, the natural map $S^{-1}(M \otimes A) \to S^{-1}M \otimes_{S^{-1}A} S^{-1}N$ is an isomorphism, and the natural map

$$S^{-1}\left(\mathrm{Hom}_A\left(M, N\right)\right) \to \mathrm{Hom}_{S^{-1}A}\left(S^{-1}M, S^{-1}N\right)$$

is an isomorphism if M is finitely generated.

Radical of an ideal and the nilradical of a ring

If I is an ideal in a ring A, then the set $\sqrt{I} = \{a \in A \mid a^n \in I\}$ is an ideal of A, called the *radical of* I. We have

$$\sqrt{I} = \bigcap_{\mathcal{P} \supset I,\, \mathcal{P} \text{ prime}} \mathcal{P}.$$

Note that $I \subset \sqrt{I}$. The ideal I is called *radical* or *reduced* if $\sqrt{I} = I$. Prime ideals are examples of radical ideals.

The set of nilpotent elements in a (commutative) ring A is an ideal denoted $\mathrm{nil}(A)$ ($a \in A$ is nilpotent if there exists some $n \geq 1$ such that $a^n = 0$). Note that $\mathrm{nil}(A) = \sqrt{(0)}$, and $\mathrm{nil}(A/I) = \sqrt{I}/I$. The ring A is said to be *reduced* if $\mathrm{nil}(A) = (0)$.

If S is a multiplicative set, then

$$\sqrt{I \cdot S^{-1}A} = \sqrt{I} \cdot S^{-1}A.$$

In particular, a ring A is reduced if and only if the full ring of fractions of A is reduced.

Transcendence degree

Let K be a field extension of k. The elements $a_1, \ldots, a_d \in K$ are said to be *algebraically independent over* k if no nonzero polynomial $f(x_1, \ldots, x_d)$ over k satisfies $f(a_1, \ldots, a_d) = 0$. A maximal subset of algebraically independent elements (over k) in K is called a *transcendence basis of* K/k. If K is a finitely generated extension of k, say $K = k(a_1, \ldots a_n)$, then a transcendence basis can be chosen from among the a_i's, say $a_1, \ldots, a_d$. The integer d (which is independent of the choice of the transcendence basis) is called the *transcendence degree of* K *over* k, and is denoted $\mathrm{tr.deg}_k K$. We have $k(a_1, \ldots a_d)$ is purely transcendental over k, and K is algebraic over $k(a_1, \ldots a_d)$.

Integral closure

Let $B \supset A$ be an extension of rings. An element $b \in B$ is said to be *integral* over A if there exists a monic polynomial $f(x) \in A[x]$ having b as a zero. The *integral closure of* A *in* B is the subring of B consisting of elements of B which are integral over A. The ring B is said to be *integral over* A if every element of B is integral over A. An integral domain A is said to be *integrally closed* if A is the integral closure of A in F, F being the field of fractions of A. (Such a domain is also said to be a *normal* domain.)

Example 1.1.2. $\mathbb{Z}$ is integrally closed.

Now let $A = k[a_1, \ldots a_n]$ be a finitely generated k-algebra; further, let A be an integral domain with quotient field K. Then $\mathrm{tr.deg}_k A$ is defined as

$$\mathrm{tr.deg}_k A = \mathrm{tr.deg}_k K.$$

Proposition 1.1.3 (Noether normalization lemma, cf. [67])**.** *Let $A = k[a_1, \ldots, a_n]$ be a finitely generated k-algebra. Further, let A be an integral domain. Let $d = \mathrm{tr.deg}_k A$. Then there exist elements $x_1, \ldots, x_d \in A$ algebraically independent over k such that A is integral over $k[x_1, \ldots, x_d]$.*

Krull dimension of a ring

Let A be a Noetherian ring. The *Krull dimension* of A is defined to be the maximal length t of a strictly increasing chain of prime ideals $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_t \subset A$, and is denoted $\dim A$. We list below some facts on $\dim A$.

1. $\dim A = \sup\{\dim A_{\mathfrak{m}} \mid \mathfrak{m}$ being a maximal ideal in $A\}$.
2. Let $A = k[x_1, \ldots, x_n]$, the polynomial algebra. Then $\dim A = n$.
3. Let A be an integral domain which is a finitely generated k-algebra. Then $\dim A = \mathrm{tr.deg}_k A$.
4. Let A be an integral domain which is a finitely generated k-algebra. Let $\mathcal{P}$ be a prime ideal in A. Then

$$\dim A/\mathcal{P} + \dim A_{\mathcal{P}} = \dim A.$$

Defining the *height* (resp. *coheight*) of a prime ideal $\mathcal{P}$ as $\dim A_{\mathcal{P}}$ (resp. $\dim A/\mathcal{P}$), the above formula is often expressed as

$$\text{height}\mathcal{P} + \text{coheight}\mathcal{P} = \dim A.$$

Cohen-Macaulay rings and modules

Let A be a Noetherian ring, and M a finitely generated A-module.

Definition 1.1.4. An element $a \in A$ is said to be a *zero divisor in* M if $am = 0$, for some nonzero element $m \in M$.

Note that a is not a zero divisor if and only if the map $M \to M$, $m \mapsto am$ is a monomorphism.

Definition 1.1.5. A sequence $a_1, \ldots, a_r$ of elements of A is said to be an *M-regular sequence*, or just an *M-sequence* if

1. $(a_1, \ldots, a_r)M \neq M$,
2. a_i is not a zero divisor in $M_{i-1} := M/(a_1, \ldots, a_{i-1})M$, $1 \leq i \leq r$, (here $a_0 = 0$).

Let now A be local with $\mathfrak{m}$ as its maximal ideal.

Proposition 1.1.6 (cf. [17]). *Any two maximal M-sequences of elements in $\mathfrak{m}$ have the same length.*

Definition 1.1.7. Let $A, M, \mathfrak{m}$ be as above. The length of a maximal M-sequence of elements in $\mathfrak{m}$ is defined as the *depth of* M, and is denoted $\text{depth}M$.

Proposition 1.1.8 (cf. [17]). *Let A, M be as above. Then $depthM \leq \dim M$, where $\dim M = \dim(A/annM)$, $annM$ being the annihilator of M.*

Definition 1.1.9. Let A, M be as above. Then M is said to be *Cohen-Macaulay* if either $M = (0)$, or $\text{depth}M = \dim M$. The ring A is said to be *Cohen-Macaulay* if A is Cohen-Macaulay as an A-module.

Regular rings

Let $(A, \mathfrak{m})$ be a local Noetherian ring, and let $K = A/\mathfrak{m}$.

Proposition 1.1.10 (Nakayama's Lemma, cf. [17]). *Let M be as above. Suppose that $\mathfrak{m}M = M$. Then $M = (0)$.*

Remark 1.1.11. The following are consequences of Nakayama's Lemma.

1. Let A, M be as above. Let N be a submodule of M such that $M = \mathfrak{m}M + N$. Then $M = N$. (This follows by applying Nakayama's Lemma to M/N; note that $\mathfrak{m} \cdot M/N = (\mathfrak{m}M + N)/N = M/N$.)

2. Let $V = M/\mathfrak{m}M$; note that V is a finite dimensional vector space over $K(= A/\mathfrak{m})$. Let $\dim_K M/\mathfrak{m}M = n$. Suppose that $x_1, \ldots, x_n$ are elements in M such that $\bar{x}_1, \ldots, \bar{x}_n$ in $M/\mathfrak{m}M$ is a K-basis for V. Then $x_1, \ldots, x_n$ generate M. (To see this, let N be the submodule of M generated by $x_1, \ldots, x_n$. Then under the canonical map $M \to M/\mathfrak{m}M$, N maps onto $M/\mathfrak{m}M$. Hence $N + \mathfrak{m}M = M$, and (1) implies $N = M$.)

Let us now take $M = \mathfrak{m}$. Let $x_1, \ldots, x_n \in \mathfrak{m}$ such that $\bar{x}_1, \ldots, \bar{x}_n$ is a K-basis for $\mathfrak{m}/\mathfrak{m}^2$. Then $x_1, \ldots, x_n$ generate $\mathfrak{m}$; further, n is the smallest number of elements in a generating set for $\mathfrak{m}$. Let $\dim A = r$. In general, we have $r \leq n$.

Definition 1.1.12. $(A, \mathfrak{m})$ is a *regular local ring* if $r = n$, or equivalently, $\mathfrak{m}$ is generated by r elements.

Definition 1.1.13. A Noetherian ring A is said to be *regular* if $A_{\mathcal{P}}$ is a regular local ring for all prime ideals $\mathcal{P}$.

Example 1.1.14. The polynomial ring $k[x_1, \ldots, x_n]$ is a regular ring.

Theorem 1.1.15. *Let A be a regular ring. Then*

1. *A is a normal domain.*
2. *A is a U.F.D. - unique factorization domain, (recall that a U.F.D. is a domain in which every nonzero non-unit element has an expression as a product of prime elements; here, an element $a \in A$ is said to be a prime element if the principal ideal Aa is a prime ideal).*

1.2 Affine Varieties

Let K be the base field, which we suppose to be algebraically closed of arbitrary characteristic.

The affine space $\mathbb{A}^n$

We shall denote by $\mathbb{A}^n_K$ or just $\mathbb{A}^n$, the affine n-space, consisting of $(a_1, \ldots, a_n)$, $a_i \in K$. For $P = (a_1, \ldots, a_n) \in \mathbb{A}^n$, the a_i's are called the *affine coordinates* of P.

Affine varieties

Given an ideal I in the polynomial algebra $K[x_1, \ldots, x_n]$, let

$$V(I) = \{(a_1, \ldots, a_n) \in \mathbb{A}^n \mid f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\}.$$

The set $V(I)$ is called an *affine variety*. Clearly $V(I) = V(\sqrt{I})$. Fixing a (finite) set of generators $\{f_1, \ldots, f_r\}$ for I, $V(I)$ can be thought of as the set of common zeros of $f_1, \ldots, f_r$.

Conversely, given a subset $X \subset \mathbb{A}^n$, let

$$\mathcal{I}(X) = \{f \in K[x_1, \ldots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}.$$

Zariski topology on $\mathbb{A}^n$

Define a topology on $\mathbb{A}^n$ by declaring $\{V(I) \mid I \text{ an ideal in } K[x_1,\ldots,x_n]\}$ as the set of closed sets. We now check that this defines a topology on $\mathbb{A}^n$. We have

1. $\mathbb{A}^n = V((0))$, $\emptyset = V(K[x_1,\ldots,x_n])$.
2. $V(I) \cup V(J) = V(I \cap J)$.
3. $\bigcap_\alpha V(I_\alpha) = V\left(\sum_\alpha I_\alpha\right)$.

Statements (1) and (3) are clear. (2): The inclusion $V(I) \cup V(J) \subset V(I \cap J)$ is clear. To see the reverse inclusion, let $a \in V(I \cap J)$. If possible, let us assume that $a \notin V(I)$, $a \notin V(J)$. Assumption implies that there exist $f \in I$, $g \in J$ such that $f(a) \neq 0$, $g(a) \neq 0$. Since $fg \in I \cap J$, we have $f(a)g(a) = 0$, a contradiction. Hence our assumption is wrong, and the reverse inclusion follows.

Remark 1.2.1.

1. $\mathbb{A}^n$ is a T_1-space for the Zariski topology, i.e. points are closed subsets.
2. $\mathbb{A}^n$ is *not* Hausdorff for the Zariski topology. For, consider $\mathbb{A}^1$. The closed subsets are precisely the finite sets, and hence no two nonempty open sets can be disjoint.
3. The d.c.c. on closed sets implies the a.c.c. on open sets, i.e. any nonempty collection of open sets has a maximal element. Hence $\mathbb{A}^n$ is quasi-compact, i.e. every open cover admits a finite subcover (the term "quasi" is used since $\mathbb{A}^n$ is not Hausdorff).
4. If $K = \mathbb{C}$, then the zero-set of a polynomial $f \in \mathbb{C}[x_1,\ldots,x_n]$ is closed in the usual topology of $\mathbb{C}^n$, being the inverse image of the closed set $\{0\}$ in $\mathbb{C}$ under the continuous map $\mathbb{C}^n \to \mathbb{C}$, $a \mapsto f(a)$. The set of common zeros of a collection of polynomials is also closed in the usual topology, being the intersection of closed sets. Of course, the complex n-space $\mathbb{C}^n$ has plenty of other closed sets which are not obtained this way (as is clear in the case $n = 1$). Thus the usual topology is stronger than the Zariski topology.

We have

$$X \subset V(\mathcal{I}(X)), \ I \subset \mathcal{I}(V(I)).$$

Fact: $V(\mathcal{I}(X)) = \bar{X}$, the closure of X.

Theorem 1.2.2 (Hilbert's Nullstellensatz, cf. [67]). *Let I be an ideal in $K[x_1,\ldots,x_n]$. Then $\sqrt{I} = \mathcal{I}(V(I))$.*

As a consequence of the fact above and Hilbert's Nullstellensatz we obtain an inclusion-reversing bijection

$$\begin{array}{rcl} \{\text{radical ideals in } K[x_1,\ldots,x_n]\} & \longleftrightarrow & \{\text{affine varieties in } \mathbb{A}^n\}, \\ I & \longmapsto & V(I) \\ \mathcal{I}(X) & \longleftarrow\!\shortmid & X. \end{array}$$

The Noetherian property of $K[x_1, \ldots, x_n]$ implies d.c.c. (descending chain condition) on the set of affine varieties in $\mathbb{A}^n$. Now, if $\mathfrak{m}$ is a maximal ideal, then by Hilbert's Nullstellensatz $V(\mathfrak{m})$ is non-empty. Let $a \in V(\mathfrak{m})$. We have $\mathfrak{m} \subset \mathcal{I}(\{a\}) \subset K[x_1, \ldots, x_n]$. Hence we get $\mathfrak{m} = \mathcal{I}(\{a\})$ (since $\mathfrak{m}$ is a maximal ideal). Conversely, if $a \in \mathbb{A}^n$, consider the homomorphism $K[x_1, \ldots, x_n] \to K$, $f \mapsto f(a)$. Its kernel is precisely $\mathcal{I}(\{a\})$, and is a maximal ideal (since K is a field). Thus, under the above bijection, points of $\mathbb{A}^n$ correspond 1-1 to the maximal ideals of $K[x_1, \ldots, x_n]$. (The maximal ideal corresponding to $(a_1, \ldots, a_n)$ is the ideal $(x_1 - a_1, \ldots, x_n - a_n)$.)

Irreducible components

A topological space X is said to be *irreducible* if X cannot be written as the union of two proper nonempty closed sets in X, or equivalently, any two nonempty open sets in X have a nonempty intersection, or equivalently, any nonempty open set is dense. It is easily seen that a subspace $Y \subset X$ is irreducible if and only if its closure $\bar{Y}$ is irreducible. By Zorn's lemma, every irreducible subspace of X is contained in a maximal one; the maximal irreducible subsets are closed, and are called the *irreducible components of X*.

Noetherian spaces

A topological space is said to be *Noetherian* if every open set in X is quasi-compact, or equivalently, if open sets satisfy the maximal condition, or equivalently, if each nonempty collection of open sets has a maximal element, or equivalently, if open sets satisfy a.c.c., or equivalently, if closed sets satisfy d.c.c.

Proposition 1.2.3 (cf. [5, 36]). *Let X be Noetherian. Then X has only finitely many irreducible components X_i, and $X = \bigcup X_i$.*

Irreducible affine varieties

Proposition 1.2.4. *A closed set X in $\mathbb{A}^n$ is irreducible if and only if $\mathcal{I}(X)$ is prime. In particular, $\mathbb{A}^n$ is irreducible.*

Proof. Denote $\mathcal{I}(X)$ by I. Let X be irreducible. Let $f_1 f_2 \in I$. Then each $a \in X$ is a zero of f_1 or f_2. Thus $X \subset V(I_1) \cup V(I_2)$, where $I_1 = (f_1)$, $I_2 = (f_2)$. The irreducibility of X implies that either $X \subset V(I_1)$ or $X \subset V(I_2)$. Hence we obtain that either f_1 or f_2 belongs to I. Thus I is prime.

Now let I be prime. If possible, let us assume that $X = X_1 \cup X_2$, with $X_i \subset X$, closed in X for $i = 1, 2$. Then we can find $f_i \in \mathcal{I}(X_i)$ such that $f_i \notin I$. Now, for $z \in X$, $a \in X_1$ or $a \in X_2$. Hence $f_1(a) = 0$ or $f_2(a) = 0$. This implies $f_1 f_2(a) = 0$ for all $a \in X$, i.e. $f_1 f_2 \in I$, with $f_i \notin I$, $i = 1, 2$. This contradicts the primality of I. Hence our assumption is wrong and the result follows. □

Corollary 1.2.5. *Under the bijection following Theorem 1.2.2, the prime ideals correspond to irreducible affine varieties.*

The affine algebra $K[X]$

A finitely generated K-algebra is also called an *affine K-algebra* (or simply an affine algebra, the field K being fixed). Let X be an affine variety in $\mathbb{A}^n$. The affine algebra $K[x_1, \ldots, x_n]/\mathcal{I}(X)$ is called the *affine algebra of* X, and is denoted $K[X]$. Now each $\bar{f} \in K[X]$ defines a function $X \to K$, $a \mapsto f(a)$ (note that this is well-defined). Thus each element of $K[X]$ may be thought of as a polynomial function on X (with values in K). For this reason, $K[X]$ is also called the *algebra of polynomial functions on* X, or also the *algebra of regular functions on* X, or the *coordinate ring of* X. If X is irreducible, then $K[X]$ is an integral domain (since $\mathcal{I}(X)$ is prime), and the quotient field $K(X)$ of $K[X]$ is called the *function field of* X (or the *field of rational functions on* X).

As in the case of $\mathbb{A}^n$, we see that we have a bijection between closed subsets of X and the radical ideals of $K[X]$, under which the irreducible closed subsets of X correspond to the prime ideals of $K[X]$. In particular, the points of X are in one to one correspondence with the maximal ideals of $K[X]$. Further, X is a Noetherian topological space, and the *principal open subsets* $X_f = \{x \in X \mid f(x) \neq 0\}$, $f \in K[X]$, give a base for the Zariski topology.

Morphisms

Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be two affine varieties. A *morphism* $\varphi : X \to Y$ is a mapping of the form $\varphi(a) = (\psi_1(a), \ldots, \psi_m(a))$, where $a \in X$, and for $i = 1, \ldots, m$, $\psi_i \in K[x_1, \ldots, x_n]$. A morphism $\varphi : X \to Y$ defines a K-algebra morphism $\varphi^* : K[Y] \to K[X]$ given by $\varphi^*(f) = f \circ \varphi$. We have

Theorem 1.2.6 (cf. [28])**.** *The map $X \mapsto K[X]$ defines a (contravariant) equivalence of the category of affine varieties (with morphisms as defined above) and the category of affine K-algebras (i.e. finitely generated K-algebras) without non-zero nilpotents (with K-algebra morphisms as morphisms). Further, irreducible affine varieties correspond to affine K-algebras which are integral domains.*

Products of affine varieties

The product of the Zariski topologies on $\mathbb{A}^n$ and $\mathbb{A}^m$ does not give the Zariski topology on $\mathbb{A}^{n+m}$. For example, in $\mathbb{A}^1 \times \mathbb{A}^1$ the only closed sets in the product topology are finite unions of horizontal and vertical lines, while $\mathbb{A}^2$ has many more sets that are closed in the Zariski topology.

To arrive at a correct definition (so that we will have $\mathbb{A}^n \times \mathbb{A}^m \cong \mathbb{A}^{n+m}$), one takes the general category-theoretical definition, and defines the product as a triple (Z, p, q), where Z is an affine variety and $p : Z \to X$, $q : Z \to Y$ are morphisms such that given a triple (M, α, β), where M is an affine variety and $\alpha : M \to X$, $\beta : M \to Y$ are morphisms, there exists a unique morphism

$\theta : M \to Z$ such that the following diagram is commutative:

$$\begin{tikzcd} & X \\ M \arrow[ur, "\alpha"] \arrow[r, "\theta"] \arrow[dr, "\beta"'] & Z \arrow[u, "p"'] \arrow[d, "q"] \\ & Y \end{tikzcd}$$

Theorem 1.2.7 (Existence of products)**.** *Let X, Y be two affine varieties with coordinate rings R, S respectively. Then the affine variety Z with coordinate ring $R \otimes_K S$ together with the canonical maps $p : Z \to X$, $q : Z \to Y$ (induced by $R \to R \otimes S$, $r \mapsto r \otimes 1$, and $S \to R \otimes S$, $s \mapsto 1 \otimes s$ respectively) is a product of X and Y.*

The uniqueness up to isomorphism of a product follows from the universal mapping property of a product. The product of X and Y is denoted by $(X \times Y, p, q)$.

Remark 1.2.8. Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be two affine varieties. Then the product variety $X \times Y$ (as defined above) is nothing but the set $X \times Y \subset \mathbb{A}^{n+m}$, together with the induced topology.

1.3 Projective Varieties

The projective space $\mathbb{P}^n$

We shall denote by $\mathbb{P}^n_K$, or just $\mathbb{P}^n$, the set $\left(\mathbb{A}^{n+1} \setminus \{0\}\right) / \sim$, where $\sim$ is the equivalence relation $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$ if there exists $\lambda \in K^*$ such that $(a_0, \ldots, a_n) = \lambda\,(b_0, \ldots, b_n)$. Thus a point $P \in \mathbb{P}^n$ is determined by an equivalence class $[a_0, \ldots, a_n]$, and for any $(n+1)$-tuple $(b_0, \ldots, b_n)$ in this equivalence class, the b_i's will be referred as the *projective* (or *homogeneous*) *coordinates of* P.

Sometimes we write $\mathbb{P}^n$ also as $\mathbb{P}(V)$, where V is an $(n+1)$-dimensional K-vector space, the points of $\mathbb{P}^n$ being identified with 1-dimensional subspaces of V. Let $f(x_0, \ldots, x_n) \in K[x_0, \ldots, x_n]$. Further, let f be homogeneous of degree d. The homogeneity of f implies $f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n)$, $\lambda \in K^*$. Hence it makes sense to talk about f being zero or nonzero at a point $P \in \mathbb{P}^n$.

Projective varieties

Let I be a homogeneous ideal in $K[x_0, \ldots, x_n]$, i.e. for each $f \in I$, the homogeneous parts of f belong to I, or equivalently, I is generated by some homogeneous polynomials. Let

$$V(I) = \{P \in \mathbb{P}^n \mid f(P) = 0,\ f \text{ a homogeneous element of } I\}.$$

The set $V(I)$ is called a *projective variety*. Conversely, given a subset of $X \subset \mathbb{A}^n$, let $\mathcal{I}(X)$ be the ideal generated by

$$\{f \in K[x_0, \ldots, x_n],\ f \text{ homogeneous} \mid f(P) = 0 \text{ for all } P \in X\}.$$

As in the affine case, $\mathcal{I}(X)$ is a radical ideal. We have a similar version (as in the affine case) of the Nullstellensatz with one minor adjustment, namely the ideal I_0 in $K[x_0, \ldots, x_n]$ generated by $x_0, \ldots, x_n$ is a proper radical ideal, but clearly has no zeros in $\mathbb{P}^n$. Deleting I_0, we have a similar formulation given by the following.

Theorem 1.3.1. *The maps $I \mapsto V(I)$, $X \mapsto \mathcal{I}(X)$ define an inclusion-reversing bijection between the set of homogeneous radical ideals of $K[x_0, \ldots, x_n]$ other than I_0 and the projective varieties in $\mathbb{P}^n$.*

Zariski topology on $\mathbb{P}^n$

The Zariski topology on $\mathbb{P}^n$ is defined in exactly the same way as in the affine case, by declaring

$$\{V(I) \mid I \text{ homogeneous radical ideal in } K[x_0, \ldots, x_n] \text{ other than } I_0\}$$

as closed sets. As in the affine case, under the above bijection, the homogeneous prime ideals (other than I_0) correspond to irreducible projective varieties. Let $U_i = \{[a] \in \mathbb{P}^n \mid a_i \neq 0\}$, $0 \leq i \leq n$. (These are some special open sets.) The map $U_i \to \mathbb{A}^n$,

$$[a] \mapsto \left(\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i}\right)$$

defines an isomorphism of affine varieties. The quotients

$$\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i}$$

are called the *affine coordinates on* U_i, $0 \leq i \leq n$. Note that $\{U_i,\ 0 \leq i \leq n\}$ is an open cover for $\mathbb{P}^n$.

1.4 Schemes - Affine and Projective

Presheaves

Let X be a topological space. Let $\operatorname{top}(X)$ be the category whose objects are open sets in X, and whose morphisms are inclusions. Let $\mathcal{C}$ be a category. A *$\mathcal{C}$-valued presheaf on X* is a contravariant functor $U \mapsto F(U)$ from $\operatorname{top}(X)$ to $\mathcal{C}$. Thus, if $V \subset U$ are open sets in X, then we have a $\mathcal{C}$-morphism

$$\operatorname{res}^U_V : F(U) \longrightarrow F(V)$$

called the *restriction map*.

A *morphism of presheaves* $\varphi : F \to F'$ is a morphism of functors. Suppose $\mathcal{C}$ is a category of "sets with structure," like groups, rings, modules, etc. Then we say that F is a presheaf of groups, rings, modules, etc., respectively.

If $x \in X$, then the collection $\mathcal{U}_x = \{F(U), U$ open neighborhood of $x\}$ is a directed system, and $F_x = \varinjlim_{F(U) \in \mathcal{U}_x} F(U)$ is called the *stalk of F at x*.

Sheaves

Let F be a $\mathcal{C}$-valued presheaf on X, where $\mathcal{C}$ is some category of "sets with structure." Then F is called a *sheaf* if it satisfies the following "sheaf axioms": For every collection $\{U_i\}$ of open sets in X with $U = \bigcup U_i$,

(S1) if $f, g \in F(U)$ are such that $f|_{U_i} = g|_{U_i}$ for all i, then $f = g$.

(S2) if $\{f_i \in F(U_i)\}$ is a collection such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i and j, then there exists an $f \in F(U)$ such that $f|_{U_i} = f_i$.

Example 1.4.1. Let X be a topological space, and for U open in X, let $F(U)$ be the ring of continuous real valued functions on U. The assignment $U \mapsto F(U)$, for U open, defines a sheaf.

Example 1.4.2. Let $X \in \mathbb{A}^n$ be an irreducible affine variety, with function field $K(X)$. Let $R = K[X]$. Define

$$\mathcal{O}_{X,x} = \{f \in K(X) \mid f \text{ is regular at } x\},$$

(note that f is regular at x if $f = g/h$, with $g, h \in R$, $h(x) \neq 0$). We have, $\mathcal{O}_{X,x}$ is simply $R_{\mathcal{P}}$, where $\mathcal{P}$ is the prime ideal $\{f \in R \mid f(x) = 0\}$; in particular, $\mathcal{O}_{X,x}$ is a local ring. The assignment $U \mapsto \bigcap_{x \in U} \mathcal{O}_{X,x}$, for U open defines a sheaf called the *structure sheaf*, and denoted $\mathcal{O}_X$.

Example 1.4.3. Let now X be an affine variety with irreducible components X_i. Let $U \subset X$ open, and $x \in U$. A function $f : U \to K$ is said to be *regular at x* if there exist $g, h \in K[X]$ and an open set $V \subset U$, $x \in V$, such that for all $y \in V$, $h(y) \neq 0$, $f(y) = \frac{g(y)}{h(y)}$. Set

$$\mathcal{O}_X(U) = \{f : U \to K \mid f \text{ regular at all } x \in U\}.$$

Then $\mathcal{O}_X$ is a sheaf, again called the *structure sheaf.* Note that $\mathcal{O}_X(X) = K[X]$.

Sheafification

Let F be a $\mathcal{C}$-valued presheaf on X, where $\mathcal{C}$ is some category of "sets with structure." Then there is a sheaf F', called the *sheafification of F*, or the *sheaf associated with F*, and a morphism $f : F \to F'$ such that the map $\mathrm{Mor}(F', G) \to \mathrm{Mor}(F, G)$ (induced by f) is bijective whenever G is a sheaf.

Construction of F'

Let $E = \bigcup_{x \in X} F_x$ (a disjoint union of sets). Let $p : E \to X$ be the "projection map," namely $p(a) = x$, if $a \in F_x$. For U open in X, and $\sigma \in F(U)$, we have a canonical map (also denoted by σ) $\sigma : U \to E$, $\sigma(x) = \overline{\sigma_x}$, where $\overline{\sigma_x}$ is the image of σ under the canonical map $F(U) \to F_x$. Equip E with the strongest topology which makes $\sigma : U \to E$ continuous for all $\sigma \in F(U)$, and all open U. It can be seen easily that a set G in E is open if and only if for every open U in X and $\sigma \in F(U)$, the set $W = \{x \in U \mid \overline{\sigma_x} \in G\}$ is open. The space (E, p) is called the *etale space of* F.

Let F' be the sheaf of continuous sections of p, i.e. for U open in X,

$$F'(U) = \{s : U \to E \text{ continuous such that } p \circ s = Id_U\}.$$

Remark 1.4.4. Given a sheaf $\mathcal{F}$ on X, $\mathcal{F}(X)$ is usually denoted by $\Gamma(X, \mathcal{F})$, and its elements are called *global sections* of $\mathcal{F}$.

Geometric spaces

A *geometric space* is a topological space X together with a sheaf $\mathcal{O}_X$ of rings (commutative with identity element) on X whose stalks $\mathcal{O}_{X,x}$ are local rings. The sheaf $\mathcal{O}_X$ is called the structure sheaf of X. We denote the maximal ideal of $\mathcal{O}_{X,x}$ by $\mathfrak{m}_x$, and the residue class field by $K(x)$.

A morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of geometric spaces consists of a continuous map $f : X \to Y$ together with ring homomorphisms

$$f_U^V : \mathcal{O}_Y(V) \to \mathcal{O}_X(U)$$

for $U \subset X$, $V \subset Y$ open sets such that $f(U) \subset V$. These maps are required to be compatible with the respective restriction maps in $\mathcal{O}_X$ and $\mathcal{O}_Y$. Then it is easy to see that if $x \in X$ and $y = f(x)$, then f induces a local homomorphism $f_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, i.e. $f_x(\mathfrak{m}_y) \subset \mathfrak{m}_x$.

1.5 The Scheme Spec(A)

Let A be a commutative ring with identity element, and let Spec(A) be the set of all prime ideals in A. Define a topology on Spec(A) (called the *Zariski topology on* Spec(A)) by declaring the closed sets as $V(I) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \supset I\}$, for I any ideal of A. For $Y \subset X = \mathrm{Spec}(A)$, let $\mathcal{I}(Y) = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$. Then $V(\mathcal{I}(Y)) = \bar{Y}$. Further, we have $\mathcal{I}(V(I)) = \sqrt{I}$. Thus we have an inclusion-reversing bijection between the set of closed sets in Spec(A), and the set of radical ideals in A, under which irreducible closed sets correspond to prime ideals. If A is Noetherian, then Spec(A) is a Noetherian topological space, and the irreducible components of X correspond to the minimal primes in A.

The principal open sets

Let $f \in A$, and $\mathfrak{p} \in \mathrm{Spec}(A)$. Let $f(\mathfrak{p})$ be the image of f in the residue class field of $A_{\mathfrak{p}}$ (which is simply the field of fractions of $A/\mathfrak{p}$). Let $X = \mathrm{Spec}(A)$, and $X_f = X \setminus V((f)) = \{\mathfrak{p} \in X \mid f(\mathfrak{p}) \neq 0\}$. The set X_f is called a *principal open set*. For any ideal I, we have $V(I) = \bigcap_{f \in I} V((f))$. Thus the principal open sets form a base of the Zariski topology on X.

Geometric space structure on Spec(A)

Let $X = \mathrm{Spec}(A)$, and let $\mathcal{O}_X$ be the sheaf associated to the presheaf $U \mapsto S_U^{-1}A$, where S_U (for U open in X) is the set of all $f \in A$ vanishing nowhere on U, i.e. $f(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in U$. It is easily seen that the stalk $\mathcal{O}_{X,x}$ is simply A_x. Thus we have the following theorem.

Theorem 1.5.1. $\left(\mathit{Spec}(A), \mathcal{O}_{\mathit{Spec}(A)}\right)$ *is a geometric space.*

Remark 1.5.2. If A is an integral domain with quotient field K, then the A_x's are subrings of K, and $\mathcal{O}_X$ can be defined directly by $\mathcal{O}_X(U) = \bigcap_{x \in U} A_x$.

Let $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. Then it is seen easily that a morphism $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ induces a ring homomorphism $B \to A$. Conversely, a ring homomorphism $B \to A$ induces a morphism $X \to Y$, (cf. [67]).

Affine schemes

Definition 1.5.3. An *affine scheme* is a geometric space $(X, \mathcal{O}_X)$ which is isomorphic to $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$.

Theorem 1.5.4. *The map* $A \mapsto (\mathit{Spec}(A), \mathcal{O}_{\mathit{Spec}(A)})$ *defines a (contravariant) equivalence of the category of commutative rings and the category of affine schemes.*

Definition 1.5.5. A *prescheme* is a geometric space $(X, \mathcal{O}_X)$ which has a finite cover by open sets U such that $(U, \mathcal{O}|_U)$ is an affine scheme.

Definition 1.5.6. A prescheme X is called a *scheme* if the diagonal $\Delta(X)$ ($= \{(x,x) \in X \times X\}$) is closed in $X \times X$.

Remark 1.5.7. An affine scheme is a scheme in the sense of Definition 1.5.6.

1.6 The Scheme Proj(S)

Let $S = \bigoplus_{d \geq 0} S_d$ be a *graded ring*, i.e., S_d is an abelian group and $S_d S_e \subset S_{d+e}$. Let $S_+ = \bigoplus_{d > 0} S_d$. Define Proj($S$) to be the set of all homogeneous prime ideals of S not containing S_+. For a homogeneous ideal $\mathfrak{a}$ of S, set

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{p} \supseteq \mathfrak{a}\}.$$

In view of the following lemma, we define $V(\mathfrak{a})$ to be a closed set for any homogeneous ideal $\mathfrak{a}$ of S, and we obtain the Zariski topology on $\mathrm{Proj}(S)$.

Lemma 1.6.1 (cf. [28]).

1. *If $\mathfrak{a}$ and $\mathfrak{b}$ are two homogeneous ideals in S, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.*
2. *If $\{\mathfrak{a}_i\}$ is any family of homogeneous ideals in S, then $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$.*

The structure sheaf $\mathcal{O}$ on Proj(S)

For $\mathfrak{p} \in \mathrm{Proj}(S)$, let $S_{(\mathfrak{p})}$ denote the *homogeneous localization* of S at $\mathfrak{p}$ consisting of elements of degree 0 in $S_{\mathfrak{p}}$, i.e.,

$$S_{(\mathfrak{p})} = \{\frac{f}{g} \in S_{\mathfrak{p}} \mid f, g \text{ homogeneous of the same degree}\}.$$

For an open set $U \subseteq \mathrm{Proj}(S)$, define $\mathcal{O}(U)$ as the set of functions $s : U \to \coprod S_{(\mathfrak{p})}$ such that the following hold:

1. For each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$.
2. s is locally a quotient of elements of S, i.e., for each $\mathfrak{p}$ in U, there exists a neighborhood V of $\mathfrak{p}$ in U, and homogeneous elements $g, f \in S$ of the same degree, such that for all $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, $s(\mathfrak{q}) = \frac{g}{f}$ in $S_{(\mathfrak{q})}$.

It is clear that $\mathcal{O}$ is a presheaf of rings, with the natural restrictions, and it is also clear from the local nature of the definition that $\mathcal{O}$ is in fact a sheaf.

Definition 1.6.2. We define $(\mathrm{Proj}(S), \mathcal{O})$ to be $\mathrm{Proj}(S)$ with the sheaf of rings constructed above.

Proposition 1.6.3 (cf. [28]). *$(Proj(S), \mathcal{O})$ is a scheme. Further, for $\mathfrak{p} \in Proj(S)$, the stalk $\mathcal{O}_{\mathfrak{p}}$ is isomorphic to the local ring $S_{(\mathfrak{p})}$.*

The special open subsets $D_+(f)$

For a homogeneous element $f \in S_+$, define $D_+(f) = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid f \notin \mathfrak{p}\}$. Then $D_+(f)$ is open in $\mathrm{Proj}(S)$. Further, these open sets cover $\mathrm{Proj}(S)$, and we have an isomorphism of geometric spaces:

$$\left(D_+(f), \mathcal{O}|_{D_+(f)}\right) \cong \mathrm{Spec}(S_{(f)}),$$

where $S_{(f)}$ is the subring of elements of degree 0 in S_f. See [28] for details.

1.7 Sheaves of $\mathcal{O}_X$-Modules

Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf on X is said to be a sheaf of $\mathcal{O}_X$-modules if for $U \subset X$ open, $F(U)$ is an $\mathcal{O}_X(U)$-module.

Example 1.7.1. Let M be an A-module. Then M defines a presheaf $\bar{M}$, namely for all $f \in A$, $\bar{M}(X_f) = M_f$ (note that for any open U, we have $\bar{M}(U) = \varprojlim_{X_f \subset U} M_f$). Let $\tilde{M}$ be the sheaf associated to $\bar{M}$. Then $\tilde{M}$ is a sheaf of $\mathcal{O}_X$-modules.

The sheaf $\tilde{M}$ on Proj(S)

Let S be a graded ring, and M a graded S-module, i.e., M is an S-module together with a decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ such that $S_d \cdot M_r \subseteq M_{d+r}$. The sheaf $\tilde{M}$on Proj(S) is defined as follows. For $\mathfrak{p} \in \mathrm{Proj}(S)$, let $M_{(\mathfrak{p})}$ denote the group of elements of degree 0 in $M_{\mathfrak{p}}$. For an open set $U \subseteq \mathrm{Proj}(S)$, define $\tilde{M}(U)$ as the set of functions $s : U \to \coprod M_{(\mathfrak{p})}$ such that the following holds:

1. For each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_{(\mathfrak{p})}$.

2. s is locally a quotient, i.e., for each $\mathfrak{p}$ in U, there exists a neighborhood V of $\mathfrak{p}$ in U, and homogeneous elements $m \in M$, $f \in S$ of the same degree, such that for all $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, $s(\mathfrak{q}) = \frac{m}{f}$ in $M_{\mathfrak{q}}$.

We make $\tilde{M}$ into a sheaf with the natural restriction maps. We have the following facts:

1. For $\mathfrak{p} \in \mathrm{Proj}(S)$, the stalk $\tilde{M}_{\mathfrak{p}} \cong M_{(\mathfrak{p})}$.

2. For a homogeneous element $f \in S_+$, we have, $\tilde{M}|_{D_+(f)} \cong \left(\widetilde{M_{(f)}}\right)$ via the isomorphism $D_+(f) \cong \mathrm{Spec}(S_{(f)})$, where $M_{(f)}$ denotes the group of elements of degree 0 in the localized module M_f.

See [28] for details.

The twisting sheaf

Let $X = \mathrm{Proj}(S)$. For $n \in \mathbb{Z}$, set $M(n)$ to be the graded S-module with $M(n)_d = M_{n+d}$ for all $d \in \mathbb{Z}$. Define the sheaf $\mathcal{O}_X(n)$ to be $\widetilde{S(n)}$. The sheaf $\mathcal{O}_X(1)$ is called the *twisting sheaf of Serre*. For any sheaf F of $\mathcal{O}_X$-modules, we define

$$F(n) = F \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

The sheaf $f_*\mathcal{F}$

Let $f : X \to Y$ be a morphism between two schemes. Let $\mathcal{F}$ be any sheaf on X. We define the direct image sheaf $f_*\mathcal{F}$ on Y by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ for any open subset V of Y.