Progress in Probability
73

Daniel Hernández-Hernández
Juan Carlos Pardo
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XII Symposium of Probability and Stochastic Processes

Merida, Mexico, November 16–20, 2015

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Introduction

The present volume contains contributions and lecture notes of the XII Symposium on Probability and Stochastic Processes, held at the Universidad Autónoma de Yucatán (UAdY), Mexico in November 16–20, 2015.

The traces of this symposium reach back to December 1988 at CIMAT, when it was held for the first time. The symposium is one of the main events in the field, and it takes place every 2 years at different academic institutions in Mexico. During these 27 years and up until today, this series of symposia has readily accomplished its main goal of exchanging ideas and discussing the latest developments in the field by gathering both national and international researchers as well as graduate students.

The symposium in 2015 gathered scholars from over seven countries and covered a wide range of topics that highlight the interaction between applied and theoretical probability. The scientific programme included two courses: *Optimality of two-parameter strategies in stochastic control* organized by Kazutoshi Yamazaki, and *Scaling limits of large random trees* organized by Bénédicte Haas. The event also benefited from nine plenary talks that were delivered by José Blanchet, Loïc Chaumont, Alex Cox, Takis Konstantopoulos, Andreas Kyprianou, Hubert Lacoin, Mihai Sirbu, Gerónimo Uribe and Hasnaa Zidani. Another four thematic sessions and fourteen contributed talks completed the outline of the symposium.

This volume is split into two main parts: first the lectures notes of the two courses provided by Bénédicte Haas and Kazutoshi Yamazaki, followed by research contributions of some of the participants. The lecture notes of Bénédicte Haas and Kazutoshi Yamazaki give an overview of the recent progress on describing the large-scale structure of random trees, and on stochastic control problems where the optimal strategies are described by two parameters under a setting that is driven by a spectrally one-sided Lévy process, respectively. The research contributions start with an illustrative article written by Ekaterina Kolkovska and Ehyter Martín-González, in which they investigate a classical risk process with two-sided jumps that is perturbed by a spectrally negative α-stable process, where the gain size distribution has a rational Laplace transform. The contribution of Daniel Hernández-Hernández and Leonel Pérez-Hernández analyses the minimality of the penalty function associated with a convex risk measure. By considering dynamic programming, Laurent series and the study of sensitive discount optimality, Beatris

Escobedo-Trujillo, Héctor Jasso-Fuentes and José Daniel López-Barrientos analyse Blackwell-Nash equilibria for a general class of zero-sum stochastic differential games. Γ-convergence of monotone functionals is discussed in the contribution written by Erick Treviño-Aguilar, where a criterion is presented under which a functional that is defined on vectors of non-decreasing functions is the Γ-limit of a functional that is defined on vectors of continuous non-decreasing functions. A criterion for the blow-up of a system of one-dimensional reaction-diffusion equations in a finite time is proposed by Eugenio Guerrero and José Alfredo López-Mimbela, where the criterion depends on the drift terms of the system of partial differential equations and on some measures which turn out to be invariant distributions of some diffusions that are associated with the system. Finally, Arno Siri-Jégousse and Linglong Yuan study the asymptotic behaviour, for small times, of the largest block size of Beta-n-coalescents as n increases.

In summary, the high quality and variety of these contributions give a broad panorama of the rich academic programme of the symposium and of its impact. It is worth noting that all papers, including the lecture notes of the invited courses, were subject to a strict peer review process with high international standards. We are very grateful to the referees, many of whom are leading experts in their fields, for their diligent and useful reports. Their comments were implemented by the authors and considerably improve the material presented herein.

We would also like to express our gratitude to all the authors whose original contributions are published in this book, as well as to all the speakers and session organizers of the symposium for their stimulating talks and support. Their valuable contributions show the interest and activity in the area of probability and stochastic processes in Mexico.

We hold in high regard the editors of the book series *Progress in Probability*, Steffen Dereich, Davar Khoshnevisan, Andreas E. Kyprianou and Sidney I. Resnick, for giving us the opportunity to publish the symposium volume in this prestigious series.

Special thanks to the symposium venue Universidad Autónoma de Yucatán and its staff for their great hospitality and for providing excellent conference facilities. We are also indebted to Rosy Davalos, whose outstanding organizational work permitted us to focus on the academic aspects of the conference.

The symposium as well as this volume would not have been possible without the generous support of our sponsors: Centro de Investigación en Matemáticas, RED-CONACYT Matemáticas y Desarrollo, Laboratorio Internacional Solomon Lefschetz CNRS-CONACYT, Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas and Instituto de Matemáticas at UNAM as well as Universidad Autónoma de Yucatán.

Finally, we hope that the reader of this volume will enjoy learning about the various topics that are treated therein, as much as we did editing it.

Guanajuato, Mexico — Daniel Hernández-Hernández
Guanajuato, Mexico — Juan Carlos Pardo
Guanajuato, Mexico — Victor Rivero

Previous Volumes from the Symposium on Probability and Stochastic Processes

- M. E. Caballero and L. G. Gorostiza, editors. *Simposio de Probabilidad y Procesos Estocásticos*, volume 4 of *Aportaciones Matemáticas: Notas de Investigación [Mathematical Contributions: Research Notes]*. Sociedad Matemática Mexicana, México, 1989.
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  Held at Centro de Investigación en Matemáticas, Guanajuato, México, November 18–22, 2013.

Contents

Part I
Courses

Scaling Limits of Markov-Branching Trees and Applications

Lecture Notes of the XII Simposio de Probabilidad y Procesos Estocásticos 16–20 Novembre 2015, Mérida, Yucatán

Bénédicte Haas

Abstract The goal of these lecture notes is to survey some of the recent progress on the description of large-scale structure of random trees. We use the framework of Markov-Branching sequences of trees and discuss several applications.

Keywords Random trees · Scaling limits · Self-similar fragmentations · self-similar Markov processes

Mathematics Subject Classification 05C05, 60F17, 60J05, 60J25, 60J80

1 Introduction

The goal of these lecture notes is to survey some of the recent progress on the description of large-scale structure of random trees. Describing the structure of large (random) trees, and more generally large graphs, is an important goal of modern probabilities and combinatorics. Beyond the purely probabilistic or combinatorial aspects, motivations come from the study of models from biology, theoretical computer science or mathematical physics.

The question we will typically be interested in is the following. For $(T_n, n \geq 1)$ a sequence of random unordered (i.e. non-planar) trees, where, for each n, T_n is a tree of size n (the size of a tree may be its number of vertices or its number of leaves, for example): does there exist a deterministic sequence $(a_n, n \geq 1)$ and a *continuous*

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random tree $\mathcal{T}$ such that

$$\frac{T_n}{a_n} \underset{n\to\infty}{\longrightarrow} \mathcal{T}?$$

To make sense of this question, we will view T_n as a metric space by "replacing" its edges with segments of length 1, and then use the notion of Gromov-Hausdorff distance to compare compact metric spaces. When such a convergence holds, the continuous limit highlights some properties of the discrete objects that approximate it, and vice-versa.

As a first example, consider T_n a tree picked uniformly at random in the set of trees with n vertices labelled by $\{1, \dots, n\}$. The tree T_n has to be understood as a *typical element* of this set of trees. In this case the answer to the previous question dates back to a series of works by Aldous in the beginning of the 1990s [8–10]: Aldous showed that

$$\frac{T_n}{2\sqrt{n}} \overset{(\mathrm{d})}{\underset{n\to\infty}{\longrightarrow}} \mathcal{T}_{\mathrm{Br}} \tag{1}$$

where the limiting tree is called the Brownian Continuum Random Tree (CRT), and can be constructed from a standard Brownian excursion. This result has various interesting consequences, e.g. it gives the asymptotics in distribution of the diameter, the height (if we consider rooted versions of the trees) and several other statistics related to the tree T_n. Consequently it also gives the asymptotic proportion of trees with n labelled vertices that have a diameter larger than $x\sqrt{n}$ or/and a height larger than $y\sqrt{n}$, etc. Some of these questions on statistics of uniform trees were already treated in previous works, the strength of Aldous's result is that it describes the asymptotics of the *whole* tree T_n.

Aldous has actually established a version of the convergence (1) in a much broader context, that of *conditioned Galton–Watson trees* with finite variance. In this situation, to fit to our context, T_n is an unordered version of the genealogical tree of a Galton–Watson process (with a given, fixed offspring distribution with mean one and finite variance) conditioned on having a total number of vertices equal to $n, n \geq 1$. Multiplied by $1/\sqrt{n}$, this tree converges in distribution to the Brownian CRT multiplied by a constant that only depends on the variance of the offspring distribution. This should be compared with (and is related to) the convergence of rescaled sums of i.i.d. random variables towards the normal distribution and its functional analog, the convergence of rescaled random walks towards the Brownian motion. It turns out that the above sequence of uniform labelled trees can be seen as a sequence of conditioned Galton–Watson trees (when the offspring distribution is a Poisson distribution) and more generally that several sequences of *combinatorial* trees reduce to conditioned Galton–Watson trees. In the early 2000s, Duquesne [44] extended Aldous's result to conditioned Galton–Watson trees with offspring distributions in the domain of attraction of a stable law. We also refer to [46, 70] for related results. In most of these cases the scaling sequences (a_n) are asymptotically much smaller, i.e. $a_n \ll \sqrt{n}$, and other continuous trees arise in the limit, the so-called family of *stable Lévy trees*. All these results on conditioned Galton–Watson trees are now well established, and have a lot of applications in the study of large random graphs (see e.g. Miermont's book [78] for the connections with random maps and Addario-Berry et al. [4] for connections with Erdős–Rényi random graphs in the critical window).

The classical proofs to establish the scaling limits of Galton–Watson trees consist in considering specific ordered versions of the trees and rely on a careful study of their so-called *contour functions*. It is indeed a common approach to encode trees into functions (similarly to the encoding of the Brownian tree by the Brownian excursion), which are more familiar objects. It turns out that for Galton–Watson trees, the contour functions are closely related to random walks, whose scaling limits are well known. Let us also mention that another common approach to study large random combinatorial structures is to use technics of analytic combinatorics, see [54] for a complete overview of the topic. None of these two methods will be used here.

In these lecture notes, we will focus on another point of view, that of sequences of random trees that satisfy a certain *Markov-Branching property*, which appears naturally in a large set of models and includes conditioned Galton–Watson trees. This property is a sort of discrete fragmentation property which roughly says that in each tree of the sequence, the subtrees above a given height are independent with a law that depends only on their total size. Under appropriate assumptions, we will see that Markov-Branching sequences of trees, suitably rescaled, converge to a family of continuous fractal trees, called the *self-similar fragmentation trees*. These continuous trees are related to the self-similar fragmentation processes studied by Bertoin in the 2000s [14], which are models used to describe the evolution of objects that randomly split as time passes. The main results on Markov-Branching trees presented here were developed in the paper [59], which has its roots in the earlier paper [63]. Several applications have been developed in these two papers, and in more recent works [15, 60, 89]: to Galton–Watson trees with arbitrary degree constraints, to several combinatorial trees families, including the Pólya trees (i.e. trees uniformly distributed in the set of rooted, unlabelled, unordered trees with n vertices, $n \geq 1$), to several examples of dynamical models of tree growth and to sequence of *cut-trees*, which describe the genealogy of some deletion procedure of edges in trees. The objective of these notes is to survey and gather these results, as well as further related results.

In Sect. 2 below, we will start with a series of definitions related to discrete trees and then present several classical examples of sequences of random trees. We will also introduce there the Markov-Branching property. In Sect. 3 we set up the topological framework in which we will work, by introducing the notions of real trees and Gromov–Hausdorff topology. We also recall there the classical results of Aldous [9] and Duquesne [44] on large conditioned Galton–Watson trees. Section 4 is the core of these lecture notes. We present there the results on scaling limits of Markov-Branching trees, and give the main ideas of the proofs. The key ingredient is the study of an integer-valued Markov chain describing the sizes of the subtrees containing a typical leaf of the tree. Section 5 is devoted to the applications mentioned above. Last, Sect. 6 concerns further perspectives and related models (multi-type trees, local limits, applications to other random graphs).

All the sequences of trees we will encounter here have a power growth. There is however a large set of random trees that naturally arise in applications that do not have such a behavior. In particular, many models of trees arising in the analysis of algorithms have a logarithmic growth. See e.g. Drmota's book [42] for an overview of the most classical models. These examples do not fit into our framework.

2 Discrete Trees, Examples and Motivations

2.1 Discrete Trees

Our objective is mainly to work with unordered trees. We give below a precise definition of these objects and mention nevertheless the notions of ordered or/and labelled trees to which we will sometimes refer.

A **discrete tree** (or graph-theoretic tree) is a finite or countable graph (V, E) that is connected and has no cycle. Here V denotes the set of vertices of the graph and E its set of edges. Note that two vertices are then connected by exactly one path and that $\#V = \#E + 1$ when the tree is finite.

In the following, we will often denote a (discrete) tree by the letter t, and for $\mathsf{t} = (V, E)$ we will use the slight abuse of notation $v \in \mathsf{t}$ to mean $v \in V$.

A tree t can be seen as a metric space, when endowed with the **graph distance** d_{gr}: given two vertices $u, v \in \mathsf{t}$, $d_{\mathrm{gr}}(u, v)$ is defined as the number of edges of the unique path from u to v.

A **rooted tree** (t, ρ) is an ordered pair where t is a tree and $\rho \in \mathsf{t}$. The vertex ρ is then called the root of t. This gives a genealogical structure to the tree. The root corresponds to the generation 0, its neighbors can be interpreted as its children and form the generation 1, the children of its children form the generation 2, etc. We will usually call the **height** of a vertex its generation, and denote it by $\mathsf{ht}(v)$ (the height of a vertex is therefore its distance to the root). The height of the tree is then

$$\mathsf{ht}(\mathsf{t}) = \sup_{v \in \mathsf{t}} \mathsf{ht}(v)$$

and its **diameter**

$$\mathsf{diam}(\mathsf{t}) = \sup_{u,v \in \mathsf{t}} d_{\mathrm{gr}}(u, v).$$

The **degree of a vertex** $v \in \mathsf{t}$ is the number of connected components obtained when removing v (in other words, it is the number of neighbors of v). A vertex v different from the root and of degree 1 is called a **leaf**. In a rooted tree, the **out-degree of a vertex** v is the number of children of v. Otherwise said, out-degree(v)=degree(v)-$\mathbb{1}_{\{v \neq \mathrm{root}\}}$. A (full) **binary** tree is a rooted tree where all vertices but the leaves have out-degree 2. A **branch-point** is a vertex of degree at least 3.

In these lecture notes, we will mainly work with rooted trees. Moreover we will consider, unless specifically mentioned, that two isomorphic trees are equal, or, when the trees are rooted, that **two root-preserving isomorphic trees are equal**. Such trees can be considered as *unordered unlabelled* trees, in opposition to the following definitions.

Ordered or/and Labelled Trees In the context of rooted trees, it may happen that one needs to order the children of the root, and then, recursively, the children of each vertex in the tree. This gives an **ordered** (or planar) tree. Formally, we generally see such a tree as a subset of the infinite Ulam–Harris tree

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

where $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}^0 = \{\varnothing\}$. The element $\varnothing$ is the root of the Ulam–Harris tree, and any other $u = u_1 u_2 \ldots u_n \in \mathcal{U}\backslash\{\varnothing\}$ is connected to the root via the unique shortest path

$$\varnothing \to u_1 \to u_1 u_2 \to \ldots \to u_1 \ldots u_n.$$

The height (or generation) of such a sequence u is therefore its length, n. We then say that $\mathsf{t} \subset \mathcal{U}$ is a (finite or infinite) rooted *ordered* tree if:

- $\varnothing \in \mathsf{t}$
- if $u = u_1 \ldots u_n \in \mathsf{t}\backslash\{\varnothing\}$, then $u = u_1 \ldots u_{n-1} \in \mathsf{t}$ (the parent of an individual in t that is not the root is also in t)
- if $u = u_1 \ldots u_n \in \mathsf{t}$, there exists an integer $c_u(\mathsf{t}) \geq 0$ such that the element $u_1 \ldots u_n j \in \mathsf{t}$ if and only if $1 \leq j \leq c_u(\mathsf{t})$.

The number $c_u(\mathsf{t})$ corresponds to the number of children of u in t, i.e., its out-degree.

We will also sometimes consider **labelled** trees. In these cases, the vertices are labelled in a bijective way, typically by $\{1, \ldots, n\}$ if there are n vertices (whereas in an unlabelled tree, the vertices but the root are indistinguishable). Partial labelling is also possible, e.g. by labelling only the leaves of the tree.

In the following we will always specify when a tree is ordered or/and labelled. When not specified, it is implicitly unlabelled, unordered.

Counting Trees It is sometimes possible, but not always, to have explicit formulæ for the number of trees of a specific structure. For example, it is known that the number of trees with n labelled vertices is

$$n^{n-2} \quad \text{(Cayley formula)},$$

and consequently, the number of rooted trees with n labelled vertices is

$$n^{n-1}.$$

The number of rooted ordered binary trees with $n + 1$ leaves is

$$\frac{1}{n+1}\binom{2n}{n}$$

(this number is called the nth Catalan number) and the number of rooted ordered trees with n vertices is

$$\frac{1}{n}\binom{2n-2}{n-1}.$$

On the other hand, there is no explicit formula for the number of rooted (unlabelled, unordered) trees. Otter [79] shows that this number is asymptotically proportional to

$$c\kappa^n n^{-3/2}$$

where $c \sim 0.4399$ and $\kappa \sim 2.9557$. This should be compared to the asymptotic expansion of the nth Catalan number, which is proportional (by Stirling's formula) to $\pi^{-1/2}4^n n^{-3/2}$.

We refer to the book of Drmota [42] for more details and technics, essentially based on generating functions.

2.2 *First Examples*

We now present a first series of classical families of random trees. Our goal will be to describe their scaling limits when the sizes of the trees grow, as discussed in the Introduction. This will be done in Sect. 5. Most of these families (but not all) share a common property, the Markov-Branching property that will be introduced in the next section.

Combinatorial Trees Let $\mathbb{T}_n$ denote a finite set of trees with n vertices, all sharing some structural properties. E.g. $\mathbb{T}_n$ may be the set of all rooted trees with n vertices, or the set of all rooted ordered trees with n vertices, or the set of all binary trees with n vertices, etc. We are interested in the asymptotic behavior of a "typical element" of $\mathbb{T}_n$ as $n \to \infty$. That is, we pick a tree *uniformly* at random in $\mathbb{T}_n$, denote it by T_n and study its scaling limit. The global behavior of T_n as $n \to \infty$ will represent some of the features shared by most of the trees. For example, if the probability that the height of T_n is larger than $n^{\frac{1}{2}+\varepsilon}$ tends to 0 as $n \to \infty$, this means that the proportion of trees in the set that have a height larger than $n^{\frac{1}{2}+\varepsilon}$ is asymptotically negligible, etc. We will more specifically be interested in the following cases:

- T_n is a uniform rooted tree with n vertices
- T_n is a uniform rooted ordered tree with n vertices

- T_n is a uniform tree with n labelled vertices
- T_n is a uniform rooted ordered binary tree with n vertices (n odd)
- T_n is a uniform rooted binary tree with n vertices (n odd),

etc. Many variations are of course possible, in particular one may consider trees picked uniformly amongst sets of trees with a given structure and n *leaves*, or more general degree constraints. Some of these uniform trees will appear again in the next example.

Galton–Watson Trees Galton–Watson trees are random trees describing the genealogical structure of Galton–Watson processes. These are simple mathematical models for the evolution of a population that continue to play an important role in probability theory and in applications. Let η be a probability on $\mathbb{Z}_+$ (η is called the *offspring distribution*) and let $m := \sum_{i \geq 1} i\eta(i) \in [0, \infty]$ denote its mean. Informally, in a Galton–Watson tree with offspring distribution η, each vertex has a random number of children distributed according to η, independently. We will always assume that $\eta(1) < 1$ in order to avoid the trivial case where each individual has a unique child. Formally, an η-Galton–Watson tree T^η is usually seen as an ordered rooted tree and defined as follows (recall the Ulam–Harris notation $\mathcal{U}$):

- $c_\varnothing(T^\eta)$ is distributed according to η
- conditionally on $c_\varnothing(T^\eta) = p$, the p ordered subtrees $\tau_i = \{u \in \mathcal{U} : iu \in T^\eta\}$ descending from $i = 1, \ldots, p$ are independent and distributed as T^η.

From this construction, one sees that the distribution of T^η is given by:

$$\mathbb{P}\left(T^\eta = \mathsf{t}\right) = \prod_{v \in \mathsf{t}} \eta_{c_v(\mathsf{t})} \tag{2}$$

for all rooted ordered tree t. This definition of Galton–Watson trees as ordered trees is the simplest, avoiding any symmetry problems. However in the following we will mainly see these trees up to isomorphism, which roughly means that we can "forget the order".

Clearly, if we call Z_k the number of individuals at height k, then $(Z_k, k \geq 1)$ is a Galton–Watson process starting from $Z_0 = 1$. It is well known that the extinction time of this process,

$$\inf\{k \geq 0 : Z_k = 0\}$$

if finite with probability 1 when $m \leq 1$ and with a probability $\in [0, 1)$ when $m > 1$. The offspring distribution η and the tree T^η are said to be *subcritical* when $m < 1$, *critical* when $m = 1$ and *supercritical* when $m > 1$. From now on, we assume that

$$m = 1$$

and for integers n such that $\mathbb{P}(\#T^\eta = n) > 0$, we let $T_n^{\eta,\mathrm{v}}$ denote a *non-ordered* version of the Galton–Watson tree T^η conditioned to have n vertices. Sometimes, we will need to keep the order and we will let $T_n^{\eta,\mathrm{v},\mathrm{ord}}$ denote this ordered conditioned version. We point out that in most cases, *but not all*, a subcritical or a supercritical Galton–Watson tree conditioned to have n vertices is distributed as a critical Galton–Watson tree conditioned to have n vertices with a different offspring distribution. So the assumption $m = 1$ is not too restrictive. We refer to [66] for details on that point.

It turns out that conditioned Galton–Watson trees are closely related to combinatorial trees. Indeed, one can easily check with (2) that:

- if $\eta = \mathrm{Geo}(1/2)$, $T_n^{\eta,\mathrm{v},\mathrm{ord}}$ is uniform amongst the set of rooted ordered trees with n vertices
- if $\eta = \mathrm{Poisson}(1)$, $T_n^{\eta,\mathrm{v}}$ is uniform amongst the set of rooted trees with n labelled vertices
- if $\eta = \frac{1}{2}(\delta_0 + \delta_2)$, $T_n^{\eta,\mathrm{v},\mathrm{ord}}$ is uniform amongst the set of rooted ordered binary trees with n vertices.

We refer e.g. to Aldous [9] for additional examples.

Hence, studying the large-scale structure of conditioned Galton–Watson trees will also lead to results in the context of combinatorial trees. As mentioned in the Introduction, the scaling limits of large conditioned Galton–Watson trees are now well known. Their study has been initiated by Aldous [8–10] and then expanded by Duquesne [44]. This will be reviewed in Sect. 3. However, there are some sequences of combinatorial trees that *cannot* be reinterpreted as Galton–Watson trees, starting with the example of the uniform rooted tree with n vertices or the uniform rooted binary tree with n vertices. Studying the scaling limits of these trees remained open for a while, because of the absence of symmetry properties. These scaling limits are presented in Sect. 5.2.

In another direction, one may also wonder what happens when considering versions of Galton–Watson trees conditioned to have n leaves, instead of n vertices, or more general degree constraints. This is discussed in Sect. 5.1.2.

Dynamical Models of Tree Growth We now turn to several sequences of finite rooted random trees that are built recursively by adding at each step new edges on the pre-existing tree. We start with a well known algorithm that Rémy [88] introduced to generate uniform binary trees with n leaves.

Rémy's Algorithm The sequence $(T_n(\mathrm{R}), n \geq 1)$ is constructed recursively as follows:

- Step 1: $T_1(\mathrm{R})$ is the tree with one edge and two vertices: one root, one leaf
- Step n: given $T_{n-1}(\mathrm{R})$, choose uniformly at random one of its edges and graft on "its middle" one new edge-leaf. By this we mean that the selected edge is split into two so as to obtain two edges separated by a new vertex, and then a new edge-leaf is glued on the new vertex. This gives $T_n(\mathrm{R})$.

It turns out (see e.g. [88]) that the tree $T_n(\mathrm{R})$, to which has been subtracted the edge between the root and the first branch point, is distributed as a binary critical Galton–Watson tree conditioned to have $2n - 1$ vertices, or equivalently n leaves (after forgetting the order in the GW-tree). As so, we deduce its asymptotic behavior from that of Galton–Watson trees. However this model can be extended in several directions, most of which are not related to Galton–Watson trees. We detail three of them.

Ford's α-Model [55] Let $\alpha \in [0, 1]$. We construct a sequence $(T_n(\alpha), n \geq 1)$ by modifying Rémy's algorithm as follows:

- Step 1: $T_1(\alpha)$ is the tree with one edge and two vertices: one root, one leaf
- Step n: given $T_{n-1}(\alpha)$, give a weight $1-\alpha$ to each edge connected to a leaf, and α to all other edges (the internal edges). The total weight is $n-1-\alpha$. Now, if $n \neq 2$ or $\alpha \neq 1$, choose an edge at random with a probability proportional to its weight and graft on "its middle" one new edge-leaf. This gives $T_n(\alpha)$. When $n = 2$ and $\alpha = 1$ the total weight is 0 and we decide to graft anyway on the middle of the edge of T_1 one new edge-leaf.

Note that when $\alpha = 1/2$ the weights are the same on all edges and we recover Rémy's algorithm. When $\alpha = 0$, the new edge is always grafted uniformly on an edge-leaf, which gives a tree $T_n(0)$ known as the *Yule tree* with n leaves. When $\alpha = 1$, we obtain a deterministic tree called the *comb tree*. This family of trees indexed by $\alpha \in [0, 1]$ was introduced by Ford [55] in order to interpolate between the Yule, the uniform and the comb models. His goal was to propose new models for phylogenetic trees.

k-Ary Growing Trees [60] This is another extension of Rémy's algorithm, where now several edges are added at each step. Consider an integer $k \geq 2$. The sequence $(T_n(k), n \geq 1)$ is constructed recursively as follows:

- Step 1: $T_1(k)$ is the tree with one edge and two vertices: one root, one leaf
- Step n: given $T_{n-1}(k)$, choose uniformly at random one of its edges and graft on "its middle" $k-1$ new edges-leaf. This gives $T_n(k)$.

When $k = 2$, we recover Rémy's algorithm. For larger k, there is no connection with Galton–Watson trees.

Marginals of Stable Trees: Marchal's Algorithm In [73], Marchal considered the following algorithm, that attributes weights also to the vertices. Fix a parameter $\beta \in (1, 2]$ and construct the sequence $(T_n(\beta), n \geq 1)$ as follows:

- Step 1: $T_1(\beta)$ is the tree with one edge and two vertices: one root, one leaf
- Step n: given $T_{n-1}(\beta)$, attribute the weight
  - $\beta - 1$ on each edge
  - $d - 1 - \beta$ on each vertex of degree $d \geq 3$.

  The total weight is $n\beta - 1$. Then select at random an edge or vertex with a probability proportional to its weight and graft on it a new edge-leaf. This gives $T_n(\beta)$.

The reason why Marchal introduced this algorithm is that $T_n(\beta)$ is actually distributed as the shape of a tree with edge-lengths that is obtained by sampling n leaves at random in the stable Lévy tree with index β. The class of stable Lévy trees plays in important role in the theory of random trees. It is introduced in Sect. 3.2 below.

Note that when $\beta = 2$, vertices of degree 3 are never selected (their weight is 0). So the trees $T_n(\beta), n \geq 1$ are all binary, and we recover Rémy's algorithm.

Of course, several other extensions of trees built by adding edges recursively may be considered, some of which are mentioned in Sects. 5.3.3 and 6.1.

Remark In these dynamical models of tree growth, we build *on a same probability space* the sequence of trees, contrary to the examples of Galton–Watson trees or combinatorial trees that give sequences of *distributions* of trees. In this situation, one may expect to have more than a convergence in distribution for the rescaled sequences of trees. We will see in Sect. 5.3 that it is indeed the case.

2.3 The Markov-Branching Property

Markov-Branching trees were introduced by Aldous [11] as a class of random binary trees for phylogenetic models and later extended to non-binary cases in Broutin et al. [30], and Haas et al. [63]. It turns out that many natural models of sequence of trees satisfy the **Markov-Branching property** (**MB-property** for short), starting with the example of conditioned Galton–Watson trees and most of the examples of the previous section.

Consider

$$\left(T_n, n \geq 1\right)$$

a sequence of trees where T_n is a rooted (unordered, unlabelled) tree with *n leaves*. The MB-property is a property of the sequence of *distributions* of $T_n, n \geq 1$. Informally, the MB-property says that for each tree T_n, *given that*

the root of T_n splits it in p subtrees with respectively $\lambda_1 \geq \ldots \geq \lambda_p$ leaves,

then T_n is distributed as the tree obtained by gluing on a common root p independent trees with respective distributions those of $T_{\lambda_1}, \ldots, T_{\lambda_p}$. The way the leaves are distributed in the sub-trees above the root, in each T_n, for $n \geq 1$, will then allow to fully describe the distributions of the $T_n, n \geq 1$.

We now explain rigorously how to build such sequences of trees. We start with a sequence of probabilities $(q_n, n \geq 1)$, where for each n, q_n is a probability on the set of partitions of the integer n. If $n \geq 2$, this set is defined by

$$\mathcal{P}_n := \left\{ \lambda = (\lambda_1, \ldots, \lambda_p), \lambda_i \in \mathbb{N}, \lambda_1 \geq \ldots \geq \lambda_p \geq 1 : \sum_{i=1}^{p} \lambda_i = n \right\},$$

whereas if $n = 1$, $\mathcal{P}_1 := \{(1), \emptyset\}$ (we need to have a cemetery point). For a partition $\lambda \in \mathcal{P}_n$, we denote by $p(\lambda)$ its length, i.e. the number of terms in the sequence λ. The probability q_n will determine how the n leaves of T_n are distributed into the subtrees above its root. We call such a probability a *splitting distribution*. In order that effective splittings occur, we will always assume that

$$q_n((n)) < 1, \quad \forall n \geq 1.$$

We need to define a notion of *gluing* of trees. Consider $\mathrm{t}_1, \ldots, \mathrm{t}_p$, p discrete rooted (unordered) trees. Informally, we want to glue them on a same common root in order to form a tree $\langle \mathrm{t}_1, \ldots, \mathrm{t}_p \rangle$ whose root splits into the p subtrees $\mathrm{t}_1, \ldots, \mathrm{t}_p$. Formally, this can e.g. be done as follows. Consider first ordered versions of the trees $\mathrm{t}_1^{\mathrm{ord}}, \ldots, \mathrm{t}_p^{\mathrm{ord}}$ seen as subsets of the Ulam–Harris tree $\mathcal{U}$ and then define a new ordered tree by

$$\langle \mathrm{t}_1^{\mathrm{ord}}, \ldots, \mathrm{t}_p^{\mathrm{ord}} \rangle := \{\emptyset\} \cup_{i=1}^{p} i\mathrm{t}_i^{\mathrm{ord}}.$$

The tree $\langle \mathrm{t}_1, \ldots, \mathrm{t}_p \rangle$ is then defined as the unordered version of $\langle \mathrm{t}_1^{\mathrm{ord}}, \ldots, \mathrm{t}_p^{\mathrm{ord}} \rangle$.

Definition 2.1 For each $n \geq 1$, let q_n be a probability on $\mathcal{P}_n$ such that $q_n((n)) < 1$. From the sequence $\mathbf{q} = (q_n, n \geq 1)$ we construct recursively a sequence of distributions $(\mathcal{L}_n^{\mathbf{q}})$ such that for all $n \geq 1$, $\mathcal{L}_n^{\mathbf{q}}$ is carried by the set of rooted trees with n leaves, as follows:

- $\mathcal{L}_1^{\mathbf{q}}$ is the distribution of a line-tree with $G + 1$ vertices and G edges where G is a geometric distribution:

$$\mathbb{P}(G = k) = q_1(\emptyset)(1 - q_1(\emptyset))^k, \quad k \geq 0,$$

- for $n \geq 2$, $\mathcal{L}_n^{\mathbf{q}}$ is the distribution of

$$\langle T_1, \ldots, T_{p(\Lambda)} \rangle$$

where Λ is a partition of n distributed according to q_n, and given Λ, the trees $T_1, \ldots, T_{p(\Lambda)}$ are independent with respective distributions $\mathcal{L}_{\Lambda_1}^{\mathbf{q}}, \ldots, \mathcal{L}_{\Lambda_{p(\Lambda)}}^{\mathbf{q}}$.

A sequence $(T_n, n \geq 1)$ of random rooted trees such that $T_n \sim \mathcal{L}_n^{\mathbf{q}}$ for each $n \in \mathbb{N}$ is called a ***MB-sequence of trees*** **indexed by the leaves**, ***with splitting distributions*** $(\boldsymbol{q_n}, \boldsymbol{n} \geq \mathbf{1})$.

This construction may be re-interpreted as follows: we start from a collection of n *indistinguishable* balls, and with probability $q_n(\lambda_1, \ldots, \lambda_p)$, split the collection into p sub-collections with $\lambda_1, \ldots, \lambda_p$ balls. Note that there is a chance $q_n((n)) < 1$ that the collection remains unchanged during this step of the procedure. Then, re-iterate the splitting operation independently for each sub-collection using this time the probability distributions $q_{\lambda_1}, \ldots, q_{\lambda_p}$. If a sub-collection consists of a single

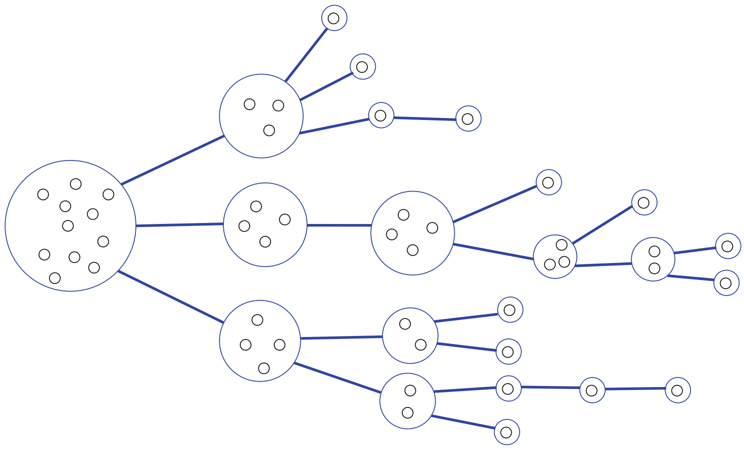

Fig. 1 A sample tree T_{11}. The first splitting arises with probability $q_{11}(4, 4, 3)$

ball, it can remain single with probability $q_1((1))$ or get wiped out with probability $q_1(\emptyset)$. We continue the procedure until all the balls are wiped out. The tree T_n is then the genealogical tree associated with this process: it is rooted at the initial collection of n balls and its n leaves correspond to the n isolated balls just before they are wiped out, See Fig. 1 for an illustration.

We can define similarly MB-sequences of (distributions of) trees indexed by their number of *vertices*. Consider here a sequence $(p_n, n \geq 1)$ such that p_n is a probability on $\mathcal{P}_n$ with no restriction but

$$p_1((1)) = 1.$$

Mimicking the previous balls construction and starting from a collection of n indistinguishable balls, we first remove a ball, split the $n-1$ remaining balls in sub-collections with $\lambda_1, \ldots, \lambda_p$ balls with probability $p_{n-1}((\lambda_1, \ldots, \lambda_p))$, and iterate independently on sub-collections until no ball remains. Formally, this gives:

Definition 2.2 For each $n \geq 1$, let p_n be a probability on $\mathcal{P}_n$, such that $p_1((1)) = 1$. From the sequence $(p_n, n \geq 1)$ we construct recursively a sequence of distributions $(\mathcal{V}_n^{\mathbf{p}})$ such that for all $n \geq 1$, $\mathcal{V}_n^{\mathbf{p}}$ is carried by the set of trees with n vertices, as follows:

- $\mathcal{V}_1^{\mathbf{p}}$ is the deterministic distribution of the tree reduced to one vertex,
- for $n \geq 2$, $\mathcal{V}_n^{\mathbf{p}}$ is the distribution of

$$\langle T_1, \ldots, T_{p(\Lambda)} \rangle$$

where Λ is a partition of $n-1$ distributed according to p_{n-1}, and given Λ, the trees $T_1, \ldots, T_{p(\Lambda)}$ are independent with respective distributions $\mathcal{V}_{\Lambda_1}^{\mathbf{p}}, \ldots, \mathcal{V}_{\Lambda_{p(\Lambda)}}^{\mathbf{p}}$.

A sequence $(T_n, n \geq 1)$ of random rooted trees such that $T_n \sim \mathcal{V}_n^{\mathbf{p}}$ for each $n \in \mathbb{N}$ is called a ***MB-sequence of trees indexed by the vertices, with splitting distributions*** $(\boldsymbol{p_n}, \boldsymbol{n} \geq \mathbf{1})$.

More generally, the MB-property can be extended to sequences of trees $(T_n, n \geq 1)$ with arbitrary degree constraints, i.e. such that for all n, T_n has n vertices in A, where A is a given subset of $\mathbb{Z}_+$. We will not develop this here and refer the interested reader to [89] for more details.

Some Examples

1. **A deterministic example.** Consider the splitting distributions on $\mathcal{P}_n$

$$q_n(\lceil n/2 \rceil, \lfloor n/2 \rfloor) = 1, \quad n \geq 2,$$

as well as $q_1(\emptyset) = 1$. Let $(T_n, n \geq 1)$ the corresponding MB-sequence indexed by leaves. Then T_n is a deterministic discrete binary tree, whose root splits in two subtrees with both $n/2$ leaves when n is even, and respectively $(n+1)/2$, $(n-1)/2$ leaves when n is odd. Clearly, when $n = 2^k$, the height of T_n is exactly k, and more generally for large n, $\mathrm{ht}(T_n) \sim \ln(n)/\ln(2)$.

2. **A basic example.** For $n \geq 2$, let q_n be the probability on $\mathcal{P}_n$ defined by

$$q_n((n)) = 1 - \frac{1}{n^\alpha} \quad \text{and} \quad q_n(\lceil n/2 \rceil, \lfloor n/2 \rfloor) = \frac{1}{n^\alpha} \quad \text{for some } \alpha > 0,$$

and let $q_1(\emptyset) = 1$. Let $(T_n, n \geq 1)$ be an MB-sequence indexed by leaves with splitting distributions (q_n). Then T_n is a discrete tree with vertices with degrees $\in \{1, 2, 3\}$ where the distance between the root and the first branch point (i.e. the first vertex of degree 3) is a Geometric distribution on $\mathbb{Z}_+$ with success parameter $n^{-\alpha}$. The two subtrees above this branch point are independent subtrees, independent of the Geometric r.v. just mentioned, and whose respective distances between the root and first branch point are Geometric distributions with respectively $(\lceil n/2 \rceil)^{-\alpha}$ and $(\lfloor n/2 \rfloor)^{-\alpha}$ parameters. Noticing the weak convergence

$$\frac{\mathrm{Geo}(n^{-\alpha})}{n^\alpha} \xrightarrow[n \to \infty]{(\mathrm{d})} \mathrm{Exp}(1)$$

one may expect that $n^{-\alpha} T_n$ has a limit in distribution. We will later see that it is indeed the case.

3. **Conditioned Galton–Watson trees**. Let $T_n^{\eta,1}$ be a Galton–Watson tree with offspring distribution η, conditioned on having n leaves, for integers n for which this is possible. The branching property is then preserved by conditioning and the sequence $(T_n^{\eta,1}, n : \mathbb{P}(\#_{\mathrm{leaves}} T^\eta) > 0)$ is Markov-Branching, with splitting