Jian-Qiao Sun · Fu-Rui Xiong
Oliver Schütze · Carlos Hernández

Cell Mapping Methods

Algorithmic Approaches and Applications

Springer

Cell Mapping Methods

Jian-Qiao Sun · Fu-Rui Xiong
Oliver Schütze · Carlos Hernández

Cell Mapping Methods

Algorithmic Approaches and Applications

Springer

Jian-Qiao Sun
School of Engineering
University of California, Merced
Merced, CA, USA

Fu-Rui Xiong
Nuclear Power Institute of China
Chengdu, China

Oliver Schütze
Computer Science Department
CINVESTAV-IPN
Mexico City, Mexico

Carlos Hernández
Computer Science Department
CINVESTAV-IPN
Mexico City, Mexico

ISBN 978-981-13-0456-9 ISBN 978-981-13-0457-6 (eBook)
https://doi.org/10.1007/978-981-13-0457-6

Library of Congress Control Number: 2018944348

Printed on acid-free paper

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

We dedicate this book to the creator of the cell mapping methods, the beloved late Prof. C. S. Hsu (May 27, 1922–July 25, 2014).

Moreover, we dedicate this book to our families and parents who have supported us in this luxury endeavor of academic research.

Preface

Recently, the cell mapping methods invented by C. S. Hsu of UC Berkeley in the 1980s have become popular again in the research community. As we enter the age of big data and high power computing, we have found applications outside the traditional area of nonlinear dynamics for the cell mapping methods, and have just realized that much more potential of the methods is waiting for further exploration. Because of these reasons, we are anxious to let the readers know the recent algorithm studies of the cell mapping methods and their applications in various research areas. We hope that the book will help readers to learn the cell mapping methods and to explore untapped applications in science and engineering.

This book consists of two parts. The first part reports the algorithm studies for the cell mapping methods. One of the objectives of the algorithm studies is to push the cell mapping method to the state space with much higher dimensions than two. Parallel computing is a tool for us to reach this objective. Hence, various algorithms discussed in this book have a strong component of parallel implementation. The common tasks in applying the cell mapping methods include computing the mappings as a database, sorting the mapping for solution discovery, subdividing the cells for solution refinement, interpolating the solutions of simple cell mappings, and identifying the normal form of the transition probability matrix for global analysis. The algorithms for all these tasks now have parallel implementation.

The second part of the book presents a series of engineering application of the cell mapping methods. In addition to the global analysis of nonlinear dynamical systems, the cell mapping methods have been applied to search for Pareto optimal solutions of multi-objective optimization problems, designing nonlinear controls, and designing optimal structures for minimum sound radiation and optimal airfoils. All the computations done for the examples reported in the book have made use of desktop computers except for the airfoil design. As we have more access to supercomputing facilities, the scale of engineering applications with the help of the cell mapping methods will undoubtedly grow.

We shall deposit all the codes for the examples and research papers in a server for the readers to download. A users' manual for the codes will be available also. The link to the code depository will be available from Springer after the book is printed.

Merced, USA Jian-Qiao Sun
Chengdu, China Fu-Rui Xiong
Mexico City, Mexico Oliver Schütze
Mexico City, Mexico Carlos Hernández
February 2018

Acknowledgements

The new materials presented in this book were created during the course of research projects funded by various agencies. We deeply appreciate all the supports.

- Two grants from the University of California Institute for Mexico and the United States (UC MEXUS) and the Consejo Nacional de Ciencia y Tecnologa de México (CONACyT) through the projects (a) “Cell-to-cell Mapping for Global Multi-objective Optimization” and (b) “Hybridizing Set Oriented Methods and Evolutionary Strategies to Obtain Fast and Reliable Multi-objective Optimization Algorithms”.
- The Conacyt project (No. 285599) “Toma de decisiones multiobjetivo para sistemas altamente complejos”.
- The graduate student fellowships from the China Scholarship Council (CSC) for sponsoring several exchange Chinese students studying at University of California, Merced.
- The scholarship from CONACyT to support graduate studies at the Computer Science Department of CINVESTAV-IPN in Mexico City.
- Three grants (11172197, 11332008, and 11572215) from the National Science Foundation of China awarded to our Chinese collaborators whose work also contributes to this book.
- A key-project grant (12JCZDJC30400) from the Natural Science Foundation of Tianjin.

We also thank our former graduate students and visiting scholars for their contributions to developing new algorithms for and to new applications of the cell mapping methods. The former graduate students at University of California, Merced, are Dr. Yousef Sardahi and Dr. Yousef Naranjani. The visiting scholars from China are Prof. Ronghua Huan of Zhejiang University, Prof. Longxiang Chen of Shanghai Jiaotong University, and Prof. Yuan Yao of Southwest Jiaotong University. The former graduate student at CINVESTAV-IPN, Jesús Fernández, also made contributions to various algorithms. The graduate students in the Department of Mechanics at Tianjin University in China made significant contributions to the work included in the book. They are Dr. Zhichang Qin, Dr. Xiang Li, and Mr. Mengxin He.

Contents

Acronyms

BOP	Bi-objective optimization problem
CTA	Continuous time approximation
CUDA	Compute unified device architecture
DFS	Depth-first search
DM	Decision-maker
DOF	Degree of freedom
EA	Evolutionary algorithm
FOPTD	First-order plus time delay
FPK	Fokker–Planck–Kolmogorov
GA	Generic algorithm
GCM	Generalized cell mapping
GPU	Graphics processing unit
IAE	Integrated absolute error
ICM	Interpolated cell mapping
IGD	Inverted generational distance
KKT	Karush, Kuhn, and Tucker
LTI	Linear time-invariant
LTV	Linear time-varying
MOEA	Multi-objective evolutionary algorithm
MOP	Multi-objective optimization problem
MOPSO	Multi-objective particle swarm optimization
MP	Mathematical programming
PID	Proportional–integral–derivative
PU	Processing unit
SCC	Strongly connected components
SCM	Simple cell mapping
SIMD	Single instruction multiple data
SOP	Single-objective optimization problem
SPEA	Strength Pareto evolutionary algorithm
STGA	Short-time Gaussian approximation

Part I
Cell Mapping Methods

Chapter 1
Introduction

The cell mapping methods were originated by Hsu in the 1980s for the global analysis of nonlinear dynamical systems that can have multiple steady-state responses including equilibrium states, periodic motions, chaotic attractors as well as domains of attraction of these steady-state responses. The cell mapping methods have been applied to deterministic, stochastic and fuzzy dynamical systems. Two important extensions of the cell mapping methods have been developed to improve the accuracy of the solutions obtained in the cell state space. The first one is the interpolated cell mapping which uses the cell mappings as a foundation or a database to calculate point-wise solutions without further numerical integrations of differential equations. The second one is the subdivision technique of the set-oriented method for improving the accuracy of the invariant solutions obtained with the simple cell mapping method. For a long time, the cell mapping methods have been applied to dynamical systems with low dimensions until now. With the advent of inexpensive computer memories and massively parallel computing technologies such as the graphics processing units (GPUs), global analysis of moderate- to high-dimensional nonlinear dynamical systems becomes feasible.

The cell mapping methods propose to discretize the continuum state space and the time. The discrete space consists of a finite collection of cells. The dynamical systems that originally obey ordinary or partial differential equations are now represented by the mappings in the cell state space, called the cell-to-cell mapping, or cell mapping for short. The cell mappings describe the system evolution over a short time in a finite region of interest in the cell state space. More importantly, long-term system responses such as periodic motion, equilibrium points, limit cycle, chaotic motion, domains of attraction, and stable and unstable manifolds of saddle points can all be obtained from the cell mappings.

The cell mapping methods have been applied to a range of mathematical and engineering problems. From these applications, the cell mapping methods have gained a new life and their computational algorithms have experienced significant advancement. In the following, we shall briefly discuss the recent applications and algorithm advancements of the cell mapping methods. Since the publication of the seminal book on the cell mapping methods by Hsu [1] and a recent edited book on the new

J.-Q. Sun et al., *Cell Mapping Methods*,
https://doi.org/10.1007/978-981-13-0457-6_1

applications and algorithms of the cell mapping method [2], there has not been a book dedicated to the algorithmic studies and applications of the cell mapping methods. This book fills the gap.

The cell mapping methods presented in this book primarily focus on the domains of nonlinear dynamics, root finding of algebraic equations and multi-objective optimization problems (MOPs). Global study in either state space (for nonlinear dynamics) or parameter space (for zero finding and MOPs) is the core task in these domains. Cell mapping offers a powerful tool to carry out such computational intensive global studies.

1.1 Global Analysis of Nonlinear Dynamics

Nonlinear dynamical systems are the underpinning of a variety of disciplines including biology, chemistry, physics, and engineering. The global analysis of nonlinear dynamical systems includes the discovery of steady state solutions, domains of attraction, boundaries of the domains of attractions and unstable solutions.

Hsu studied nonlinear mappings and discovered their rich and complete dynamical behaviors with the simple cell mapping (SCM) method [3]. Such a study would have been a difficult task with point-wise methods. The SCM method was also employed to study multiple challenging engineering applications including the motion analysis of ship [4], impact dynamical analysis with discontinuity [5], global analysis of fault gear system [6], airfoil flutter analysis [7]. Xu and his colleagues have studied the stochastic response and bifurcation of various nonlinear oscillators with the generalized cell mapping (GCM) method [8–10]. The GCM based upon the short term Gaussian solution of the Fokker-Planck-Kolmogorov (FPK) equation was proposed by Sun [11, 12] to accurately compute the probability distribution of nonlinear stochastic systems with much less computational effort. By casting the GCM into the canonical form of a Markov chain represented by the probability matrix, the invariant sets, domains of attraction boundary of domains of attraction and even unstable solutions can all be extracted from the matrix in an unified manner. Similar to the case of stochastic systems, all the information about the global dynamics of nonlinear fuzzy systems is stored in the canonical form of the transition possibility matrix. The master equation that governs the possibility evolution of membership function is represented by a GCM based on Zadeh's extension principle [13, 14]. Interesting phenomena such as crises and chaotic transition have been found in fuzzy nonlinear systems [15]. The bifurcation in fuzzy Duffing and Mathieu systems were studied in [16] by means of the fuzzy generalized cell mapping. A comprehensive review of the global analysis with the cell mapping method in [2] provides rich content on engineering applications and algorithm development.

As an extension of cell mapping, Dellnitz and colleagues introduced the set-oriented method with the subdivision technique that is capable of obtaining the invariant sets of nonlinear dynamical systems with high accuracy [17]. The set-oriented method starts with relatively large cells and removes the cells that don't contain a part of the invariant set by sampling a number of initial conditions from each cell. The subdivision is then applied to the cells that are retained. This is how the set-oriented method gains computational efficiency. There have been publised many studies of the set-oriented method. An adaptive subdivision algorithm was developed [18] that allows the existence of multiple different cell sizes to cover the solution. A study of non-smooth mechanical system was carried out by Neumann et al. with the set-oriented method to find global attractors [19]. The algorithm for extracting unstable manifolds and saddle solutions was introduced in [20]. The set-oriented method is also a robust tool for designing optimal controls [21, 22], especially for multi-objective optimal controls [23, 24]. The set-oriented method with subdivision has not been applied to investigate the transient dynamics of the system such as the domains of attraction and basin boundary. On the other hand, the cell mapping methods were developed for comprehensive global analyses of nonlinear dynamical systems including the discovery of invariant sets and transient dynamics.

The subdivision techniques generate a sequence of smaller and smaller cells, and therefore, the accuracy of the solution for invariant sets improves during the run of the algorithm. At some point, the subdivision has to stop. This is when another important extension of the cell mapping methods comes in: the interpolated cell mapping (ICM) method [25–27]. The ICM uses the simple cell mappings to interpolate the image of a point without integrating the differential equation using this point as an initial condition. The simple cell mappings in the refined cells provide a database for interpolation. The ICM method is able to construct very fine solutions of invariant sets, which assumes that the simple cell mappings are on a sufficiently small grid and that the underlying dynamics of the system is smooth enough for interpolation. The local interpolation error of ICM is of order $O(h^2)$ with the linear interpolation, where h is the cell size, whereas the accuracy of SCM is of order $O(h)$ [28]. More adjacent cells around the point of interest can be used to construct high order interpolations to further improve the accuracy [29]. A modified ICM by introducing the sampling idea of GCM was proposed to further increase the capability of ICM to capture the boundaries of domains of attraction [30]. Several typical nonlinear systems have been studied with the ICM method including the Lorenz system [31], a forced beam with cubic nonlinearity [32] and a spring-pendulum system [28]. If we put the set-oriented method with subdivision and the ICM method in the framework of the cell mapping methods, it becomes apparent that the ICM method represents a post processing step to extract point mappings from the cell mappings on a refined partition of the cell state space.

Both the set-oriented method and ICM method represent efforts to increase the computational efficiency for finding invariant sets of nonlinear dynamical systems so that the cell mapping methods can attack dynamical systems in higher dimensional space. Nevertheless, the curse of dimensionality still prevails. Fortunately, the cell mapping methods treat one-step mapping of each cell as an independent event. The computations of cell mappings as well as other search algorithms over the cell mappings are perfect for parallel computing.

1.2 Zero Finding of Nonlinear Equations

Finding zeros of multi-variable nonlinear functions is a common problem existing in many scientific and engineering fields. In the area of dynamics, finding equilibrium states of nonlinear systems, bifurcation and stability analysis of the system all lead to zero finding of nonlinear functions. In control systems, the stability region in the controller parameter space can also be transformed to a zero finding problem. General zero finding problems can be expressed as $\mathbf{f}(\mathbf{x}) = 0$ with $\mathbf{f} : \mathbf{R}^m \to \mathbf{R}^n$ and $\mathbf{x} \in \mathbf{U} \subset \mathbf{R}^m$ where $\mathbf{U}$ is in a bounded region in $\mathbf{R}^m$. The study in [33] presents an algorithm using the simple cell mapping and generalized cell mapping that can find zeros of multi-variable nonlinear functions in an efficient manner.

Since analytical solutions for zeros of nonlinear functions are in general difficult to obtain, there have been many studies of numerical methods for zero finding. Classical Newton's methods with second-order derivative information have been successfully applied to various problems for a long time [34]. A number of novel variations of Newton's method are popular choices for many applications [35, 36]. Other algorithms are focused on the non-smooth or complex functions where the derivatives are not at hand [37].

To address the problem of finding global solutions in a certain domain, intensive studies have been carried out to take both gradient based or gradient-free algorithms as underlying dynamics and study their long term evolutionary status in parameter space. The homotopy continuation method [38], cell mapping [35] and set-oriented method [39, 40] have been applied by many scholars to attack the problem of global searching. The homotopy continuation method is performed in continuous point-wise parameter space while the latter two methods are performed in discrete cellular space. For problems with moderate to high dimensions, the point-wise methods become less feasible due to the increasing need of computational efforts. The set-oriented method and its predecessor, the cell mapping method, are computationally more effective.

A point-to-point iterative search algorithm—either using gradient information or not—can be viewed as a dynamical system that ideally evolves to the potential solutions. Hence, finding function zeros can be equivalently treated as finding global invariant sets of such iterative dynamical systems. Both the cell mapping and set-oriented methods were originally developed for finding global invariant sets. The set-oriented method has shown great performance with the capability of locating all solutions of nonlinear algebraic equations in both real and complex domains [39, 40].

1.3 Multi-objective Optimization in Control Engineering

Full state feedback control is an important part of the modern control theory. Because of its vast applications in industries, there have been many studies to develop design or tuning techniques of the control. The well-known linear quadratic regulator (LQR) is the most popular optimal controller design in the modern control theory [41]. Since feedback controls are often designed to meet multiple and possibly conflicting performance goals, comprehensive studies are usually carried out to tune control gains in order to achieve the best overall performance [42, 43].

The history of proportional-integral-derivative (PID) control can be traced back to the 1930s [44]. Because of vast applications of PID controls in industries, there have been many studies to develop design or tuning techniques of the control. Two classic designs are a heuristic tuning method due to Ziegler and Nichols [45] and the Smith predictor due to Smith [46]. Because feedback controls inherently are designed to meet multiple and often conflicting performance goals, comprehensive studies are usually carried out to tune control gains in order to achieve best performances [42, 43].

In the last decades, a large number of multi-objective optimal designs of full state feedback controls and PID controls have been proposed. Different from the traditional single objective optimization problems (SOPs), the multi-objective optimization problems (MOPs) no longer have unique solutions consisting of a single point in the design space, but rather a set, called *Pareto set*. The corresponding objective function values form the so-called *Pareto front*.

Multi-objective optimal control design can be carried out in time domain or frequency domain. A time domain approach uses the time domain specifications of the closed-loop response as the objective functions such as overshoot, peak time, settling time and tracking error [47]. On the other hand, a frequency domain design uses phase and gain margins as the objectives, and can consider robustness issues such as model uncertainty, load disturbance and measurement noise. Multi-objective optimization with robustness often involves the optimization among several norms. Vroemen and Jager reviewed the multi-objective design of robust controls for linear systems [48]. They examined different combinations of H_2, H_∞ and L_2 norms to formulate the robust control synthesis problems. A more recent overview by Gambier and Badreddin summarized most available methods for multi-objective optimal control design in both time and frequency domain [49]. They stated that despite the significant development of multi-objective optimization in control engineering, on-line design methods with multi-objective optimization are still at the beginning phase.

Even though there have been many studies of multi-objective optimization control designs for linear systems, only a handful references are available for nonlinear systems, and are scattered in different disciplines. Since the concept of frequency domain in nonlinear systems is not as well studied as in linear systems, the control design for nonlinear systems was usually done in time domain. A nonlinear fuzzy controller based on Pareto rule-based design is carried out by examining the temporal response in [50]. A variable complexity modeling technique with multi-objective optimization design was studied by Silva et al. to tune the multivariable PI control of a nonlinear thermodynamic model in gas turbine [51]. A more theoretical research of multi-objective nonlinear control is presented in reference [52] where the multi-objective optimization algorithm is combined with the classical variational method.

1.4 Methods for Multi-objective Optimization Problems

Many algorithms for obtaining the Pareto set and Pareto front of MOPs have been developed. There are biologically inspired optimization algorithms such as Genetic Algorithm (GA) [53], Ant Colony Optimization [54], Immune Algorithm [55] and Particle Swarm Optimization (PSO) [56]. All these methods have been successfully applied to feedback control designs including PID controls to meet multiple objectives. Fliege and Svaiter have developed several gradient-based algorithms by converting MOP to SOP for point-wise iteration and step length determination of the steepest descend search for MOP solutions [57]. Bosman [58] expands the concept of gradient by introducing novel geometric transformations and combines it with the Genetic Algorithm for MOPs. A gradient-free approach is introduced by Zhong et al. [59] to address MOPs with undifferentiable objective functions. In the work of Custodio et al. [60], methods for pattern searching are adopted to direct gradient-free search.

The Pareto set has been investigated by the set-oriented methods with subdivision techniques [61–63]. The advantage of the set-oriented methods is that they generate an approximation of the global Pareto set in one single run of the algorithm. The cell mapping method in this study is the predecessor of the set-oriented methods, and was proposed by Hsu [1] for global analysis of nonlinear dynamical systems. Two cell mapping methods have been extensively studied, namely, the simple cell mapping (SCM) and the generalized cell mapping (GCM) to study the global dynamics of nonlinear systems [1, 3]. The cell mapping methods have been applied to optimal control problems of deterministic and stochastic dynamic systems [64–66]. Other interesting applications of the cell mapping methods include optimal space craft momentum unloading [67], single and multiple manipulators of robots [68], optimum trajectory planning in robotic systems by [69], tracking control of the read-write head of computer hard disks [70], and airfoil flutter analysis [7]. Sun and his group studied the fixed final state optimal control problems with the simple cell mapping method [71, 72], and applied the cell mapping methods to the optimal control of deterministic systems described by Bellman's principle of optimality [73]. The SCM method can discover the global Pareto fronts with fine structures in a quite effective manner for low and moderate dimensional problems [74, 75].

Sun and his colleagues studied the multi-objective optimal control design using both SCM and GCM [75, 76]. Recent application of parallel computing with cell mapping technique has been reported in [77] where multi-core CPU architecture is used to speed up the global analysis of nonlinear systems.

1.5 Outline of the Book

The book consists of two parts. The first part presents the cell mapping methods and their recent algorithmic developments. The second part introduces the applications of the cell mapping methods.

In Chap. 2, we review the dynamical systems that are appropriate for the cell mapping methods. We briefly discuss equilibrium points, periodic orbits and chaotic motion. In particular, we discuss their geometric features and their representation in the cell mapping methods. Chapter 3 introduces the simple cell mapping (SCM) and its computational algorithms. An n-dimensional interpolation scheme to post-process the invariant sets identified by the SCM method and its error analysis are discussed. Chapter 4 introduces the generalized cell mapping (GCM). Chapter 5 introduces the subdivision techniques. Chapter 6 presents the algorithms for parallel computing of the cell mapping methods. Chapter 7 discusses the hybrid implementation of SCM and GCM.

Chapter 8 starts the application part of the book, and presents a study of finding zeros of nonlinear algebraic equations and computing stability boundaries in a parameter space by using the cell mapping methods. Chapter 9 discusses multi-objective optimization problems (MOPs), their solution properties and algorithms. Applications to multi-objective optimal control designs, vibro-acoustic optimal designs for beam structures and airfoil profiles are presented in Chaps. 10–12. Finally, Chap. 13 presents applications of the cell mapping methods to the global analysis of nonlinear dynamics.

Chapter 2
Dynamical Systems

This chapter describes the classes of dynamical systems that can be studied with the cell mapping methods. We also discuss various features of the responses of dynamical systems and their cell mapping representation in the discrete time-state space.

2.1 Discrete Time Systems

The first class of dynamical systems of interest is described by the point mapping, i.e. the finite difference equation such as

$$\mathbf{x}_{k+1} = \mathbf{G}(\mathbf{x}_k, k), \tag{2.1}$$

where $\mathbf{x}_k \in \mathbf{R}^n$, $k \in \mathbf{N}_+$ is a positive integer, and $\mathbf{G} : \mathbf{R}^n \times \mathbf{N}^+ \longmapsto \mathbf{R}^n$. When the map $\mathbf{G}$ is an explicit function of time k, the discrete time system is non-autonomous. The image of the state $\mathbf{x}_k$ changes with time. Hence, the long term behavior of the system as $k \to \infty$ is more difficult to predict, and may often have to be computed iteratively.

When the map $\mathbf{G}$ is not an explicit function of time k, the discrete time system is autonomous. The image of the state $\mathbf{x}_k$ does not change with time. In this case, the long term behavior of the system as $k \to \infty$ is fully determined by the one step mapping

$$\mathbf{x}_{k+1} = \mathbf{G}(\mathbf{x}_k), \tag{2.2}$$

where $\mathbf{G} : \mathbf{R}^n \longmapsto \mathbf{R}^n$.

In the following we state some basic notations and definitions that will be frequently used throughout this book. For a more thorough discussion we refer e.g. to [1, 78, 79]. For simplicity, we state all notations and definitions for autonomous maps, however, all the statements hold analogously for non-autonomous maps.

The mapping $\mathbf{G}$ applied $k \in \mathbf{N}^+$ times is denoted by $\mathbf{G}^k$, and $\mathbf{G}^0$ denotes the identity mapping. Starting with an initial point $\mathbf{x}_0 \in \mathbf{R}^n$, the iteration (2.2) defines a sequence of points $\{\mathbf{x}_k\}_{k \in \mathbf{N}_0}$ with $\mathbf{x}_k = \mathbf{G}^k(\mathbf{x}_0)$. This sequence is called the *discrete trajectory* or simply *trajectory* of the system with initial solution $\mathbf{x}_0$.

J.-Q. Sun et al., *Cell Mapping Methods*,
https://doi.org/10.1007/978-981-13-0457-6_2

A point $\bar{\mathbf{x}}$ is called a *limit point* of the initial point $\mathbf{x}_0$ under $\mathbf{G}$ if there exists a sequence of integers k_j such that $k_j \to \infty$ and $\mathbf{G}^{k_j}(\mathbf{x}_0) \to \bar{\mathbf{x}}$ as $j \to \infty$. The *limit set* $\Omega(\mathbf{x})$ of $\mathbf{x}_0$ under $\mathbf{G}$ is the set of all limit points of $\mathbf{x}_0$ under $\mathbf{G}$. For a set $\mathbf{D} \subset \mathbf{R}^n$ it is $\mathbf{G}(\mathbf{D}) := \{\mathbf{G}(\mathbf{x}) \ : \ \mathbf{x} \in \mathbf{D}\}$. A set $\mathbf{S}$ is called *positively (negatively) invariant* under $\mathbf{G}$ if $\mathbf{G}(\mathbf{D}) \subset \mathbf{D}$ ($\mathbf{D} \subset \mathbf{G}(\mathbf{D})$). $\mathbf{D}$ is called *invariant* under $\mathbf{G}$ if

$$\mathbf{G}(\mathbf{D}) = \mathbf{D}. \tag{2.3}$$

We say that an invariant set $\mathbf{D}$ is an *attracting set* if there exists a neighborhood $\mathbf{U}$ of $\mathbf{D}$ such that for every open set $\mathbf{V} \supset \mathbf{D}$ there is a $N \in \mathbf{N}^+$ such that $\mathbf{G}^k(\mathbf{U}) \subset \mathbf{V}$ for all $k \geq N$. We stress that for every invariant set also its closure is invariant. Hence, we can restrict the consideration to closed invariant sets $\mathbf{D}$, and in this case we obtain

$$\mathbf{D} = \bigcap_{k \in \mathbf{N}_0} \mathbf{G}^k(\mathbf{U}). \tag{2.4}$$

All points $\mathbf{u} \in \mathbf{U}$ are attracted by $\mathbf{D}$ under iteration of $\mathbf{G}$. Hence, $\mathbf{D}$ is called an *attractor* and $\mathbf{U}$ the *basin of attraction* of $\mathbf{D}$. If $\mathbf{U} = \mathbf{R}^n$, then $\mathbf{D}$ is called the *global attractor* of the dynamical system $\mathbf{G}$. This set is of particular interest as it contains all the potentially interesting dynamics [17].

In the following, we investigate fixed points and periodic points that represent special but very important cases of invariant sets.

Definition 2.1 (*Fixed point, periodic point*)

(a) A point $\mathbf{x}^*$ is called a *fixed point* of $\mathbf{G}$ if $\mathbf{G}(\mathbf{x}^*) = \mathbf{x}^*$.
(b) A point $\mathbf{x}^*$ is called a *periodic point of period* $m \in \mathbf{N}^+$ if $\mathbf{G}^m(\mathbf{x}^*) = \mathbf{G}(\mathbf{x}^*)$. The least integer m for which $\mathbf{G}^m(\mathbf{x}^*) = \mathbf{G}(\mathbf{x}^*)$ is called the prime period of $\mathbf{x}^*$.

If $\mathbf{x}^*$ is a fixed point of period m, then there exist points $\mathbf{x}_1 := \mathbf{x}^*$ and $\mathbf{x}_2, \ldots, \mathbf{x}_m \in \mathbf{R}^n$ such that

$$\begin{aligned} \mathbf{G}(\mathbf{x}_i) &= \mathbf{x}_{i+1}, \quad i = 1, \ldots, m-1, \quad \textit{and} \\ \mathbf{G}(\mathbf{x}_m) &= \mathbf{x}_1. \end{aligned} \tag{2.5}$$

That is, all points $\mathbf{x}_i$, $i = 1, \ldots, m$, are fixed points of period m. The set $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ is called an *m-periodic orbit* of G.

Definition 2.2 (*Attracting and repelling fixed point*). Let $\mathbf{x}^*$ be a fixed point of $\mathbf{G}$.

(a) $\mathbf{x}^*$ is called an *attracting fixed point* of $\mathbf{G}$ if there exists a neighborhood N of $\mathbf{x}^*$ such that for all $\mathbf{x}_0 \in N$ it holds

$$\lim_{k \to \infty} \mathbf{G}^k(\mathbf{x}_0) = \mathbf{x}^*. \tag{2.6}$$

(b) $\mathbf{x}^*$ is called a *repelling fixed point* of $\mathbf{G}$ if there exists a neighborhood N of $\mathbf{x}^*$ such that for all $\mathbf{x}_0 \in N \backslash \{\mathbf{x}^*\}$ there exists an integer k such that $\mathbf{G}^k(\mathbf{x}_0) \notin N$.