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Walter Dittrich

# Reassessing Riemann's Paper On the Number of Primes Less Than a Given Magnitude

 Springer

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# Reassessing Riemann's Paper

On the Number of Primes Less Than a Given  
Magnitude

 Springer

Walter Dittrich  
Institut für Theoretische Physik  
Universität Tübingen  
Tübingen  
Germany

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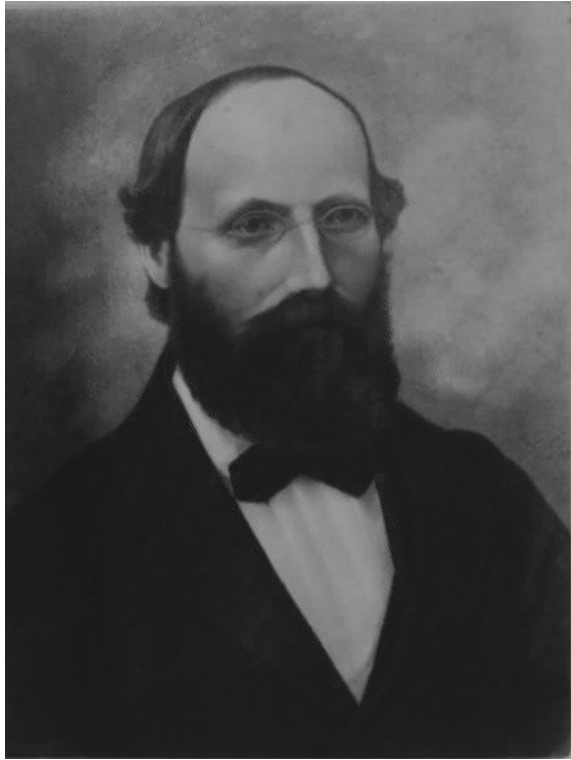
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**Georg Friedrich Bernhard Riemann (1826 – 1866)**

# Preface

This book is devoted to one of the members of the Göttingen triumvirate, Gauß, Dirichlet, and Riemann. It is the latter to whom I wish to pay tribute, and especially to his world-famous article of 1859, which he presented in person at the Berlin Academy upon his election as a corresponding member. His article entitled, “Über die Anzahl der Primzahlen unter einer gegebenen Größe” (“On the Number of Primes Less Than a Given Magnitude”), revolutionized mathematics worldwide. Included in the present book is a detailed analysis of Riemann’s article, including such novel concepts as analytical continuation in the complex plane; the product formula for entire functions; and, last but not least, a detailed study of the zeros of the so-called Riemann zeta function and its close relation to determining the number of primes up to a given magnitude, i.e., an explicit formula for the prime number counting function.

Tübingen, Germany

Walter Dittrich

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