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To our families:
Gizem; Reyhan, and Altay
Shahzad; Dorsa, Parisa, and Golsa
Fiona

Preface

This is a book about a beautiful subject that begins with the topic of Möbius transformations. Indeed, Möbius transformations

$$z \mapsto \frac{az + b}{cz + d}$$

are studied in complex analysis since their mapping properties demonstrate wonderful connections with geometry. These transformations map extended circles to extended circles, enjoy the symmetry principle, come in several types yielding different behavior depending on their fixed point(s), and, through an identification with 2×2 matrices, make connections to group theory and projective geometry. Finite Blaschke products, the focus of this book, are products of certain types of Möbius transformations, the automorphisms of the open unit disk $\mathbb{D}$, namely

$$z \mapsto \xi \frac{w - z}{1 - \overline{w}z},$$

where $|w| < 1$ and $|\xi| = 1$ are fixed. These products have an uncanny way of appearing in many areas of mathematics such as complex analysis, linear algebra, group theory, operator theory, and systems theory. This book covers finite Blaschke products and is designed for advanced undergraduate students, graduate students, and researchers who are familiar with complex analysis but who want to see more of its connections to other fields of mathematics. Much of the material in this book is scattered throughout mathematical history, often only appearing in its original language, and some of it has never seen a modern exposition. We gather up these gems and put them together as a cohesive whole, taking a leisurely pace through the subject and leaving plenty of time for exposition and examples. There are plenty of exercises for the reader who not only wants to appreciate the beauty of the subject but to gain a working knowledge of it as well.

In the early twentieth century, the study of infinite products of the form

$$B(z)=\prod_{k\geqslant 1}\frac{|z_k|}{z_k}\frac{z_k-z}{1-\overline{z_k}z},$$

in which $z_1, z_2, \ldots$ is a sequence in $\mathbb{D}$, was initiated in 1915 by Wilhelm Blaschke (1885–1962). This product converges uniformly on compact subsets of $\mathbb{D}$ if and only if the zero sequence z_k satisfies $\sum_{k\geqslant 1}(1-|z_k|)<\infty$. These *Blaschke products* are analytic on $\mathbb{D}$ and have the additional property that the radial limit $\lim_{r\to 1^-} B(re^{i\theta})$ exists and is of unit modulus for almost every $\theta \in [0, 2\pi)$. In other words, B is an inner function. Blaschke products have been studied intensely since they were first introduced and they appear in many contexts throughout complex analysis and operator theory.

This book is concerned with *finite Blaschke products*, in which the zero sequence $z_1, z_2, \ldots, z_n$ is finite and the product terminates. Although the skeptical reader might think this focus is too narrow, there are many fascinating connections with geometry, complex analysis, and operator theory that demand attention.

There are already some excellent texts that cover infinite Blaschke products and, more generally, inner functions [38, 61]. However, as the reader will see, there are many beautiful theorems involving finite Blaschke products that have no clear analogues in the infinite case. Finite Blaschke products are not often discussed in the standard texts on function spaces or complex variables since the focus there is often on inner functions as part of the broader theory of Hardy spaces. This book focuses on finite Blaschke products and the many results that pertain only to the finite case.

The book begins with an exposition of the *Schur class* $\mathscr{S}$, the set of analytic functions from $\mathbb{D}$ to $\mathbb{D}^-$, the closure of $\mathbb{D}$, and an introduction to hyperbolic geometry. We develop this material from scratch, assuming only that the reader has had a basic course in complex variables. We characterize the finite Blaschke products in several different ways. First, a rational function is a finite Blaschke product if and only if it is of the form

$$\frac{\alpha_0+\alpha_1 z+\cdots+\alpha_n z^n}{\overline{\alpha_n}+\overline{\alpha_{n-1}}z^{n-1}+\cdots+\overline{\alpha_0}z^n},$$

in which the numerator is a polynomial whose n roots lie in $\mathbb{D}$. Second, a finite Blaschke product maps $\mathbb{D}$ onto $\mathbb{D}$ (and the unit circle $\mathbb{T}$ onto itself) precisely n times and a theorem of Fatou confirms that these are the only functions that are continuous on $\mathbb{D}^-$ and analytic on $\mathbb{D}$ with this property. Third, each finite Blaschke product B satisfies

$$\lim_{|z|\to 1^-}|B(z)|=1$$

and another result of Fatou shows that the finite Blaschke products are the only analytic functions on $\mathbb{D}$ that do this. Whether as rational functions whose defining polynomials enjoy certain symmetries, as n-to-1 analytic functions on $\mathbb{D}$, or as analytic functions with unimodular boundary values, the finite Blaschke products distinguish themselves as special elements of the Schur class.

The approximation of a given analytic function by well-understood functions from a fixed class is a standard technique in complex analysis. For example, there are the well-known approximation theorems of Runge, Mergelyan, and Weierstrass. We examine a few results of this type that involve finite Blaschke products. More specifically, a celebrated theorem of Carathéodory ensures that any function in the Schur class $\mathscr{S}$ can be approximated, uniformly on compact subsets of $\mathbb{D}$, by a sequence of finite Blaschke products. In fact, one can even take the approximating Blaschke products to have simple zeros. After Carathéodory's theorem, we discuss Fisher's theorem, which says that any function in $\mathscr{S}$ that extends continuously to $\mathbb{D}^-$ can be approximated uniformly on $\mathbb{D}^-$ by convex combinations of finite Blaschke products. As another example, a theorem of Helson and Sarason states that any continuous function from $\mathbb{T}$ to $\mathbb{T}$ can be uniformly approximated by a sequence of quotients of finite Blaschke products.

One might think there is not much to say about the zeros of a finite Blaschke product. After all, the location of the zeros is part of the definition! However, there are some beautiful gems here. The famed Gauss–Lucas theorem asserts that if P is a polynomial, then the zeros of P', the derivative of P, are contained in the convex hull of the zeros of P. There are theorems that say that the zeros of a finite Blaschke product B are contained in the convex hull of the solutions to the equation $B(z) = 1$ (or indeed the solutions to $B(z) = e^{i\theta}$ for any $\theta \in [0, 2\pi)$). Moreover, the hyperbolic analogue of the Gauss–Lucas theorem says that the zeros of B' (the critical points of B) are contained in the hyperbolic convex hull of the zeros of B. For Blaschke products of low degree, these results are even more explicit and can be stated in terms of classical geometry involving ellipses. There is also a result of Heins which says that one can create a finite Blaschke product with any desired set of critical points in $\mathbb{D}$. Finally, for analytic functions on $\mathbb{D}^-$, one can state, in terms of finite Blaschke products, a curious converse (the Challener–Rubel theorem) to Rouché's theorem.

Interpolation is another important topic in complex analysis. The most basic result in this direction is the Lagrange interpolation theorem, which guarantees that for distinct $z_1, z_2, \ldots, z_n$ and any $w_1, w_2, \ldots, w_n$ there is a polynomial P for which $P(z_j) = w_j$ for all j. The connection finite Blaschke products make with interpolation comes from Pick's theorem: given distinct $z_1, z_2, \ldots, z_n \in \mathbb{D}$ and any $w_1, w_2, \ldots, w_n \in \mathbb{D}$, then there is an $f \in \mathscr{S}$ for which $f(z_j) = w_j$ for all j if and only if the *Pick matrix*

$$\left[\frac{1 - \overline{w_j} w_i}{1 - \overline{z_j} z_i}\right]_{1 \leqslant i, j \leqslant n}$$

is positive semidefinite. Furthermore, when the interpolation is possible, it can be done with a finite Blaschke product. A more involved boundary interpolation result is the Cantor–Phelps theorem (for which we provide two distinct proofs, one abstract and another constructive), which says that given distinct $\zeta_1, \zeta_2, \ldots, \zeta_n \in \mathbb{T}$ and any $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{T}$ there is a finite Blaschke product B with $B(\zeta_j) = \xi_j$ for all j.

So far we have discussed finite Blaschke product themselves and their connections to well-studied topics in complex analysis (zeros, critical points, residues, valence, approximation, and interpolation). However, as mentioned earlier, finite Blaschke products appear in many other places.

For example, Bohr's inequality asserts that if $f = \sum_{n \geqslant 0} a_n z^n \in \mathscr{S}$, then

$$\sum_{n \geqslant 0} |a_n| r^n \leqslant 1, \qquad r \in [0, \tfrac{1}{3}].$$

The number $\frac{1}{3}$ is optimal and is called the *Bohr radius* for the Schur class. Using finite Blaschke products, we explore a Bohr-type inequality for subclasses of Schur functions that vanish at certain points of $\mathbb{D}$ and for the Schur class functions whose first several derivatives vanish at zero. It turns out that the extremal functions for these extended Bohr problems are finite Blaschke products.

Next we cover two connections finite Blaschke products make with group theory. For a fixed finite Blaschke product B, consider the set G_B of continuous functions $u : \mathbb{T} \to \mathbb{T}$ for which $B \circ u = B$. One can see that G_B is a semigroup under function composition. A theorem of Chalendar and Cassier reveals that G_B is a cyclic group and that one can identify a generator by considering the previously mentioned n-to-1 mapping properties of B on $\mathbb{T}$. We also cover, via Cowen's unpublished exposition, an old theorem of Ritt that examines when we can write B as a composition $B = C \circ D$, in which C and D are finite Blaschke products. The answer is in terms of the monodromy group of B^{-1}. We also give an equivalent formulation of Ritt's theorem in terms of certain subgroups of G_B.

Finite Blaschke products also make connections to operator theory. For example, if T is a contraction on a Hilbert space and B is a finite Blaschke product with n zeros, then $B(T)$ is also a contraction. Moreover, a theorem of Gau and Wu says that $\|B(T)\| = 1$ if and only if $\|T^n\| = 1$. Another connection is with the numerical range of an operator. The spectral mapping theorem says that $\sigma(p(T)) = p(\sigma(T))$, in which $\sigma(T)$ is the spectrum of a bounded Hilbert space operator T and p is a polynomial. One may wonder whether or not a similar identity $W(p(T)) = p(W(T))$ holds for the *numerical range*

$$W(T) = \{\langle T\mathbf{x}, \mathbf{x}\rangle : \|\mathbf{x}\| = 1\}.$$

Although the desired identity is not true in general, there are some suitable substitutes. In fact, Halmos asked whether or not $W(T) \subseteq \mathbb{D}^-$ implies that $W(T^n) \subseteq \mathbb{D}^-$ for every $n \geqslant 1$. Progress was made when it was shown that if $W(T) \subseteq \mathbb{D}^-$ and B is a finite Blaschke product with $B(0) = 0$, then $W(B(T)) \subseteq \mathbb{D}^-$. A theorem of Berger and Stampfli extends this result from finite Blaschke products that vanish at the origin to the Schur functions that are continuous on $\mathbb{D}^-$ and vanish at the origin. However, without the condition $f(0) = 0$, there are contractions T with $W(T) \subseteq \mathbb{D}^-$ for which $W(f(T))$ intersects the complement of $\mathbb{D}^-$. A suitable replacement here is a theorem of Drury which says that though $W(f(T))$ may intersect the complement of $\mathbb{D}^-$, it is contained in a certain "teardrop" region, a slight "bulge" of $\mathbb{D}$. Moreover, the use of finite Blaschke products indicates the sharpness of Drury's theorem.

Still another connection to finite Blaschke products comes with models of linear transformations. In linear algebra, or more broadly in operator theory, one often wants to create a model for certain types of linear transformations. For example, there is the classical spectral theorem from linear algebra which says that any normal matrix is unitarily equivalent to a diagonal matrix. One can show that any contractive matrix T with $\operatorname{rank}(I - T^*T) = 1$ and whose eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are contained in $\mathbb{D}$ is unitarily equivalent to the compression of the shift operator $f \mapsto zf$ on the Hardy space H^2 to the model space

$$\operatorname{span}\left\{\frac{1}{1 - \overline{\lambda_j} z} : 1 \leqslant j \leqslant n\right\}.$$

Along with this result, one obtains a function-theoretic characterization of the invariant subspaces of these operators as well. In fact, this model space is the vector space of rational functions f with no poles in $\mathbb{D}^-$ for which

$$\int_0^{2\pi} f(e^{i\theta})\overline{B(e^{i\theta})}e^{-in\theta}\frac{d\theta}{2\pi} = 0, \qquad n \geqslant 0,$$

in which B is the finite Blaschke product whose zeros are the eigenvalues λ_j. The finite-dimensional approach undertaken in this book is intuitive and prepares interested readers for the more advanced text [59].

Finite Blaschke products can also be used to explore rational functions f that are analytic on $\mathbb{D}$ and for which $f(e^{i\theta})$ is an extended real number for all $\theta \in [0, 2\pi]$. These functions are sometimes called the real rational functions. Examples include

$$f(z) = i\frac{1+z}{1-z},$$

and, more generally,

$$f = i\frac{B_1 + B_2}{B_1 - B_2},$$

in which B_1 and B_2 are finite Blaschke products such that $B_1 - B_2$ has no zeros on $\mathbb{D}$. In fact, a theorem of Helson says these are all of the real rational functions. We will discuss various properties of real rational functions such as a characterization of those that are zero free on $\mathbb{D}$, the valence of these functions, as well as a factorization of a real rational function f as $f = FG$, where F and G are real rational functions, F has the same zeros of f, and G is zero free.

Finally, there is the connection Blaschke products make with the Darlington synthesis problem from electrical engineering. Here, in its simplest realization, one is given a rational function a with no poles in $\mathbb{D}^-$ and one needs to find rational functions b, c, d on $\mathbb{D}$ with no poles in $\mathbb{D}^-$ so that the matrix-valued analytic function

$$M(z) = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}$$

is such that $M(e^{i\theta})$ is a unitary matrix for every $\theta \in [0, 2\pi)$. The determinant of such a matrix M is a finite Blaschke product B and the model space associated with B determines the structure of and relations between the unknown functions b, c, d. Most curiously, we see that every rational matrix inner function $M(z)$ enjoys a peculiar quaternionic structure.

This book is mostly self-contained and should be accessible to a student with a background in basic real and complex analysis along with linear algebra. The proofs are detailed and dozens of illustrations are provided. We thank Zach Glassman for his assistance with `Tikz` and for producing many of our illustrations. At the end of each chapter, we include exercises so that the reader can gain greater technical fluency with the material. An appendix contains some background information about operator theory and function spaces that is relevant for a few results in the later chapters.

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Contents

Notation

$\delta_{j,k}$	Kronecker delta
$\mathbb{N} := \{1, 2, \ldots\}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{C}^n$	Complex n-space
A^*	Conjugate transpose of a matrix A
$\operatorname{Re} z$	Real part of a complex number z
$\operatorname{Im} z$	Imaginary part of a complex number z
$\equiv$	Identically equals
E^-	Closure of E
∂E	Boundary of E
$\lvert E \rvert$	Cardinality of a set E
$\operatorname{Res}(f, z_0)$	Residue of f at z_0
$\operatorname{diag}(z_1, \ldots, z_n)$	$n \times n$ diagonal matrix with $z_1, \ldots, z_n$ on the diagonal
$\mathbb{D} := \{z \in \mathbb{C} : \lvert z \rvert < 1\}$	Open unit disk (p. 1)
$\mathbb{D}^- := \{z \in \mathbb{C} : \lvert z \rvert \leqslant 1\}$	Closed unit disk (p. 1)
$\mathbb{T} := \{z \in \mathbb{C} : \lvert z \rvert = 1\}$	Unit circle (p. 1)
$\mathscr{S}$	Schur class (p. 1)
$\operatorname{Aut}(\mathbb{D})$	The automorphism group of $\mathbb{D}$ (p. 2)
id	Identity function (p. 2)
$\rho_\gamma(z) = \gamma z$	A rotation by $\arg \gamma$ (p. 2)
$\tau_w(z) = (w - z)/(1 - \overline{w}z)$	A special disk automorphism (p. 2)
$\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$	Riemann sphere (p. 7)
$S_C(\alpha)$	Stolz domain anchored at α with constant C (p. 12)
$\angle \lim_{z \to \alpha} f(z)$	Nontangential limit (p. 12)
$\mathbb{D}_- := \mathbb{D} \cap \{z : \operatorname{Im} z < 0\}$	Lower half of unit disk (p. 16)
$\mathbb{D}_+ := \mathbb{D} \cap \{z : \operatorname{Im} z > 0\}$	Upper half of unit disk (p. 16)
$\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$	Upper half plane (p. 17)
$\mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$	Lower half plane (p. 17)

d	Euclidean metric on $\mathbb{C}$ (p. 21)
ϱ	Pseudohyperbolic metric (p. 21)
$\varDelta(z, \rho)$	Pseudohyperbolic disk of radius ρ centered at z (p. 22)
$D(c, r)$	Euclidean disk of radius r centered at c (p. 22)
$\wp$	Poincaré metric (p. 27)
$\ell(\varGamma)$	Hyperbolic length of a curve $\varGamma$ (p. 27)
d_μ	Metric induced by μ (p. 31)
$\triangle$	Laplace operator (p. 32)
κ_μ	Curvature of the metric μ (p. 32)
$\mathbb{D}_e = \{z \in \mathbb{C} : \lvert z\rvert > 1\} \cup \{\infty\}$	The complement of $\mathbb{D}^-$ in $\widehat{\mathbb{C}}$ (p. 39)
deg	Degree of a rational functional (p. 43)
$P^\#(z) := z^n \overline{P(1/\overline{z})}$	In which P is a polynomial of degree n (p. 44)
$P^{\#_n}$	$P^\#$ when n needs to be specified (p. 44)
$\mathscr{A}(\mathbb{D})$	The disk algebra (p. 49)
H^∞	The bounded analytic functions on $\mathbb{D}$ (p. 59)
$\lVert \cdot \rVert_\infty$	Norm on H^∞ (p. 59)
conv	Convex hull (p. 61)
$Z_f(\varGamma)$	Number of zeros of f inside a curve $\varGamma$ (p. 69)
$P_f(\varGamma)$	Number of poles of f inside a curve $\varGamma$ (p. 69)
$\beta(a, z)$	(p. 109)
$\mathscr{B}_d$	(p. 110)
$\varSigma_d$	(p. 112)
$\varPhi$	(p. 114)
R_f	The distance-ratio function for f (p. 116)
$\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$	Extended real line (p. 131)
M_n	the set of $n \times n$ complex matrices (p. 135)
$\mathfrak{B}_0$	Bohr radius (p. 156)
$m(f, r)$	(p. 156)
$M(\mathscr{F}, r)$	(p. 157)
$\mathfrak{B}_k$	The kth Bohr radius (p. 163)
$\mathfrak{B}_k(\lambda)$	A generalized Bohr radius (p. 165)
$\mathscr{S}_\lambda$	(p. 165)
G_B	(p. 183)
$\mathscr{S}_B$	Critical values of a Blaschke product (p. 183)
$\widetilde{B}$	Normalized form of a finite Blaschke product B (p. 186)
$\mathscr{G}_B$	Monodromy group for a Blaschke product (p. 191)
S_n	Group of permutations of $\{1, 2, \ldots, n\}$ (p. 192)
Hol$(\mathbb{D}^-)$	The set of functions analytic on a neighborhood of $\mathbb{D}^-$ (p. 215)
$\sigma(T)$	Spectrum of an operator T (p. 219)
$W(T)$	Numerical range of an operator T (p. 219)
$w(T)$	Numerical radius of an operator T (p. 220)
td(α)	Drury's teardrop region (p. 235)

$\mathfrak{R}$	The set rational of functions with real boundary values (p. 245)
$\mathfrak{R}^+$	The set of $\mathfrak{R}$ functions which are analytic on $\mathbb{D}$ (p. 245)
$\mathscr{K}_B$	Model space associated with a Blaschke product B (p. 261)
$b_w = (z - w)/(1 - \overline{w}z)$	a special disk automorphism (p. 267)
$k_\lambda(z)$	Reproducing kernel for $\mathscr{K}_B$ (p. 268)
$c_\lambda(z)$	Cauchy kernel (p. 268)
$\widetilde{c}_\lambda(z)$	Normalized Cauchy kernel (p. 268)
S_B	Compressed shift (p. 276)
i_F	Finite Blaschke product associated with F (p. 294)
$\lVert T \rVert$	Operator norm of an operator T (p. 313)
$\oplus$	Orthogonal direct sum (p. 314)
T_ϕ	Toeplitz operator with symbol ϕ (p. 316)

Chapter 1
Geometry of the Schur Class

This chapter will cover some basic facts about the Schur class. In what follows,

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \quad \mathbb{D}^- = \{z \in \mathbb{C} : |z| \leqslant 1\}, \quad \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}.$$

Definition 1.0.1 The *Schur class* $\mathscr{S}$ is

$$\mathscr{S} := \{f : \mathbb{D} \to \mathbb{D}^- : \ f \text{ is analytic}\}. \tag{1.0.2}$$

The Maximum Modulus Principle ensures that $f(z) \in \mathbb{T}$ for some $z \in \mathbb{D}$ if and only if f is a constant function of unit modulus. Thus, $\mathscr{S}$ consists of the nonconstant analytic functions $f : \mathbb{D} \to \mathbb{D}$ along with the constant functions with values in $\mathbb{D}^-$.

1.1 The Schwarz Lemma

The Schwarz Lemma is one of the cornerstones of complex analysis. Despite its deceptive simplicity, it has many profound consequences [31]. Schwarz proved this lemma for injective functions. Carathéodory proved the general version.

Lemma 1.1.1 (Schwarz [125]) *If* $f \in \mathscr{S}$ *and* $f(0) = 0$*, then*

(a) $|f(z)| \leqslant |z|$ *for all* $z \in \mathbb{D}$*, and*
(b) $|f'(0)| \leqslant 1$.

Moreover, if $|f(w)| = |w|$ *for some* $w \in \mathbb{D}\backslash\{0\}$ *or if* $|f'(0)| = 1$*, then there is a* $\zeta \in \mathbb{T}$ *so that* $f(z) = \zeta z$ *for all* $z \in \mathbb{D}$.

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Proof (Carathéodory [15]) Define $g : \mathbb{D} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & \text{if } z \neq 0, \\ f'(0) & \text{if } z = 0, \end{cases}$$

and observe that g is analytic on $\mathbb{D}\backslash\{0\}$. The singularity at 0 is removable since

$$\lim_{z \to 0} g(z) = f'(0)$$

and hence g is analytic on all of $\mathbb{D}$. For $r \in [0, 1)$, an application of the Maximum Modulus Principle to the disk $|z| \leqslant r$ yields a $\zeta \in \mathbb{T}$ so that

$$|g(rz)| \leqslant |g(r\zeta)| = \frac{|f(r\zeta)|}{|r\zeta|} \leqslant \frac{1}{r}, \quad z \in \mathbb{D}.$$

Now let $r \to 1^-$ to obtain statements (a) and (b).

Suppose that $|f(w)| = |w|$ for some $w \in \mathbb{D}\backslash\{0\}$ or that $|f'(0)| = 1$. Then $|g(w)| = 1$ for some $w \in \mathbb{D}$. Since $|g| \leqslant 1$ on $\mathbb{D}$, the Maximum Modulus Principle provides a $\zeta \in \mathbb{T}$ such that $g(z) = \zeta$ for all $z \in \mathbb{D}$. Thus, $f(z) = \zeta z$ for all $z \in \mathbb{D}$. □

1.2 Automorphisms of the Disk

Definition 1.2.1 A bijective analytic function $f : \mathbb{D} \to \mathbb{D}$ is an *automorphism* of $\mathbb{D}$.

Since most of our work concerns the unit disk $\mathbb{D}$, we often say "f is an automorphism" without explicit reference to $\mathbb{D}$. The set of all automorphisms of $\mathbb{D}$, denoted by $\operatorname{Aut}(\mathbb{D})$, is a subset of the Schur class $\mathscr{S}$.

If f is an automorphism, then the inverse bijection $f^{-1} : \mathbb{D} \to \mathbb{D}$ is analytic and hence f^{-1} is also an automorphism. The *identity function* $\operatorname{id} : \mathbb{D} \to \mathbb{D}$ defined by

$$\operatorname{id}(z) = z$$

is an automorphism satisfying $f \circ f^{-1} = f^{-1} \circ f = \operatorname{id}$ for every $f \in \operatorname{Aut}(\mathbb{D})$. Since the composition of two automorphisms is also an automorphism, and since function composition is an associative operation, $\operatorname{Aut}(\mathbb{D})$ is a group under function composition.

We now focus on two special automorphisms. For $w \in \mathbb{D}$ and $\gamma \in \mathbb{T}$, define $\rho_\gamma : \mathbb{D} \to \mathbb{C}$ and $\tau_w : \mathbb{D} \to \mathbb{C}$ by

$$\rho_\gamma(z) = \gamma z \qquad \text{and} \qquad \tau_w(z) = \frac{w - z}{1 - \overline{w}z}. \tag{1.2.2}$$

Since $|\gamma| = 1$, we see that ρ_γ induces a rotation of $\mathbb{D}$ about the origin through an angle of $\arg \gamma$. Consequently, $\rho_\gamma \in \operatorname{Aut}(\mathbb{D})$. Moreover,

$$\rho_{\gamma_1} \circ \rho_{\gamma_2} = \rho_{\gamma_1 \gamma_2} \qquad \text{and} \qquad \rho_\gamma \circ \rho_{\overline{\gamma}} = \mathrm{id}. \tag{1.2.3}$$

The function τ_w is also an automorphism of $\mathbb{D}$, although to establish this requires a little more work. First, a computation confirms that

$$\tau_w \circ \tau_w = \mathrm{id}, \tag{1.2.4}$$

so τ_w is injective on $\mathbb{D}$ and the range of τ_w contains $\mathbb{D}$. To show that the range of τ_w is precisely $\mathbb{D}$, observe that for each $\zeta \in \mathbb{T}$ and $w \in \mathbb{D}$,

$$|\tau_w(\zeta)| = \left| \frac{w - \zeta}{1 - \overline{w}\zeta} \right| = \frac{|w - \zeta|}{|\overline{w} - \overline{\zeta}|} = 1$$

since $\zeta\overline{\zeta} = |\zeta|^2 = 1$. Since the Maximum Modulus Principle implies that

$$|\tau_w(z)| < 1, \qquad z \in \mathbb{D},$$

it follows that $\tau_w \in \operatorname{Aut}(\mathbb{D})$. Therefore, by the discussion above,

$$\{\rho_\gamma \circ \tau_w : \gamma \in \mathbb{T}, w \in \mathbb{D}\} \subseteq \operatorname{Aut}(\mathbb{D}).$$

The following theorem establishes that the preceding containment is an equality.

Theorem 1.2.5 *If $f \in \operatorname{Aut}(\mathbb{D})$, then there are unique $w \in \mathbb{D}$ and $\gamma \in \mathbb{T}$ such that $f = \rho_\gamma \circ \tau_w$. In other words,*

$$\operatorname{Aut}(\mathbb{D}) = \{\rho_\gamma \circ \tau_w : \gamma \in \mathbb{T},\ w \in \mathbb{D}\}.$$

Proof If $f \in \operatorname{Aut}(\mathbb{D})$, then there is a unique $w \in \mathbb{D}$ so that $f(w) = 0$. Then $g = f \circ \tau_w \in \operatorname{Aut}(\mathbb{D})$ and $g(0) = 0$. Hence the Schwarz Lemma (Lemma 1.1.1) ensures that

$$|g(z)| \leqslant |z|, \qquad z \in \mathbb{D}.$$

Since $g^{-1} \in \operatorname{Aut}(\mathbb{D})$ and $g^{-1}(0) = 0$, the same argument yields

$$|g^{-1}(z)| \leqslant |z|, \qquad z \in \mathbb{D}.$$

Since $g(z) \in \mathbb{D}$, we may substitute $g(z)$ in place of z in the previous inequality and obtain $|z| \leqslant |g(z)|$ for all $z \in \mathbb{D}$. Consequently,

$$|g(z)| = |z|, \qquad z \in \mathbb{D},$$

and hence another application of the Schwarz Lemma yields a unique unimodular constant γ such that $g(z) = \gamma z$. Thus, $f(\tau_w(z)) = \gamma z$ for all $z \in \mathbb{D}$. Now substitute z in place of $\tau_w(z)$ in the preceding identity and use (1.2.4) to obtain $f = \gamma \tau_w = \rho_\gamma \circ \tau_w$.

We now verify the uniqueness of the parameters γ and w in the representation $\rho_\gamma \circ \tau_w$ of a typical element of $\operatorname{Aut}(\mathbb{D})$. Suppose that

$$\rho_\gamma \circ \tau_w = \rho_{\gamma'} \circ \tau_{w'}$$

for some $\gamma, \gamma' \in \mathbb{T}$ and $w, w' \in \mathbb{D}$. Then (1.2.3) and (1.2.4) yield

$$\rho_{\gamma \overline{\gamma'}} = \tau_{w'} \circ \tau_w.$$

Evaluate the preceding identity at $z = 0$ to obtain $\tau_{w'}(w) = 0$ and so $w = w'$. Hence $\rho_{\gamma \overline{\gamma'}} = \mathrm{id}$ and thus $\gamma = \gamma'$. □

Since $\tau_0 = -\mathrm{id}$ and $\rho_1 = \mathrm{id}$, the unique representations of τ_w and ρ_γ afforded by Theorem 1.2.5 are

$$\tau_w = \rho_1 \circ \tau_w$$

and

$$\rho_\gamma = \rho_{-\gamma} \circ \tau_0. \tag{1.2.6}$$

It is also worth noting that if $f \in \operatorname{Aut}(\mathbb{D})$ and $f(0) = 0$, then $f = \rho_\gamma$ for some $\gamma \in \mathbb{T}$; that is, the only automorphisms of $\mathbb{D}$ that fix the origin are the rotations.

1.3 Algebraic Structure of Aut($\mathbb{D}$)

If $f = \rho_{\gamma_1} \circ \tau_{w_1}$ and $g = \rho_{\gamma_2} \circ \tau_{w_2}$ are automorphisms of $\mathbb{D}$, then Theorem 1.2.5 implies that $f \circ g = \rho_\gamma \circ \tau_w$ for some unique $\gamma \in \mathbb{T}$ and $w \in \mathbb{D}$. Since we often require concrete formulas that are applicable to problems in function theory, our primary goal in this section is to obtain expressions for γ and w in terms of the parameters γ_1, γ_2, w_1, and w_2. At the end of this section, however, we will briefly describe a more group-theoretic approach to $\operatorname{Aut}(\mathbb{D})$.

Lemma 1.3.1 *If $f = \rho_\gamma \circ \tau_w$, then $w = f^{-1}(0)$ and*

$$\gamma = \begin{cases} f(0)/f^{-1}(0) & \text{if } f(0) \neq 0, \\ -f'(0) & \text{if } f(0) = 0. \end{cases}$$

Proof Since

$$f(w) = \rho_\gamma(\tau_w(w)) = \rho_\gamma(0) = 0$$

and f is invertible, we conclude that $w = f^{-1}(0)$. Moreover,

$$f(0) = \rho_\gamma(\tau_w(0)) = \rho_\gamma(w) = \gamma w = \gamma f^{-1}(0),$$

which yields the desired formula when $f(0) \neq 0$. When $f(0) = 0$, we get

$$w = f^{-1}(0) = 0$$

and hence

$$f(z) = \rho_\gamma(\tau_0(z)) = \rho_\gamma(-z) = -\gamma z.$$

Thus, $\gamma = -f'(0)$ as claimed. □

The discussion below requires the following derivative formula:

$$\tau_w'(z) = -\frac{1 - |w|^2}{(1 - \overline{w}z)^2}.$$

Let $z = 0$ and $z = w$, respectively, in the preceding and obtain

$$\tau_w'(0) = -(1 - |w|^2) \tag{1.3.2}$$

and

$$\tau_w'(w) = -\frac{1}{1 - |w|^2}. \tag{1.3.3}$$

The following theorem provides an explicit realization of the group operation on Aut($\mathbb{D}$). It also yields several formulas that are needed later on.

Theorem 1.3.4 *If $\gamma_1, \gamma_2 \in \mathbb{T}$ and $w_1, w_2 \in \mathbb{D}$, then*

$$\left(\rho_{\gamma_1} \circ \tau_{w_1}\right) \circ \left(\rho_{\gamma_2} \circ \tau_{w_2}\right) = \rho_\gamma \circ \tau_w,$$

where

$$w = \tau_{w_2}(\overline{\gamma_2} w_1)$$

and

$$\gamma = \begin{cases} \gamma_1 \tau_{w_1 \overline{w_2}}(\gamma_2) & \text{if } w_2 \neq \overline{\gamma_2} w_1, \\ -\gamma_1 \gamma_2 & \text{if } w_2 = \overline{\gamma_2} w_1. \end{cases}$$

In particular, if $w_2 = \overline{\gamma_2} w_1$, then

$$\left(\rho_{\gamma_1} \circ \tau_{w_1}\right) \circ \left(\rho_{\gamma_2} \circ \tau_{w_2}\right) = \rho_{\gamma_1 \gamma_2}.$$

Proof Let $f = (\rho_{\gamma_1} \circ \tau_{w_1}) \circ (\rho_{\gamma_2} \circ \tau_{w_2})$. Lemma 1.3.1 says that w is the unique solution to the equation

$$f(w) = \left[(\rho_{\gamma_1} \circ \tau_{w_1}) \circ (\rho_{\gamma_2} \circ \tau_{w_2})\right](w) = 0.$$

Since $\rho_{\gamma_1}(0) = 0$, we see that

$$\left[\tau_{w_1} \circ (\rho_{\gamma_2} \circ \tau_{w_2})\right](w) = 0$$

and hence

$$\left(\rho_{\gamma_2} \circ \tau_{w_2}\right)(w) = \tau_{w_1}(0) = w_1$$

by (1.2.4). An application of (1.2.3) yields

$$\tau_{w_2}(w) = \rho_{\overline{\gamma_2}}(w_1) = \overline{\gamma_2} w_1, \tag{1.3.5}$$

after which another appeal to (1.2.4) provides the desired formula for w. Now observe that the preceding formula yields

$$w = 0 \quad \Longleftrightarrow \quad w_2 = \overline{\gamma_2} w_1.$$

Since $w = f^{-1}(0)$, the second formula in Lemma 1.3.1 asserts that $\gamma = f(0)/w$ when $w \neq 0$. The computation

$$\begin{aligned} f(0) &= \left[(\rho_{\gamma_1} \circ \tau_{w_1}\right) \circ \left(\rho_{\gamma_2} \circ \tau_{w_2})\right](0) \\ &= \gamma_1 \tau_{w_1}\left(\gamma_2 \tau_{w_2}(0)\right) \\ &= \gamma_1 \tau_{w_1}(\gamma_2 w_2) \end{aligned}$$

and (1.3.5) reveal that

$$\gamma = \frac{f(0)}{w} = \frac{\gamma_1 \tau_{w_1}\left(\gamma_2 w_2\right)}{\tau_{w_2}(\overline{\gamma_2} w_1)} = \gamma_1 \tau_{w_1 \overline{w_2}}(\gamma_2).$$

The final equality in the statement of the theorem is verified by direct computation.

If $w = 0$, then we need to evaluate $f'(0)$. By the chain rule and (1.3.2),

$$\begin{aligned} f'(0) &= \gamma_1 \tau'_{w_1}[(\rho_{\gamma_2} \circ \tau_{w_2})(0)] \times \gamma_2 \tau'_{w_2}(0) \\ &= -\gamma_1 \tau'_{w_1}(\gamma_2 w_2) \times \gamma_2(1 - |w_2|^2) \\ &= -\gamma_1 \gamma_2. \end{aligned}$$

□

Corollary 1.3.6 *If* $w_1, w_2 \in \mathbb{D}$ *and* $w_1 \neq w_2$, *then*

$$\tau_{w_1} \circ \tau_{w_2} = \rho_\gamma \circ \tau_w,$$

where

$$w = \tau_{w_2}(w_1) = \frac{w_2 - w_1}{1 - \overline{w_2} w_1} \quad \textit{and} \quad \gamma = \tau_{w_1 \overline{w_2}}(1) = -\frac{1 - w_1 \overline{w_2}}{1 - \overline{w_1} w_2}.$$

For the following result, let $\gamma_1 = 1$ and $w_2 = 0$, then replace γ_2 by $-\gamma$ and w_1 by w in Theorem 1.3.4. However, we admit that direct verification might be easier; see Exercise 1.1.

Corollary 1.3.7 *If $w \in \mathbb{D}$ and $\gamma \in \mathbb{T}$, then*

$$\tau_w \circ \rho_\gamma = \rho_\gamma \circ \tau_{\overline{\gamma} w}.$$

Although Theorem 1.3.4 provides an explicit description, in terms of the factorization afforded by Theorem 1.2.5, of the group operation on Aut($\mathbb{D}$), an algebraist might find our approach unsatisfactory. Let us briefly discuss a more abstract approach to Aut($\mathbb{D}$).

A *Möbius transformation* (also called a *linear fractional transformation*) is a rational function of the form

$$f(z) = \frac{az+b}{cz+d}, \tag{1.3.8}$$

in which $ad - bc \neq 0$. Each Möbius transformation is a bijective map from the *extended complex plane* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (or *Riemann sphere*) to itself. The set of all Möbius transformations is a group under composition; the identity is the function $\mathrm{id}(z) = z$ and the inverse of f is

$$f^{-1}(z) = \frac{dz-b}{-cz+a}.$$

If we multiply the numerator and denominator of (1.3.8) by a suitable constant, we may assume that $ad - bc = 1$.

The group of Möbius transformations is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$, the *projective special linear group of order* 2 *over* $\mathbb{C}$. To be more specific, $\mathrm{PSL}_2(\mathbb{C})$ is the quotient of $\mathrm{SL}_2(\mathbb{C})$, the group of 2×2 complex matrices with determinant 1, by the subgroup $\{I, -I\}$. Here I denotes the 2×2 identity matrix. The isomorphism between the group of Möbius transformations and $\mathrm{PSL}_2(\mathbb{C})$ is given by sending the function in (1.3.8), in which $ad - bc = 1$, to the coset of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in $\mathrm{SL}_2(\mathbb{C})/\{I, -I\}$.

Theorem 1.2.5 asserts that $\mathrm{Aut}(\mathbb{D}) = \{\rho_\gamma \circ \tau_w : \gamma \in \mathbb{T}, w \in \mathbb{D}\}$, in which

$$\rho_\gamma(z) = \frac{\gamma z + 0}{0z + 1} \quad \text{and} \quad \tau_w(z) = \frac{-1z + w}{-\overline{w}z + 1}.$$

The cosets in $\mathrm{SL}_2(\mathbb{C})/\{I, -I\}$ that correspond to ρ_γ and τ_w are the cosets of

$$\begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix},$$

where

$$\gamma = e^{i\theta}, \qquad \alpha = \frac{i}{\sqrt{1-|w|^2}}, \quad \text{and} \quad \beta = \frac{-iw}{\sqrt{1-|w|^2}}.$$

Consequently, $\mathrm{Aut}(\mathbb{D})$ can be identified with $\mathrm{PSU}_{1,1}(\mathbb{C})$, the quotient of

$$\mathrm{SU}_{1,1}(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ \overline{b} & \overline{a} \end{bmatrix} : |a|^2 - |b|^2 = 1 \right\}$$

by the subgroup $\{I, -I\}$. It is worth remarking that $\mathrm{SU}_{1,1}(\mathbb{C})$ is the set of 2×2 complex matrices U for which $\det U = 1$ and $U^* \Gamma U = \Gamma$, in which U^* denotes the conjugate transpose of U and

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This suggests a connection between $\mathrm{Aut}(\mathbb{D})$ and hyperbolic geometry that will be explored further in Chap. 2.

From a topological perspective, $\mathrm{Aut}(\mathbb{D})$ is homeomorphic to $\mathbb{T} \times \mathbb{D}$ via the map

$$(\gamma, w) \mapsto \rho_\gamma \circ \tau_w, \qquad \gamma \in \mathbb{T},\ w \in \mathbb{D}.$$

Thus, $\mathrm{Aut}(\mathbb{D})$ can be visualized as an open solid torus, endowed with the group structure described in Theorem 1.3.4.

1.4 The Schwarz–Pick Theorem

The hypothesis of the Schwarz Lemma (Lemma 1.1.1) involves a function that vanishes at the origin. A generalization can be obtained that removes this hypothesis. The crucial idea is to employ suitable automorphisms to reduce the general case to the classical Schwarz Lemma.

Theorem 1.4.1 (Schwarz–Pick) *For each* $f \in \mathscr{S}$,

$$\left| \frac{f(z) - f(w)}{1 - \overline{f(w)} f(z)} \right| \leqslant \left| \frac{z - w}{1 - \overline{w} z} \right|, \qquad w, z \in \mathbb{D}, \tag{1.4.2}$$

and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leqslant \frac{1}{1 - |z|^2}, \qquad z \in \mathbb{D}. \tag{1.4.3}$$

Moreover, the following are equivalent.

- *(a) Equality holds in* (1.4.2) *at two distinct* $z, w \in \mathbb{D}$.
- *(b) Equality holds in* (1.4.2) *at all* $z, w \in \mathbb{D}$ *with* $z \neq w$.
- *(c) Equality holds in* (1.4.3) *at some* $z \in \mathbb{D}$.
- *(d) Equality holds in* (1.4.3) *at all* $z \in \mathbb{D}$.
- *(e)* $f \in \operatorname{Aut}(\mathbb{D})$.

Proof Fix $w \in \mathbb{D}$. If $|f(w)| = 1$, the Maximum Modulus Principle implies that f is constant which means that (1.4.2) and (1.4.3) hold automatically. On the other hand, if $f(w) \in \mathbb{D}$, the Maximum Modulus Principle implies that $f(\mathbb{D}) \subseteq \mathbb{D}$. Let

$$g = \tau_{f(w)} \circ f \circ \tau_w \tag{1.4.4}$$

and observe that $g : \mathbb{D} \to \mathbb{D}$ is analytic and $g(0) = 0$. Since

$$g(\tau_w(z)) = \frac{f(w) - f(z)}{1 - \overline{f(w)}f(z)} \qquad \text{and} \qquad g'(0) = \frac{1 - |z|^2}{1 - |f(z)|^2} f'(z),$$

we see that (1.4.2) is equivalent to

$$|g(\tau_w(z))| \leqslant |\tau_w(z)|, \qquad w, z \in \mathbb{D} \tag{1.4.5}$$

and (1.4.3) is equivalent to

$$|g'(0)| \leqslant 1. \tag{1.4.6}$$

However, (1.4.5) and (1.4.6) hold by the Schwarz Lemma.

If any of (a)–(d) hold, then an application of the Schwarz Lemma to g confirms that $g = \rho_\gamma$ for some $\gamma \in \mathbb{T}$. Thus, (1.4.4) ensures that $f \in \operatorname{Aut}(\mathbb{D})$. Conversely, if $f \in \operatorname{Aut}(\mathbb{D})$, then (1.4.4) implies that $g \in \operatorname{Aut}(\mathbb{D})$ with $g(0) = 0$ and thus $g = \rho_\gamma$ for some $\gamma \in \mathbb{T}$. For this automorphism g, equality holds in (1.4.5) and (1.4.6) and consequently equality holds in (1.4.2) and (1.4.3). In other words, (e) implies any of (a)–(d). □

As a special case of Theorem 1.4.1, let $f = \tau_{z_0}$ to obtain

$$\left| \frac{\tau_{z_0}(z) - \tau_{z_0}(w)}{1 - \overline{\tau_{z_0}(w)}\tau_{z_0}(z)} \right| = \left| \frac{z - w}{1 - \overline{w}z} \right|, \qquad z, w \in \mathbb{D}, \tag{1.4.7}$$

and

$$|\tau'_{z_0}(z)| = \frac{1 - |\tau_{z_0}(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}. \tag{1.4.8}$$

These two identities will be useful later.

1.5 An Extremal Problem

Theorem 1.4.1 can be applied to solve certain extremal problems for $\mathscr{S}$. We briefly discuss one of them. Fix $\alpha, \beta \in \mathbb{D}$ and let

$$\mathscr{A}_{\alpha,\beta} = \{f \in \mathscr{S} : f(\alpha) = \beta\}.$$

Observe that $f = \tau_\beta \circ \tau_\alpha \in \mathscr{A}_{\alpha,\beta}$ and hence $\mathscr{A}_{\alpha,\beta} \neq \varnothing$. Our goal is to compute

$$M = \sup_{f \in \mathscr{A}_{\alpha,\beta}} |f'(\alpha)|,$$

along with the functions $f \in \mathscr{A}_{\alpha,\beta}$ for which the supremum above is attained.

Theorem 1.4.1 implies that

$$|f'(\alpha)| \leqslant \frac{1 - |f(\alpha)|^2}{1 - |\alpha|^2} = \frac{1 - |\beta|^2}{1 - |\alpha|^2}, \qquad f \in \mathscr{A}_{\alpha,\beta}.$$

A computation using (1.3.2) and (1.3.3) confirms that equality is attained when $f = \tau_\beta \circ \tau_\alpha$. Thus,

$$M = \frac{1 - |\beta|^2}{1 - |\alpha|^2}.$$

Moreover, Theorem 1.4.1 asserts that the $f \in \mathscr{A}_{\alpha,\beta}$ for which

$$|f'(\alpha)| = \frac{1 - |\beta|^2}{1 - |\alpha|^2}$$

are precisely the $f \in \operatorname{Aut}(\mathbb{D})$ that satisfy $f(\alpha) = \beta$. Let f be such an automorphism and let $g = \tau_\beta \circ f \circ \tau_\alpha$; observe that $g \in \operatorname{Aut}(\mathbb{D})$. Then

$$g(0) = \tau_\beta(f(\tau_\alpha(0))) = \tau_\beta(f(\alpha)) = \tau_\beta(\beta) = 0$$

and hence $g(z) = \gamma z$ for some $\gamma \in \mathbb{T}$; that is, $g = \rho_\gamma$. Hence the solutions to the extremal problem are given by

$$f = \tau_\beta \circ \rho_\gamma \circ \tau_\alpha,$$

in which $\gamma \in \mathbb{T}$ is a free parameter.

1.6 Julia's Lemma

The Schwarz–Pick theorem (Theorem 1.4.1) involves two points $z, w \in \mathbb{D}$. What happens if one of the points approaches $\mathbb{T}$? This situation was studied by Julia and it may be interpreted as a boundary Schwarz–Pick theorem [83, p. 87]. Julia's lemma plays an essential role in studying the behavior of the derivative of infinite Blaschke products. The proof of Julia's lemma requires the important identity

$$1-\left|\frac{\alpha-\beta}{1-\overline{\beta}\alpha}\right|^2=\frac{(1-|\alpha|^2)(1-|\beta|^2)}{|1-\overline{\beta}\alpha|^2},\qquad \alpha,\beta\in\mathbb{D},\tag{1.6.1}$$

which follows from (1.4.8).

Lemma 1.6.2 (Julia [83]) *Let $f\in\mathscr{S}$. If there is a sequence z_n in $\mathbb{D}$ such that*

$$\lim_{n\to\infty}z_n=1,\qquad \lim_{n\to\infty}f(z_n)=1,$$

and

$$\lim_{n\to\infty}\frac{1-|f(z_n)|}{1-|z_n|}=A<\infty,\tag{1.6.3}$$

then

$$\frac{|1-f(z)|^2}{1-|f(z)|^2}\leqslant A\frac{|1-z|^2}{1-|z|^2},\qquad z\in\mathbb{D}.\tag{1.6.4}$$

Proof The Schwarz–Pick theorem (Theorem 1.4.1) implies that

$$\left|\frac{f(z)-f(z_n)}{1-\overline{f(z_n)}f(z)}\right|\leqslant\left|\frac{z-z_n}{1-\overline{z}_nz}\right|,\qquad z\in\mathbb{D},$$

and hence

$$1-\left|\frac{z-z_n}{1-\overline{z}_nz}\right|^2\leqslant 1-\left|\frac{f(z)-f(z_n)}{1-\overline{f(z_n)}f(z)}\right|^2.$$

The identity (1.6.1), applied to both sides of the above, yields

$$\frac{(1-|z|^2)(1-|z_n|^2)}{|1-\overline{z}_nz|^2}\leqslant\frac{(1-|f(z)|^2)(1-|f(z_n)|^2)}{|1-\overline{f(z_n)}f(z)|^2}.$$

Rewrite the preceding inequality as

$$\frac{|1-\overline{f(z_n)}f(z)|^2}{1-|f(z)|^2}\leqslant\frac{1-|f(z_n)|^2}{1-|z_n|^2}\cdot\frac{|1-\overline{z}_nz|^2}{1-|z|^2}.$$

Now let $n\to\infty$ and apply (1.6.3) to complete the proof. □

In the lemma above, we assumed that $z_n\to 1$ and $f(z_n)\to 1$. However, the important issue is that the sequences z_n and $f(z_n)$ converge toward points of the unit circle $\mathbb{T}$. For the sake of completeness, here is the general version of this result.

Corollary 1.6.5 *Let $f \in \mathscr{S}$ and $\alpha, \beta \in \mathbb{T}$. If there is a sequence z_n in $\mathbb{D}$ such that*

$$\lim_{n\to\infty} z_n = \alpha, \qquad \lim_{n\to\infty} f(z_n) = \beta,$$

and

$$\lim_{n\to\infty} \frac{1 - |f(z_n)|}{1 - |z_n|} = A < \infty,$$

then

$$\frac{|\beta - f(z)|^2}{1 - |f(z)|^2} \leqslant A \frac{|\alpha - z|^2}{1 - |z|^2}, \qquad z \in \mathbb{D}.$$

Proof Apply Lemma 1.6.2 to the function $g(z) = \overline{\beta} f(\overline{\alpha} z)$. □

We can also discuss the boundary limits of functions in $\mathscr{S}$ that satisfy the hypotheses of Julia's Lemma. Let $\zeta \in \mathbb{T}$ and $C > 1$. The region

$$S_C(\zeta) = \{z \in \mathbb{D} : |z - \zeta| \leqslant C(1 - |z|)\}$$

is the *Stolz domain* anchored at α with constant C; see Fig. 1.1.

We say that $f \in \mathscr{S}$ has the *nontangential limit* L at $\zeta \in \mathbb{T}$ if, for each fixed $C > 1$,

$$\lim_{\substack{z \to \zeta \\ z \in S_C(\zeta)}} f(z) = L. \tag{1.6.6}$$

If so, we define $f(\zeta) = L$ and write

$$\angle \lim_{z\to\zeta} f(z) = f(\zeta).$$

The quantity $f(\zeta)$ is referred to as *the boundary value* of f at ζ. The restriction that z belongs to a Stolz domain $S_C(\zeta)$ in (1.6.6) ensures that z does not approach ζ along a path that is tangent to $\mathbb{T}$ at ζ. Each Schur function has non-tangential boundary values almost everywhere with respect to Lebesgue measure on $\mathbb{T}$; see Theorem A.3.1.

Corollary 1.6.7 *Let $f \in \mathscr{S}$ and let $\alpha, \beta \in \mathbb{T}$. If there is a sequence z_n in $\mathbb{D}$ such that $z_n \to \alpha$, $f(z_n) \to \beta$, and*

$$\lim_{n\to\infty} \frac{1 - |f(z_n)|}{1 - |z_n|} < \infty,$$

then

$$\angle \lim_{z\to\alpha} f(z) = \beta.$$