MATRIX Book Series 1

David R. Wood *Editor-in-chief*
Jan de Gier · Cheryl E. Praeger
Terence Tao *Editors*

2016 MATRIX Annals

MATRIX

Springer

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MATRIX is Australia's international and residential mathematical research institute. It facilitates new collaborations and mathematical advances through intensive residential research programs, each lasting 1–4 weeks.

More information about this series at http://www.springer.com/series/15890

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Editors
David R. Wood (*Editor-in-Chief*)
Monash University
Clayton, Australia

Jan de Gier
University of Melbourne
Melbourne, Australia

Cheryl E. Praeger
University of Western Australia
Crawley, Australia

Terence Tao
UCLA
Los Angeles, USA

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Preface

MATRIX is Australia's first international and residential mathematical research institute. It was established in 2015 and launched in 2016 as a joint partnership between Monash University and the University of Melbourne, with seed funding from the ARC Centre of Excellence for Mathematical and Statistical Frontiers. The purpose of MATRIX is to facilitate new collaborations and mathematical advances through intensive residential research programs, which are currently held in Creswick, a small town nestled in the beautiful forests of the Macedon Ranges, 130 km west of Melbourne.

This book, *2016 MATRIX Annals*, is a scientific record of the five programs held at MATRIX in 2016:

- *Higher Structures in Geometry and Physics*
- *Winter of Disconnectedness*
- *Approximation and Optimisation*
- *Refining C*-Algebraic Invariants for Dynamics Using KK-Theory*
- *Interactions Between Topological Recursion, Modularity, Quantum Invariants and Low-Dimensional Topology*

The MATRIX Scientific Committee selected these programs based on scientific excellence and the participation rate of high-profile international participants. This committee consists of Jan de Gier (Melbourne University, Chair), Ben Andrews (Australian National University), Darren Crowdy (Imperial College London), Hans De Sterck (Monash University), Alison Etheridge (University of Oxford), Gary Froyland (University of New South Wales), Liza Levina (University of Michigan), Kerrie Mengersen (Queensland University of Technology), Arun Ram (University of Melbourne), Joshua Ross (University of Adelaide), Terence Tao (University of California, Los Angeles), Ole Warnaar (University of Queensland), and David Wood (Monash University).

The selected programs involved organisers from a variety of Australian universities, including Federation, Melbourne, Monash, Newcastle, RMIT, Sydney, Swinburne, and Wollongong, along with international organisers and participants. Each program lasted 1–4 weeks and included ample unstructured time to encourage collaborative research. Some of the longer programs had an embedded conference or lecture series. All participants were encouraged to submit articles to the MATRIX Annals.

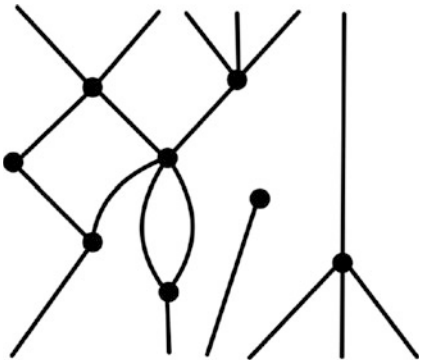
The articles were grouped into refereed contributions and other contributions. Refereed articles contain original results or reviews on a topic related to the MATRIX program. The other contributions are typically lecture notes based on talks or activities at MATRIX. A guest editor organised appropriate refereeing and ensured the scientific quality of submitted articles arising from each program. The editors (Jan de Gier, Cheryl E. Praeger, Terence Tao, and myself) finally evaluated and approved the papers.

Many thanks to the authors and to the guest editors for their wonderful work.

MATRIX has hosted eight programs in 2017, with more to come in 2018; see www.matrix-inst.org.au. Our goal is to facilitate collaboration between researchers in universities and industry, and increase the international impact of Australian research in the mathematical sciences.

David R. Wood
MATRIX Book Series Editor-in-Chief

Higher Structures in Geometry and Physics



6–17 June 2016

Organisers

Marcy Robertson (Melbourne)
Philip Hackney (Macquarie)

The inaugural program at MATRIX took place on June 6–17, 2016, and was entitled "Higher Structures in Geometry and Physics". It was both a pleasure and a privilege to take part in this first ever program at MATRIX. The excellent working conditions, cosy environment, friendly staff and energetic participants made this time both memorable and productive.

The scientific component of our program was comprised of a workshop with lecture series by several invited speakers, followed in the subsequent week by a conference featuring talks on a range of related topics from speakers from around the globe. The two events were separated by a long weekend which gave the participants free time to discuss and collaborate. Within this volume is a collection of lecture notes and articles reflecting quite faithfully the ideas in the air during these two weeks.

Our title Higher Structures (not unlike the term down under) suggests a certain fixed perspective. For the participants in our program, this perspective comes from the twentieth-century examples of algebraic and categorical constructions associated to topological spaces, possibly with geometric structures and possibly taking motivation from physical examples. From this common frame of reference stems a range of new and rapidly developing directions, activities such as this program at MATRIX play a vital role in weaving these threads into a collective understanding. The excitement of working in this rapidly developing field was felt during our time at MATRIX, and I hope it comes across in this volume as well.

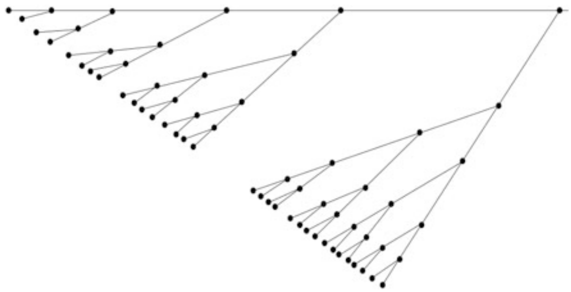
I would like to thank all of the authors who took the time to contribute to this volume. I would also like to thank the MATRIX staff and officials for hosting and facilitating this event and giving us the opportunity to share our work with this volume. Most importantly, I would like to thank the organizers of our program Marcy and Philip for all of their hard work and for giving all of us participants this unique opportunity.

Ben Ward
Guest Editor

Participants

Ramon Abud Alcala (Macquarie), Clark Barwick (Massachusetts Institute of Technology), Alexander Campbell (Macquarie), David Carchedi (George Mason), Gabriel C. Drummond-Cole (IBS Center for Geometry and Physics), Daniela Egas Santander (Freie Universitat Berlin), Nora Ganter (Melbourne), Christian Haesemeyer (Melbourne), Philip Hackney (Osnabrueck), Ralph Kauffman (Purdue), Edoardo Lanari (Macquarie), Martin Markl (Czech Academy of Sciences), Branko Nikolic (Macquarie), Simona Paoli (Leicester), Sophia Raynor (Aberdeen), Emily Riehl (Johns Hopkins), David Roberts (Adelaide), Marcy Robertson (Melbourne), Chris Rogers (Louisiana), Martina Rovelli (EPFL), Matthew Spong (Melbourne), Michelle Strumila (Melbourne), TriThang Tran (Melbourne), Victor Turchin (Kansas State), Dominic Verity (Macquarie), Raymond Vozzo (Adelaide), Ben Ward (Stony Brook), Mark Weber (Macquarie), Felix Wierstra (Stockholm), Sinan Yalin (Copenhagen), Jun Yoshida (Tokyo), Dimitri Zaganidis (EPFL)

Winter of Disconnectedness



27 June–8 July 2016

Organisers

Murray Elder (Newcastle)
Jacqui Ramagge (Sydney)
Colin Reid (Newcastle)
Anne Thomas (Sydney)
George Willis (Newcastle)

Our understanding of totally disconnected locally compact (t.d.l.c.) groups has been growing rapidly in recent years. These groups are of interest for general theoretical reasons, because half the task of describing the structure of general locally compact groups falls into the totally disconnected case, and also for purposes of specific applications, because of the significance that various classes of t.d.l.c. groups have in geometry, number theory and algebra. The workshop held at Creswick from 27 June to 8 July 2016 was the first part of a program in which leading researchers presented the most recent advances by giving short courses and individual lectures. Time was also set aside for collaboration between established researchers and students. Both general techniques and results relating to particular classes of t.d.l.c. groups were covered in the lectures.

Four courses of five lectures each were delivered at Creswick, as follows:

- Helge Glöckner (Paderborn): *Endomorphisms of Lie groups over local fields*
- George Willis (Newcastle): *The scale, tidy subgroups and flat groups*
- Anne Thomas (Sydney): *Automorphism groups of combinatorial structures*
- Phillip Wesolek (Binghamton): *A survey of elementary totally disconnected locally compact groups*

A second workshop was held in Newcastle at the end of July 2016 with the same structure.

Notes from the four courses just listed and from the courses

- Adrien Le Boudec (Louvain Le Neuve): *Groups of automorphisms and almost automorphisms of trees: subgroups and dynamics*
- Colin Reid (Newcastle): *Normal subgroup structure of totally disconnected locally compact groups*

delivered at the Newcastle workshop are published here. Although the distinction is not absolute, the notes by Reid, Wesolek and Willis cover general methods and those by Glöckner, Le Boudec and Thomas treat specific classes of t.d.l.c. groups. In several cases, they are based on notes taken by listeners at the workshops, and we are grateful for their assistance. Moreover, the notes by Glöckner, which have been refereed, are an expanded version of what was delivered in lectures and contain calculations and proofs of some results that have not previously been published.

We believe that these notes give the 2016 overview of the state of knowledge and of research directions on t.d.l.c. groups and hope that they will also serve to introduce students and other researchers new to the field to this rapidly developing subject.

Dave Robertson
Guest Editor

Participants

Benjamin Brawn (Newcastle), Timothy Bywaters (Sydney), Wee Chaimanowong (Melbourne), Murray Elder (Newcastle), Helge Glöckner (Paderborn), John Harrison

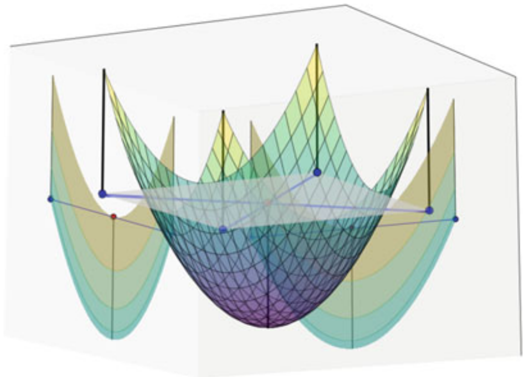
(Baylor), Waltraud Lederle (ETH Zurich), Rupert McCallum (Tübingen), Sidney Morris (Federation), Uri Onn (ANU), C.R.E. Raja (Indian Statistical Institute, Bangalore), Jacqui Ramagge (Sydney), Colin Reid (Newcastle), David Robertson (Newcastle), Anurag Singh (Utah), Simon Smith (City University of New York), George Willis (Newcastle), Thomas Taylor (Newcastle), Anne Thomas (Sydney), Stephan Tornier (ETH Zurich), Tian Tsang (RMIT), Phillip Wesolek (Universite Catholique de Louvain)

Approximation and Optimisation

10–16 July 2016

Organisers

Vera Roshchina (RMIT)
Nadezda Sukhorukova (Swinburne)
Julien Ugon (Federation)
Aris Daniilidis (Chile)
Andrew Eberhard (RMIT)
Alex Kruger (Federation)
Zahra Roshanzamir (Swinburne)



There are many open problems in the field of approximation theory where tools from variational analysis and nonsmooth optimisation show promise. One of them is the Chebyshev (also known as uniform) approximation problem. Chebyshev's work is generally considered seminal in approximation theory and remains influential to this day in this field of mathematics. His work also has many significant implications in optimisation, where uniform approximation of functions is considered an early example of an optimisation problem where the objective function is not differentiable. In fact, the problem of best polynomial approximation can be reformulated as an optimisation problem which provides very nice textbook examples for convex analysis.

The joint ancestry and connections between optimisation and approximation are still very apparent today. Many problems in approximation can be reformulated as optimisation problems. On the other hand, many optimisation methods require set and/or function approximation to work efficiently.

These connections were explored in the 1950s, 1960s and 1970s. This was also the period when the area of nonsmooth analysis emerged. Convex and nonsmooth analysis techniques can be applied to obtain theoretical results and algorithms for solving approximation problems with nonsmooth objectives. These problems include Chebyshev approximation: univariate (polynomial and fixed-knots polynomial spline) approximation and multivariate polynomial approximation. In 1972, P.-J. Laurent published his book, where he demonstrated interconnections between approximation and optimisation. In particular, he showed that many difficult (Chebyshev) approximation problems can be solved using optimisation techniques.

Despite this early work, historically the fields of optimisation and approximation have remained separate, each (re)developing their own methodology and terminology. There is a clear pattern though: advances in optimisation result in significant breakthroughs in approximation and, on the other hand, new approximation techniques and approaches advance the development of new optimisation methods and improve the performance of existing ones. Therefore, the lift-off theme of the program was multivariate and polynomial spline approximation from the perspective of optimisation theory. We invited top researchers in approximation theory, polynomial and semialgebraic optimisation and variational analysis to find new ways to attack the existing open problems and to establish major new research directions. A fresh look at approximation problems from the optimisation point of view is vital, since it enables us to solve approximation problems that cannot be solved without very advanced optimisation techniques (and vice versa) and discover beautiful interconnections between approximation and optimisation.

The morning sessions of the program consisted of lectures:

- Approximation of set-valued functions (Nira Dyn, Tel Aviv University)
- Algebraic, convex analysis and semi-infinite programming approach to Chebyshev approximation (Julien Ugon, Federation University Australia, and Nadezda Sukhorukova, Swinburne University of Technology)
- The sparse grid combination technique and optimisation (Markus Hegland, Australian National University)
- Quasi-relative interior and optimisation (Constantin Zalinescu, Al. I. Cuza University)

The afternoons were dedicated to smaller group discussions. The following discussions were vital for the research papers submitted to this volume:

1. **Compact convex sets with prescribed facial dimensions, by Vera Roshchina, Tian Sang and David Yost**

   The rich soup of research ideas that was stirred up during the workshop helped us with a breakthrough in a seemingly unrelated research direction. During the workshop, David Yost came up with a neat inductive idea that finished the proof of the dimensional sequence theorem that he and Vera Roshchina have been working on for a while. Later during the workshop the fractal ideas introduced by Markus Hegland motivated us to consider convex sets with fractal facial structure. Even more surprisingly, we discovered that a beautiful example of such a set can be obtained from the spherical gasket studied by Tian Sang in her prior research on infinite Coxeter groups.

2. **Chebyshev multivariate polynomial approximation: alternance interpretation, by Nadezda Sukhorukova, Julien Ugon and David Yost**
   The notion of alternating sequence (alternance) is central for univariate Chebyshev approximation problems. How can we extend this notion to the case of multivariate approximation, where the sets are not totally ordered? In one of the papers, namely, "Chebyshev multivariate polynomial approximation: alternance interpretation" by Sukhorukova, Ugon and Yost, the authors work on this issue and propose possible solutions, in particular a very elegant formulation for necessary and sufficient optimality conditions for multivariate Chebyshev approximation.

Julien Ugon and Nadezda Sukhorukova
Guest Editors

Participants

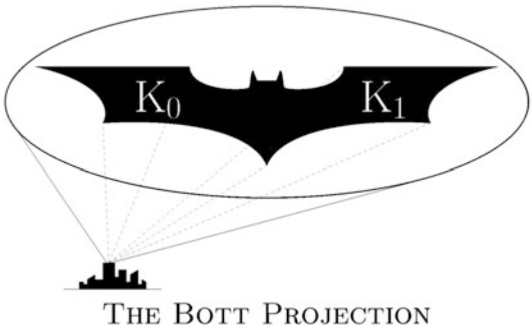
Alia Al nuaimat (Federation), Fusheng Bai (Chongqing Normal University), Yi Chen (Federation), Jeffrey Christiansen (RMIT), Brian Dandurand (RMIT), Reinier Diaz Millan (Federal Institute of Goias), Nira Dyn (Tel Aviv), Andrew Eberhard (RMIT), Gabriele Eichfelder (Technische Universitat Ilmenau), Markus Hegland (ANU), Alexander Kruger (Federation), Vivek Laha (Indian Institute of Technology, Patna), Jeffrey Linderoth (Wisconsin- Madison), Prabhu Manyem (Nanchang Institute of Technology), Faricio Oliveira (RMIT), Zahra Roshan Zamir (Swinburne), Vera Roshchina (RMIT), Tian Sang (RMIT), Jonathan Scanlan, Vinay Singh (National Institute of Technology, Mizoram), Nadezda Sukhorukova (Swinburne), Julien Ugon (Federation), Dean Webb (Federation), David Yost (Federation), Constantin Zalinescu (University "Al. I. Cuza" Iasi), Jiapu Zhang (Federation)

Refining C*-Algebraic Invariants for Dynamics Using KK-Theory

18–29 July 2016

Organisers

Magnus Goffeng
(Chalmers / Gothenburg)
Adam Rennie (Wollongong)
Aidan Sims (Wollongong)



This graduate school and workshop were motivated by intense recent interest and progress in the non-commutative geometry of dynamical systems.

The progress has been of several sorts. The assignment of C^*-algebras to dynamical systems is not new but has become much more sophisticated in recent years. The K-groups of such dynamical C^*-algebras provide invariants of the original dynamical system but are not always fine enough to capture the structural features of greatest interest. In response to this, precise characterisations of the relationships between more detailed K-theoretic invariants and equivalence classes of dynamical systems have recently been sharpened significantly.

The extra ingredient whose potential applications to such problems we hoped to highlight to attendees is recent progress in importing ideas from algebraic topology to dynamical systems theory through the computability of the Kasparov product. The Kasparov product is a far-reaching generalisation of index theory and provides an abstract composition rule for morphisms in the KK-category of C^*-algebras. The KK-category extends and refines the correspondence category of C^*-algebras; and a correspondence can be regarded as a generalised dynamical system and is closely related to the construction of dynamical C^*-algebras.

With this background in mind, we started the first week with three lecture series during the mornings, with informal Q&A and research in the afternoons. The three lecture courses were:

- Robin Deeley: *Groupoids and* C^**-algebras*
- Bram Mesland: *Kasparov's KK-theory*
- Adam Rennie and Aidan Sims: *Hilbert modules and Cuntz–Pimsner algebras*

Groupoid C^*-algebras and Cuntz–Pimsner algebras are two of the most flexible and best developed frameworks for modelling dynamical systems using C^*-algebras. The basics of Kasparov's KK-theory and the recent advances in the computability of the product proved central to the progress seen during the workshop.

The lecture series by Deeley, Mesland, Rennie, and Sims set the stage for the second week, bringing attendees, particularly the significant student cohort, together around a common language and a joint leitmotif. The talks ranged widely over the core topics, their applications and neighbouring disciplines. The five papers which follow give a good indication of the breadth of the conference.

Deeley offers refinements of Putnam's homology theory for dynamical systems, and Ruiz et al. provide strong invariants for a special class of dynamical systems. Bourne (with Schulz-Baldes) provides an application of KK-theoretic techniques to topological insulators. Goffeng and Mesland provide a detailed account of novel aspects of the non-commutative geometry of the Cuntz algebras, while Arici probes the non-commutative topology and geometry of quantum lens spaces using techniques which are perfectly in tune with the theme of the workshop.

Adam Rennie
Guest Editor

Participants

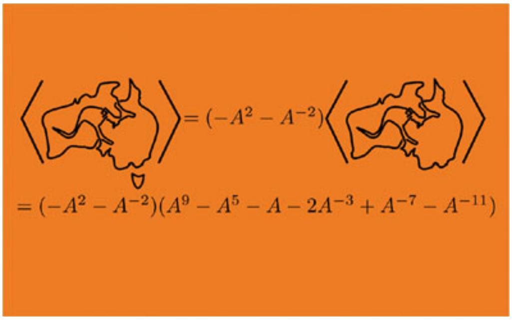
Zahra Afsar (Wollongong), Francesca Arici (Radboud University Nijmegen), Chris Bourne (Erlangen-Nurnberg), Guo Chuan Thiang (Adelaide), Robin Deeley (Hawaii), Anna Luise Duwenig (Victoria), James Fletcher (Wollongong), Iain Forsyth (Leibniz University Hannover), Elizabeth Anne Gillaspy (Universitet Manster), Magnus Goffeng (Chalmers Technology/Gothenburg), Peter Hochs (Adelaide), Marcelo Laca (Victoria), Lachlan MacDonald (Wollongong), Michael Mampusti (Wollongong), Bram Mesland (Leibniz University Hannover), Alexander Mundey (Wollongong), Adam Rennie (Wollongong), Karen Rught Strung (Polish Academy of Sciences), Efren Ruiz (Hawaii at Hilo), Thomas Scheckter (UNSW), Aidan Sims (Wollongong), Hang Wang (Adelaide), Yasuo Watatani (Kyushu)

Interactions Between Topological Recursion, Modularity, Quantum Invariants and Low-Dimensional Topology

28 November–23 December 2016

Organisers

Motohico Mulase (UC Davis)
Norman Do (Monash)
Neil Hoffman (Oklahoma State)
Craig Hodgson (Melbourne)
Paul Norbury (Melbourne)



This program contributed to the active international research effort under way at present to connect structures in mathematical physics with those in low-dimensional topology, buoyed by recent theoretical advances and a broad range of applications. It brought together people in cognate areas with common interests, including algebraic geometry, conformal field theory, knot theory, representation theory, quantum invariants and combinatorics.

The program was motivated by recent generalisations of the technique of topological recursion, as well as fundamental conjectures concerning invariants in low dimensional topology. These include the AJ conjecture, the Jones slope conjecture of Garoufalidis and the underlying topological significance of the 3D-index. Algorithmic techniques to compute these and related invariants are also featured in the program.

The program began with a week of short courses, comprising three lectures each, on the following topics:

- *Conformal field theory* (Katrin Wendland, Freiburg)
- *Hyperbolic knot theory* (Jessica Purcell, Monash)
- *Quantum invariants* (Roland van der Veen, Leiden)
- *Topological recursion* (Norman Do, Monash)

The program for the middle week was largely informal and reserved for research collaborations. The final week was dedicated to an international conference, which gathered together leading experts in the areas of mathematical physics, topological recursion, quantum invariants and low-dimensional topology to address recent advances and explore new connections between these fields.

The program attracted a total of 51 attendees. During the conference, there were 33 talks, of which 23 were delivered by international visitors. Among these talks were the following:

- Jørgen Andersen: Verlinde formula for Higgs bundles
- Feng Luo: Discrete uniformization for polyhedral surfaces and its convergence
- Rinat Kashaev: Pachner moves and Hopf algebras
- Scott Morrison: Modular data for Drinfeld doubles
- Hyam Rubinstein and Craig Hodgson: Counting genus two surfaces in 3-manifolds
- Gaëtan Borot: Initial conditions for topological recursion
- Tudor Dimofte: Counting vortices in the 3D index
- George Shabat: Counting Belyi pairs over finite fields
- Leonid Chekhov: Abstract topological recursion and Givental decomposition
- Piotr Sułkowski: Knots and BPS/super-quantum curves

The articles in these proceedings represent different aspects of the program. Kashaev's contribution describes a topological quantum field theory in four dimensions. Licata-Mathews and Spreer-Tillmann describe topological and geometric results for 3-manifolds. Shabat describes first steps towards generalising Belyi maps to finite fields. Roland van der Veen kindly contributed notes from his short course on quantum invariants of knots.

Norman Do, Neil Hoffman, Paul Norbury
Guest Editors

Participants

Jørgen Andersen (Aarhus), Vladimir Bazhanov (ANU), Gaetan Borot (Max Planck), Benjamin Burton (Queensland), Alex Casella (Sydney), Wee Chaimanowong (Melbourne), Abhijit Champanerkar (CUNY), Anupam Chaudhuri (Monash), Leonid

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Contents

Refining C* Algebraic Invariants for Dynamics Using KK-Theory

Interactions Between Topological Recursion, Modularity, Quantum Invariants and Low-dimensional Topology

PART II OTHER CONTRIBUTED ARTICLES

Higher Structures in Geometry and Physics

Winter of Disconnectedness

Interactions Between Topological Recursion, Modularity, Quantum Invariants and Low-dimensional Topology

Part I
Refereed Articles

Homotopical Properties of the Simplicial Maurer–Cartan Functor

Christopher L. Rogers

Abstract We consider the category whose objects are filtered, or complete, L_∞-algebras and whose morphisms are ∞-morphisms which respect the filtrations. We then discuss the homotopical properties of the Getzler–Hinich simplicial Maurer–Cartan functor which associates to each filtered L_∞-algebra a Kan simplicial set, or ∞-groupoid. In previous work with V. Dolgushev, we showed that this functor sends weak equivalences of filtered L_∞-algebras to weak homotopy equivalences of simplicial sets. Here we sketch a proof of the fact that this functor also sends fibrations to Kan fibrations. To the best of our knowledge, only special cases of this result have previously appeared in the literature. As an application, we show how these facts concerning the simplicial Maurer–Cartan functor provide a simple ∞-categorical formulation of the Homotopy Transfer Theorem.

1 Introduction

Over the last few years, there has been increasing interest in the homotopy theory of filtered, or complete, L_∞-algebras[1] and the role these objects play in deformation theory [10, 12], rational homotopy theory [3, 4, 13], and the homotopy theory of homotopy algebras [6, 8, 9]. One important tool used in these applications is the simplicial Maurer–Cartan functor $\mathfrak{MC}_\bullet(-)$ which produces from any filtered L_∞-algebra a Kan simplicial set, or ∞-groupoid. This construction, first appearing in the work of Hinich [12] and Getzler [10], can (roughly) be thought of as a "non-abelian analog" of the Dold–Kan functor from chain complexes to simplicial

[1] Throughout this paper, all algebraic structures have underlying $\mathbb{Z}$ graded $\Bbbk$-vector spaces with char $\Bbbk = 0$.

C.L. Rogers (✉)
Department of Mathematics and Statistics, University of Nevada, Reno, 1664 N. Virginia Street, Reno, NV 89557-0084, USA
e-mail: chrisrogers@unr.edu; chris.rogers.math@gmail.com

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vector spaces. In deformation theory, these ∞-groupoids give higher analogs of the Deligne groupoid. In rational homotopy theory, this functor generalizes the Sullivan realization functor, and has been used to study rational models of mapping spaces.

A convenient presentation of the homotopy theory of filtered L_∞-algebras has yet to appear in the literature. But based on applications, there are good candidates for what the weak equivalences and fibrations should be between such objects. One would also hope that the simplicial Maurer–Cartan functor sends these morphisms to weak homotopy equivalences and Kan fibrations, respectively. For various special cases, which are recalled in Sect. 3, it is known that this is indeed true. In joint work with Dolgushev [5], we showed that, in general, $\mathfrak{MC}_\bullet(-)$ maps any weak equivalence of filtered L_∞-algebras to a weak equivalence of Kan complexes. This can be thought of as the natural L_∞ generalization of the Goldman–Millson theorem in deformation theory.

The purpose of this note is to sketch a proof of the analogous result for fibrations (Theorem 2 in Sect. 3 below): The simplicial Maurer–Cartan functor maps any fibration between any filtered L_∞-algebras to a fibration between their corresponding Kan complexes. Our proof is not a simple generalization of the special cases already found in the literature, nor does it follow directly from general abstract homotopy theory. It requires some technical calculations involving Maurer–Cartan elements, similar to those found in our previous work [5].

As an application, we show in Sect. 4 that "∞-categorical" analogs of the existence and uniqueness statements that comprise the Homotopy Transfer Theorem [1, 2, 14, 15] follow as a corollary of our Theorem 2. In more detail, suppose we are given a cochain complex A, a homotopy algebra B of some particular type (e.g., an A_∞, L_∞, or C_∞-algebra) and a quasi-isomorphism of complexes $\phi\colon A \to B$. Then, using the simplicial Maurer–Cartan functor, we can naturally produce an ∞-groupoid $\mathfrak{F}$ whose objects correspond to solutions to the "homotopy transfer problem". By a solution, we mean a pair consisting of a homotopy algebra structure on A, and a lift of ϕ to a ∞-quasi-isomorphism of homotopy algebras $A \xrightarrow{\sim} B$. The fact that $\mathfrak{MC}_\bullet(-)$ preserves both weak equivalences and fibrations allows us to conclude that: (1) The ∞-groupoid $\mathfrak{F}$ is non-empty, and (2) it is contractible. In other words, a homotopy equivalent transferred structure always exists, and this structure is unique in the strongest possible sense.

2 Preliminaries

2.1 Filtered L_∞-Algebras

In order to match conventions in our previous work [5], we define an L_∞**-algebra** to be a cochain complex (L, ∂) for which the reduced cocommutative coalgebra $\underline{S}(L)$ is equipped with a degree 1 coderivation Q such that $Q(x) = \partial x$ for all $x \in L$ and $Q^2 = 0$. This structure is equivalent to specifying a sequence of *degree 1* multi-brackets

$$\{\ ,\ ,\dots,\ \}_m : S^m(L) \to L \quad m \geq 2 \tag{1}$$

satisfying compatibility conditions with the differential ∂ and higher-order Jacobi-like identities. (See Eq. 2.5 in [5].) More precisely, if $\mathrm{pr}_L : \underline{S}(L) \to L$ denotes the usual projection, then

$$\{x_1, x_2, \dots, x_m\}_m = \mathrm{pr}_L Q(x_1 x_2 \dots x_m)\,, \qquad \forall x_j \in L\,.$$

This definition of L_∞-algebra is a "shifted version" of the original definition of L_∞-algebra. A shifted L_∞-structure on L is equivalent to a traditional L_∞-structure on $\mathbf{s}L$, the suspension of L.

A **morphism** (or ∞-morphism) Φ from an L_∞-algebra (L, Q) to an L_∞-algebra $(\tilde{L}, \tilde{Q})$ is a dg coalgebra morphism

$$\Phi \colon \big(\underline{S}(L), Q\big) \to \big(\underline{S}(\tilde{L}), \tilde{Q}\big). \tag{2}$$

Such a morphism Φ is uniquely determined by its composition with the projection to $\tilde{L}$:

$$\Phi' := \mathrm{pr}_{\tilde{L}} \Phi.$$

Every such dg coalgebra morphism induces a map of cochain complexes, e.g., the linear term of Φ:

$$\phi := \mathrm{pr}_{\tilde{L}} \Phi|_L \colon (L, \partial) \to (\tilde{L}, \tilde{\partial}), \tag{3}$$

and we say Φ is **strict** iff it consists only of a linear term, i.e.

$$\Phi'(x) = \phi(x) \qquad \Phi'(x_1, \dots, x_m) = 0 \quad \forall m \geq 2 \tag{4}$$

A morphism $\Phi \colon (L, Q) \to (\tilde{L}, \tilde{Q})$ of L_∞-algebras is an **∞-quasi-isomorphism** iff $\phi \colon (L, \partial) \to (\tilde{L}, \tilde{\partial})$ is a quasi-isomorphism of cochain complexes.

We say an L_∞-algebra (L, Q) is a **filtered L_∞-algebra** iff the underlying cochain complex (L, ∂) is equipped with a complete descending filtration,

$$L = \mathscr{F}_1 L \supset \mathscr{F}_2 L \supset \mathscr{F}_3 L \cdots \tag{5}$$

$$L = \varprojlim_k L/\mathscr{F}_k L\,, \tag{6}$$

which is compatible with the brackets, i.e.

$$\left\{\mathscr{F}_{i_1}L, \mathscr{F}_{i_2}L, \ldots, \mathscr{F}_{i_m}L\right\}_m \subseteq \mathscr{F}_{i_1+i_2+\cdots+i_m}L \quad \forall\ m > 1.$$

A filtered L_∞-algebra in our sense is a shifted analog of a "complete" L_∞-algebra, in the sense of Berglund [3, Def. 5.1].

Remark 1 Due to its compatibility with the filtration, the L_∞-structure on L induces a filtered L_∞-structure on the quotient $L/\mathscr{F}_nL$. In particular, $L/\mathscr{F}_nL$ is a **nilpotent L_∞-algebra** [3, Def. 2.1], [10, Def. 4.2]. Moreover, when the induced L_∞-structure is restricted to the sub-cochain complex

$$\mathscr{F}_{n-1}L/\mathscr{F}_nL \subseteq L/\mathscr{F}_nL$$

all brackets of arity ≥ 2 vanish. Hence, the nilpotent L_∞-algebra $\mathscr{F}_{n-1}L/\mathscr{F}_nL$ is an **abelian L_∞-algebra**.

Definition 1 We denote by $\widehat{\mathsf{Lie}}_\infty$ the category whose objects are filtered L_∞-algebras and whose morphisms are ∞-morphisms $\Phi\colon (L, Q) \to (\tilde{L}, \tilde{Q})$ which are compatible with the filtrations:

$$\Phi'(\mathscr{F}_{i_1}L \otimes \mathscr{F}_{i_2}L \otimes \cdots \otimes \mathscr{F}_{i_m}L) \subset \mathscr{F}_{i_1+i_2+\cdots+i_m}\tilde{L}\,, \tag{7}$$

Definition 2 Let $\Phi\colon (L, Q) \to (\tilde{L}, \tilde{Q})$ be a morphism in $\widehat{\mathsf{Lie}}_\infty$.

1. We say Φ is a **weak equivalence** iff its linear term $\phi\colon (L, \partial) \to (\tilde{L}, \tilde{\partial})$ induces a quasi-isomorphism of cochain complexes
$$\phi|_{\mathscr{F}_nL}\colon (\mathscr{F}_nL, \partial) \to (\mathscr{F}_n\tilde{L}, \tilde{\partial}) \qquad \forall n \geq 1.$$
2. We say Φ is a **fibration** iff its linear term $\phi\colon (L, \partial) \to (\tilde{L}, \tilde{\partial})$ induces a surjective map of cochain complexes
$$\phi|_{\mathscr{F}_nL}\colon (\mathscr{F}_nL, \partial) \to (\mathscr{F}_n\tilde{L}, \tilde{\partial}) \qquad \forall n \geq 1.$$
3. We say Φ is an **acyclic fibration** iff Φ is both a weak equivalence and a fibration.

Remark 2 If (L, Q) is a filtered L_∞-algebra, then for each $n \geq 1$, we have the obvious short exact sequence of cochain complexes

$$0 \to \mathscr{F}_{n-1}L/\mathscr{F}_nL \xrightarrow{i_{n-1}} L/\mathscr{F}_nL \xrightarrow{p_n} L/\mathscr{F}_{n-1}L \to 0. \tag{8}$$

It is easy to see that (8) lifts to a sequence of filtered L_∞-algebras, in which all of the algebras in the sequence are nilpotent L_∞-algebras (see Remark 1), and in