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Global Aspects of Complex Geometry

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With 15 Figures

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Preface

Over the period 2000–2006 the Deutsche Forschungsgemeinschaft sponsored a special Schwerpunkt programme, entitled "Global Methods in Complex Geometry".

The articles of this volume grew out of this programme and document some of the scientific activity performed in the realm of the Schwerpunkt. They also aim at giving a broader overview of recent developments in various directions of Complex Geometry such as

- Low-dimensional geometry: surfaces of general type, Fano threefolds, Calabi-Yau threefolds;
- moduli spaces and families of varieties over curves;
- Hodge theory, motivic cohomology and characteristic p-geometry;
- moment maps and group actions on flag manifolds;
- geometry of singular varieties: vector fields, equisingular families and vector bundles;
- geometry of rational curves and pseudo-effective line bundles.

The articles are devoted to a broad spectrum of topics, which range from purely algebraic to complex-analytic aspects of our subject.

The participants of the Schwerpunkt would like to thank the Deutsche Forschungsgemeinschaft for its generous support.

Bayreuth, Essen, Bochum, Hannover, June 2006

Fabrizio Catanese, Hélène Esnault, Alan Huckleberry, Klaus Hulek,
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Contents

Complex Surfaces of General Type: Some Recent Progress

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Introduction

In this article we shall give an overview of some recent developments in the theory of complex algebraic surfaces of general type.

After the *rough* or *Enriques-Kodaira* classification of complex (algebraic) surfaces, dividing compact complex surfaces in four classes according to their Kodaira dimension $-\infty,\ 0,\ 1,\ 2$, the first three classes nowadays are quite well understood, whereas even after decades of very active research on the third class, the class of surfaces *of general type*, there is still a huge number of very hard questions left open. Of course, we made some selection, which is based on the research interest of the authors and we claim in no way completeness of our treatment. We apologize in advance for omitting various very interesting and active areas in the theory of surfaces of general type as well as for not being able to mention all the results and developments which are important in the topics we have chosen.

Complex surfaces of general type come up with certain (topological, birational) invariants, topological as for example the topological Euler number e and the self intersection number of the canonical divisor K^2 of a minimal surface, which are linked by several (in-) equalities. In the first chapter we will summarize the classically known inequalities, which force surfaces of general type in a certain region of the plane having K^2 and e as coordinates, and we shall briefly comment on the so-called *geography* problem, whether,

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given numerical invariants lying in the admissible range, i.e., fulfilling the required inequalities, does there exist a surfaces having these invariants. We shall however more broadly consider the three classical invariants K^2, p_g, q, which determine the other invariants $\chi := 1 - q + p_g, e = 12\chi - K^2$.

An important new inequality, which Severi tried without success to establish, and which has been attacked for many years with partial results by several authors, asserts that a surface of maximal Albanese dimension satisfies the inequality $K^2 \geq 4\chi$. We will report on Pardini's surprisingly simple proof of this so-called Severi's conjecture (cf. [Par05]).

The study of the pluricanonical maps is an essential technique in the classification of surface of general type. The main results concerning the m-canonical maps with $m \geq 3$ go back to an earlier period and we refer to [Cat87b] for a report on them.

We will report in the second chapter on recent developments concerning the bicanonical map; we would like to mention Ciliberto's survey (cf. [Cil97]) on this topic for the state of art ten years ago. Here instead, we combine a discussion of this topic with the closely intertwined problem of classification of surfaces with low values of the numerical invariants.

In the third chapter we report on surfaces of general type with geometric genus p_g equal to four, a class of surfaces whose investigation was started by Federigo Enriques (cf. chapter VIII of his book 'Le superficie algebriche', [Enr49]).

By Gieseker's theorem we know that for fixed K^2 and χ there exists a quasi projective coarse moduli space $\mathcal{M}_{K^2,\chi}$ for the birational equivalence classes of surfaces of general type. It is a very challenging problem to understand the geometry of these moduli spaces even for low values of the invariants. The case $p_g = 4$ is studied via the behaviour of the canonical map. While it is still possible to divide the moduli space into various locally closed strata according to the behaviour of the canonical map, it is very hard to decide how these strata patch together.

Using certain presentations of Gorenstein rings of codimension 4 introduced by M. Reid and D. Dicks, which arrange the defining equations as Pfaffians of certain matrices with many symmetries in such a way that these equations behave well under deformation, it is possible to exhibit explicit deformations, which allow to "connect" certain irreducible components of the moduli space.

Inspired by a construction of A. Beauville of a surface with $K^2 = 8$, $p_g = q = 0$, the second author defined Beauville surfaces as surfaces which are rigid and which admit an unramified covering which is isomorphic to a product of curves of genus at least 2. In this case the moduli space of surfaces orientedly homeomorphic to a given surface consists either of a unique real point, or of a pair of complex conjugate points corresponding to complex conjugate surfaces.

These surfaces, and the more general surfaces isogenous to a product, not only provide cheap counterexamples to the Friedman-Morgan speculation (which will be treated more extensively in the sixth section of this article), but provide also a wide class of surfaces quite manageable in order to test conjectures, and offer also counterexamples to various problems. The ease with which one can handle these surfaces is based on the fact that these surfaces are determined by "discrete" combinatorial data.

Beauville surfaces, their relations to group theory and to Grothendieck's theory of 'Dessins d'enfants' will be discussed in the fourth chapter.

It is a very difficult and very intriguing problem to decide whether two algebraic surfaces, which are not deformation equivalent, are in fact diffeomorphic.

The theory of Lefschetz fibrations provides an algebraic tool to prove that two surfaces are diffeomorphic. By a theorem of Kas (which holds also in the symplectic context) two Lefschetz fibrations are diffeomorphic if and only if their corresponding factorizations of the identity in the mapping class group are equivalent under the equivalence relation generated by Hurwitz moves and by simultaneous conjugation. We outline the theory, which was used with success in [CW04] in chapter five, which we end with a brief report on the status of two very old conjectures by Chisini concerning cuspidal curves and algebraic braids.

As already mentioned before, one of the fundamental problems in the theory of surfaces of general type is to understand their moduli spaces, in particular the connected components which parametrize the deformation equivalence classes of minimal surfaces of general type. By a classical result of Ehresmann, two deformation equivalent algebraic varieties are diffeomorphic. The other direction, i.e., whether two diffeomorphic minimal surfaces of general type are indeed in the same connected component of the moduli space, was an open problem since the eighties. We discuss in the last chapter the various counterexamples to the Friedman-Morgan speculation, who expected a positive answer to the question (unlike the second author, cf. [Kat83]).

Moreover, we briefly report on another equivalence relation introduced by the second author, the so-called quasi étale-deformation (Q.E.D.) equivalence relation, i.e., the equivalence relation generated by birational equivalence, by quasi étale morphisms and by deformation equivalence. For curves and surfaces of special type two varieties are Q.E.D. equivalent if and only if they have the same Kodaira dimension, whereas there are infinitely many surfaces of general type, which are pairwise not Q.E.D. equivalent.

1 Old and New Inequalities

1.1 Invariants of Surfaces

Let X be a compact complex manifold and let Ω^n_X be its canonical bundle, i.e., the line bundle of holomorphic $n-$forms (usually denoted by ω_X, since it is a dualizing sheaf in the sense of Serre duality). A corresponding canonical divisor is usually denoted by K_X.

To X one associates its *canonical ring*

$$R(X) := \oplus_{m \geq 0} H^0(\omega_X^{\otimes m}).$$

The trascendency degree over $\mathbb{C}$ of this ring leads to

- the *Kodaira dimension* $\kappa(X) := tr(R(X)) - 1$,

if $R(X) \neq \mathbb{C}$, otherwise $\kappa(X) := -\infty$. The Kodaira dimension is invariant under deformation (by Siu's theorem [Siu02], generalizing Iitaka's theorem for surfaces) and can assume the values $-\infty, 0, \ldots, n = \dim X$.

Definition 1. *X is said to be* of general type *if the Kodaira dimension is maximal, $\kappa(X) = \dim X$.*

We are interested in the case of *surfaces*, i.e., of manifolds of dimension 2, of general type.

The three principal invariants under deformations for the study of these surfaces are

- the self intersection of the canonical class K_S^2 of a minimal model,
- the geometric genus $p_g := h^0(\omega_X)$ and
- the irregularity $q := h^1(\mathcal{O}_S) = h^0(\Omega^1_S)$.

The equality $h^1(\mathcal{O}_S) = h^0(\Omega^1_S)$ follows by Hodge theory since every algebraic surface is projective.

The invariants we have introduced, with the exception of K_S^2, are not only deformation invariants but also birational invariants.

Definition 2. *A smooth surface S is called* minimal *(or* a minimal model*) iff it does not contain any exceptional curve E of the first kind (i.e. $E \cong \mathbb{P}^1$, $E^2 = -1$).*

Every surface can be obtained by a minimal one (its "minimal model") after a finite sequence of blowing ups of smooth points; this model is moreover unique if $\kappa(S) \geq 0$ (see III.4.4, III.4.5 and III.4.6 of [BHPV04]). Thus, every birational class of surfaces of general type contains exactly one minimal surface, and one classifies surfaces of general type by studying their minimal models. To each minimal surface of general type we will associate its numerical

- type (K_S^2, p_g, q),

a triple of integers given by the three invariants introduced above.

In fact these determine all other classical invariants, as

- the Euler-Poincaré characteristic of the trivial sheaf $\chi(\mathcal{O}_S) = 1 - q + p_g$;
- the topological Euler characteristic $e(S) = c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2$;
- the plurigenera $P_m(S) := h^0(\omega_X^{\otimes m}) = \chi(\mathcal{O}_S) + \binom{m}{2} K_S^2$.

The epression for c_2 is a classical theorem of M. Noether, and the expression for the plurigenera follows by Riemann-Roch and by Mumford's vanishing theorem.

By the theorems on pluricanonical maps (cf. [Bom73]), minimal surfaces S of general type with fixed invariants are birationally mapped to normal surfaces X in a fixed projective space of dimension $P_5(S) - 1$. X is uniquely determined, is called the *canonical model* of S, and is obtained contracting to points all the (-2)-curves of S (curves $E \cong \mathbb{P}^1$, with $E^2 = -2$).

Let us recall Gieseker's theorem

Theorem 1 (Gieseker [Gie77]). *There exists a quasi-projective coarse moduli scheme for canonical models of surfaces of general type S with fixed K_S^2 and $c_2(S)$.*

In particular, we can consider the subscheme $\mathcal{M}_{K_S^2, p_g, q}$ corresponding to minimal surfaces of general type of type (K_S^2, p_g, q). By the above theorem, it is a quasi projective scheme, in particular, it has finitely many irreducible components.

It is a dream ever since to completely describe $\mathcal{M}_{K_S^2, p_g, q}$ for as many types as possible.

1.2 Classical Inequalities and Geography

Obviously the first question is: for which values of (K_S^2, p_g, q) is $\mathcal{M}_{K_S^2, p_g, q}$ non empty?

For example, it is clear that $p_g(S)$ and $q(S)$ are always nonnegative, since they are dimensions of vector spaces.

In fact much more is known. In the following table we collect the well known classical inequalities holding among the invariants of minimal surfaces of general type:

$$\begin{array}{lll}
 & K_S^2 \geq 1 & \qquad\qquad \chi \geq 1 \\
(N) & K_S^2 \geq 2p_g - 4 & \text{or the weaker } K_S^2 \geq 2\chi(\mathcal{O}_S) - 6 \\
(D) & \text{if } q > 0,\ K_S^2 \geq 2p_g & \text{or the weaker if } q > 0,\ K_S^2 \geq 2\chi(\mathcal{O}_S) \\
(MY) & K_S^2 \leq 9\chi &
\end{array}$$

We have labeled by (N)= Noether, (D) = Debarre, (MY) = Miyaoka-Yau the rows, corresponding to the names of the inequalities ([Deb82], [Deb83], [Miy77], [Yau78], see also [BHPV04], chap. 7).

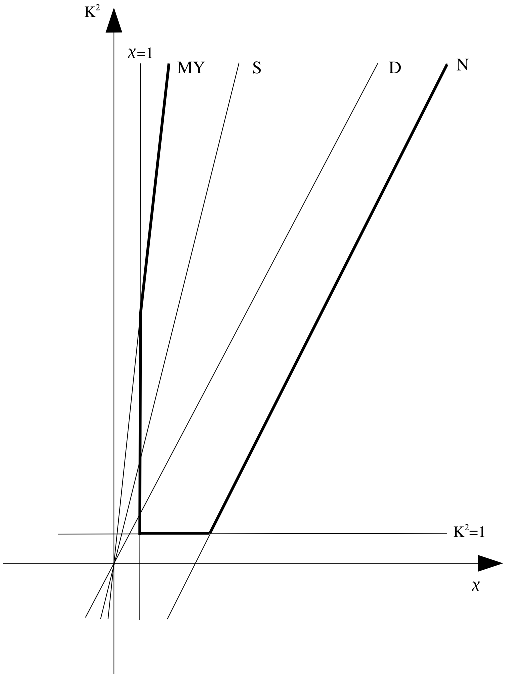

Fig. 1. The geography of minimal surfaces of general type

In figure 1 we have drawn the limit lines (i.e., where equality holds) of the various inequalities in the (χ, K_S^2)-plane.

The above listed inequalities show that the pair of invariants χ, K_S^2 of a minimal surface of general type gives a point with integral coordinates in the convex region limited by the "bold" piecewise linear curve. Moreover, if $q > 0$ this point cannot be at the "right" of the line D.

We drew one more line in our picture, labeled by S. This is the Severi line $K^2 = 4\chi$, i.e., the equality case of the Severi inequality $K^2 \geq 4\chi \Leftrightarrow K^2 \geq \frac{1}{2}e$, which will be discussed in detail at the end of this section.

1.3 Surfaces Fibred over a Curve

An important method for the study of surfaces of general type is to consider relatively minimal fibrations of surfaces over curves $f : S \to B$.

Definition 3. *A* fibration $f : S \to B$ *is a surjective morphism with connected fibres. We are interested in the case of fibrations* of surfaces to curves*, meaning that in this paper* S *and* B *will always be smooth compact complex manifolds of respective dimensions* 2 *and* 1*.*

The fibration is said to be relatively minimal *if* f *does not contract any rational curve of self intersection* -1 *to a point.*

One denotes

- by b the genus of the base curve B;
- by g the genus of a general fibre.

To avoid confusion, let us point out that a fibration is called rational or irrational according to the genus b of the base being 0 or > 0. On the other hand, the genus of the fibration is the genus g of the fibre. For example, if we say f is a genus 2 rational fibration, we intend that $g = 2$ and $b = 0$.

The classical way of saying: a genus b pencil of curves of genus g is however still the most convenient way to describe a fibration.

To a relatively minimal fibration f one associates

- its relative canonical bundle $\omega_{S|B} := \omega_S \otimes f^*(\omega_B^\vee)$ and
- the sheaves $(\forall n \geq 0)$ $V_n := f_*(\omega_{S|B}^{\otimes n})$.

The sheaves V_n are vector bundles (i.e., locally free sheaves) with very nice properties.

Theorem 2 (Fujita [Fuj78a], [Fuj78b]). *The vector bundles V_n are* semi-positive*, i.e., every locally free quotient of it has nonnegative degree.*

To be more precise, V_1 is a direct sum of an ample vector bundle with $q(S) - b$ copies of the trivial bundle and with some undecomposable stable degree 0 vector bundle without global sections. Zucconi [Zuc97] proved moreover that if one of those stable bundles has rank 1, then it is a torsion line bundle.

For $n \geq 2$ we have:

Theorem 3 (Esnault-Viehweg [EV90]). $\forall n \geq 2$ *the vector bundle V_n is ample unless f has constant moduli, which means that all the smooth fibres are isomorphic.*

Since $R^1 f_* \omega_{S|B} = \mathcal{O}_B$ by relative duality, and $R^1 f_* \omega_{S|B}^{\otimes n} = 0$ $\forall n \geq 2$ by the assumption of relative minimality, one can compute the Euler characteristic of V_n by Riemann-Roch, and consequently its degree.

We introduce the following invariants of the fibration f:

- the self intersection of the relative canonical divisor

$$K_f^2 := \omega_{S|B} \cdot \omega_{S|B} = K_S^2 - 8(g-1)(b-1),$$

- the Euler characteristic of the relative canonical divisor

$$\chi_f = \chi(\omega_{S|B}) = \chi(\mathcal{O}_S) - (g-1)(b-1),$$

- its slope $\lambda(f) := K_f^2/\chi_f$.

The slope is clearly defined only for $\chi_f \neq 0$, or equivalently (as we will see soon) if the fibration is not a holomorphic bundle.

The above mentioned computation gives

$$\deg V_n = \chi_f + \frac{n(n-1)}{2} K_f^2$$

and since by Fujita's theorem these numbers are nonnegative this gives the two inequalities $K_f^2 \geq 0$ and $\chi_f \geq 0$ respectively known as Arakelov's inequality (cf. [Ara71]) and Beauville's inequality (cf. [Bea82]).

In fact, we have the following list of inequalities

$$\begin{array}{ll} (A) & K_f^2 \geq 0 \ , \ i.e., \ K_S^2 \geq 8(g-1)(b-1), \\ (B) & \chi_f \geq 0 \ , \ i.e., \ \chi(\mathcal{O}_S) \geq (g-1)(b-1), \\ (ZS) & c_2(S) \geq 4(b-1)(g-1), \\ (NN) & q \leq b+g, \\ (X) & 4 - \frac{4}{g} \leq \lambda(f) \leq 12. \end{array}$$

Here the meaning of the labeling is the following: (A) = Arakelov's inequality, (B) = Beauville' inequality, (X) = Xiao's inequality (also known as slope inequality), (NN) = no name's inequality, (ZS) = Zeuthen-Segre. A proof of those inequalities can be found in [Bea82] with the exception of the slope inequality, proved in [Xia87] (see also [CH88] in the semistable case).

The equality cases of the first 4 inequalities are well described:

- if equality holds in (A), f has constant moduli;
- equality holds in (B) $\Leftrightarrow$ f has constant moduli and is smooth;
- for $g \geq 2$, equality holds in (ZS) $\Leftrightarrow$ f is smooth;
- $q = b + g \Leftrightarrow f$ is birationally equivalent to the projection of a product $B \times F$ to the first factor.

In particular, we see that the slope is defined whenever the fibration is not a holomorphic bundle, since the denominator χ_f vanishes iff equality holds in Beauville's inequality.

An important consequence is the following

Theorem 4 (Beauville). *If X is a minimal surface of general type, then $p_g \geq 2q - 4$. Moreover, if $p_g = 2q - 4$, then S is a product of a curve of genus 2 with a curve of genus $q - 2$.*

Note for later use (see next section) the following

Corollary 1. *If $p_g = q$ (i.e., if $\chi(\mathcal{O}_S) = 1$), then $p_g = q \leq 4$. Moreover, minimal surfaces of general type with $p_g = q = 4$ are exactly the products of two genus 2 curves.*

Proof of theorem 4. The standard wedge product on 1−forms induces a natural map

$$\wedge : \Lambda^2 H^0(\Omega_S^1) \to H^0(\Omega_S^2)$$

Recall that $q = \dim H^0(\Omega^1_S)$, $p_g = \dim H^0(\Omega^2_S)$. Let us assume $p_g \leq 2q - 4$.

By a dimension count, if $p_g \leq 2q - 4$, the projective linear subspace of $\mathbb{P}(\Lambda^2 H^0(\Omega^1_S))$ corresponding to the kernel of the above map must intersect the Plücker embedding of the Grasmannian $G_2(H^0(\Omega^1_S))$ (which has dimension $2q - 4$), and therefore there are two linearly independent $1-$forms ω_1 and ω_2 such that the following holomorphic two form is identically zero: $\omega_1 \wedge \omega_2 \equiv 0$.

By the theorem of Castelnuovo-De Franchis there is a fibration $f : S \to B$ with base of genus $b \geq 2$, and two holomorphic $1-$forms $\alpha_1,\ \alpha_2 \in H^0(\Omega^1_B)$ such that $f^*\alpha_i = \omega_i$. Since S is of general type, also $g \geq 2$.

Then

$$\chi_f \geq 0 \Rightarrow \chi(\mathcal{O}_S) \geq (b-1)(g-1) = (b-2)(g-2) + b + g - 3 \geq q - 3.$$

So we have $1 - q + p_g \geq q - 3 \Leftrightarrow p_g \geq 2q - 4$.

If $p_g = 2q - 4$, all inequalities are equalities and then, since $q = b + g$ and $(b-2)(g-2) = 0$, S is a product of two curves of genus at least 2, and one of the two must have genus exactly 2.

□

1.4 Severi's Inequality

We recall that the Albanese variety $Alb(X)$ of a compact Kähler manifold X is the cokernel of the natural map

$$\int : H_1(X, \mathbb{Z}) \to H^0(\Omega^1(X))^\vee$$

defined by integrating $1-$forms on 1-cycles.

The Albanese morphism

$$\alpha : X \to Alb(X)$$

is defined (up to translations in $Alb(X)$) by fixing a point $p_0 \in X$, and by associating to each point $p \in X$ the class in $Alb(X)$ of $\int_{p_0}^{p}$, where the integral is taken along any path between p_0 and p.

Recall that, if X is projective (as any surface of general type), $Alb(X)$ is an abelian variety (of dimension q).

The Albanese morphism is a powerful tool for studying *irregular* surfaces ($q > 0$) and in particular:

Definition 4. *A variety X is called of* maximal Albanese dimension *if the image of the Albanese morphism has the same dimension as X.*

This is the general case for surfaces, since otherwise the Albanese morphism is a fibration onto a smooth curve of genus q. We see then that for surfaces maximal Albanese dimension is equivalent to the non existence of a genus q pencil.

We can now state the theorem known as *Severi's inequality*

Theorem 5 (Pardini [Par05]). *If S is a smooth complex minimal surface of maximal Albanese dimension, then $K_S^2 \geq 4\chi$.*

This theorem was proved only very recently by R. Pardini, but it has a long story, which we briefly sketch in the following.

Severi's Conjecture

The inequality takes its name from F. Severi, since he was the first to claim the result in the 30's [Sev32].

His proof turned out to be wrong, as was pointed out in [Cat83], since it was based on the assertion that a surface with irregularity q either contains an irrational genus q fibration, or the sections of $H^0(\Omega_S^1)$ have no common zero. Counterexamples were given in [Cat84], where there were constructed bidouble covers $S \to X$ of any algebraic surface with, among other properties, $q(S) = q(X)$. If X has no irrational pencils, since the Albanese map of S factors through the cover, then also S has no irrational pencils. But any ramification point of the cover is a base point for $H^0(\Omega_S^1)$.

Therefore Severi's inequality was posed in [Cat83] as *Severi's conjecture*, a conjecture on surfaces of general type, since for surfaces with $\kappa(S) \leq 1$ it is a straightforward consequence of the Enriques-Kodaira classification. It had also been posed as a conjecture by M. Reid (conj. 4 in [Rei79]) who proved the weaker $K_S^2 \geq 3\chi$.

Proofs in Special Cases

In the 80's, Xiao's work on surfaces fibred over a curve was mainly motivated by Severi's conjecture. In [Xia87] he proved the slope inequality and Severi's conjecture for surfaces having an irrational pencil.

In the 90's Konno [Kon96] proved the conjecture in the special case of *even* surfaces, i.e., surfaces whose canonical class is $2-$ divisible in the Picard group.

Finally, at the end of the 90's, Manetti [Man03] could prove the inequality for surfaces of general type whose canonical bundle is ample.

Manetti's Proof

Manetti considers the tautological line bundle L of the $\mathbb{P}^1-$bundle $\pi : \mathbb{P}(\Omega_S^1) \to S$; standard computations give

$$3(K_S^2 - 4\chi) = L^2 \cdot (L + \pi^* K_S).$$

Then, using the fact that Ω_S^1 is generically globally generated, he can write the right hand side of the above equation as $2K_S E + (L + \pi^* K_S)C$ for an effective $1-$cycle C in $\mathbb{P}(\Omega_S^1)$, and where E is the maximal effective divisor in S such that $h^0(\Omega^1_S(-E)) = h^0(\Omega^1_S)$. Thus the problem is reduced to the nonnegativity of the term $(L + \pi^* K_S)C$. This is obvious if $\Omega^1(K_S)$ is nef, but in general it requires a very detailed and complicated analysis of the $1-$cycle C.

In fact, Pardini's proof not only does not require the ampleness of the canonical divisor, but is much easier than Manetti's.

We should however mention that Manetti's argument leads to a very detailed description of the equality case, showing that a surface of general type of maximal Albanese dimension, lying on the Severi line ($K^2 = 4\chi$), and having ample canonical class, has irregularity $q = 2$ and is a double cover of a principally polarized Abelian surface, branched on a divisor D algebraically equivalent to 2Θ.

Up to now there is no similar description of the limit case without the assumption that K be ample.

Pardini's Proof

Pardini's idea is to construct a sequence of genus g_d fibrations $f_d : Y_d \to \mathbb{P}^1$ such that

$$\lim_{d\to\infty} g_d = +\infty \text{ and } \lim_{d\to\infty} \lambda(f_d) = K_S^2/\chi(\mathcal{O}_S).$$

Then, taking the limit of the left-hand side of the slope inequality, one gets the desired inequality $K_S^2/\chi(\mathcal{O}_S) \geq 4$.

To construct these fibrations, she considers the Cartesian diagram

$$\begin{tikzcd} S' \arrow[r, "p"] \arrow[d, "\alpha'"'] & S \arrow[d, "\alpha"] \\ Alb(S) \arrow[r, "\cdot d"] & Alb(S) \end{tikzcd} ,$$

where $d : Alb(S) \to Alb(S)$ is multiplication by d.

One observes that S' is connected since we have a surjection $\pi_1(S) \to \pi_1(Alb(S)) = H_1(S, \mathbb{Z})$.

Clearly, $K_{S'}^2 = d^{2q} K_S^2$, $\chi(\mathcal{O}_{S'}) = d^{2q}\chi(\mathcal{O}_S)$.

Let L be a very ample divisor on $Alb(S)$ and set $H := \alpha^* L$, $H' := \alpha'^* L$. Then $p^*H \sim_{num} d^2 H'$, whence ${H'}^2 = d^{2q-4} H^2$ and $K_{S'} H' = d^{2q-2} K_S H$.

Let now $D_1,\ D_2 \in |H'|$ be two general curves and define $C_1 := D_1 + D_2 \in |H_1 + H_2|$. Moreover, choose $C_2 \in |2H'|$ sufficiently general such that C_1 and C_2 intersect transversally. C_1 and C_2 define a rational pencil $f_d : Y_d \to \mathbb{P}^1$, where Y_d is the blow up of S' at $C_1 \cap C_2$. The singular fibre induced by C_1 guarantees that f_d is not a holomorphic bundle, whence the slope $\lambda(f_d)$ is well defined.

For the invariants of Y_d we get

$$K_{Y_d}^2 = K_{S'}^2 - 4{H'}^2 = d^{2q} K_S^2 - 4d^{2q-4} H^2$$

$$\chi(Y_d) = \chi(S') = d^{2q}\chi(S)$$

$$g_d = 1 + K_{S'}H' + 2H'^2 = 1 + d^{2q-2}K_SH + 2d^{2q-4}H^2$$

and therefore $\lim_d g_d = +\infty$ as requested.

Moreover, $K^2_{f_d} = K^2_{Y_d} + 8(g_d - 1)$ and $\chi(f_d) = \chi(Y_d) + (g_d - 1)$ and we see that both invariants are polynomials in d of degree $2q$, whose leading terms are respectively K_S^2 and $\chi(\mathcal{O}_S)$. In particular,

$$\lim_{d\to\infty} \lambda(f_d) = \lim_{d\to\infty} K_f^2/\chi_f = K_S^2/\chi(\mathcal{O}_S)$$

2 Surfaces with $\chi = 1$ and the Bicanonical Map

2.1 The Bicanonical Map

The behaviour of the $m - th$ canonical map of S (i.e., the rational map associated to $|mK_S|$) is an essential tool in the theory of surfaces of general type.

As we mentioned in the introduction, the cases where $m \geq 3$ are solved since long (cf. the survey [Cat87b]).

The canonical map ($m = 1$) was first studied by Beauville in [Bea79], but there remain still many unresolved questions.

The case $m = 2$ was particularly studied in the last years, and we have the impression that we are very close to a complete understanding.

In order to fix the starting point, we summarize the results of several authors ([Fra91], [Rei88], [Cat81], [CC91], [CC93], [Xia85a]) in the following

Theorem 6. *Let S be a minimal surface of general type. Then*

- *the bicanonical map is generically finite unless $p_g = 0$ and $K^2 = 1$;*
- *if $K_S^2 \geq 5$ or $p_g \geq 1$, the bicanonical map is a morphism.*

Note that, if $p_g = 0$ and $K^2 = 1$, then $P_2 = 2$ and the bicanonical map is a rational ($b = 0$) fibration. In all known examples this is a genus 4 fibration, although at the moment it is only proven that its genus is 3 or 4 (see [CP05]).

These surfaces are usually called *numerical Godeaux surfaces*. Numerical Godeaux surfaces with torsion (in the Picard group) of cardinality at least 3 are classified in [Rei78], a family with torsion $\mathbb{Z}/2$ was constructed in [Bar84]. Up to last year only sporadic examples of surfaces with trivial torsion were known, but recently Schreyer [Sch05] has announced the construction of a family of the expected dimension (= 8) using a new approach based on homological algebra.

The above theorem says that in all other cases the bicanonical map maps S to a surface, and it is a morphism (except for finitely many families).

In the last years many people studied the degree of this map, in particular, trying to classify the surfaces such that the bicanonical map is not birational.

The Standard Case

It is well known that the bicanonical map of a smooth curve of general type (i.e., of genus at least 2) fails to be birational if and only if the curve has genus 2.

This exception induces a *"standard exception"* to the birationality of the bicanonical map in dimension 2.

Definition 5. *A surface S of general type* presents the standard case *(for the non birationality of the bicanonical map) if there exists a dominant rational map onto a curve $f : S \dashrightarrow B$ whose general fibre is irreducible of genus* 2.

In fact, if S presents the standard case, then the restriction of the bicanonical map of S to a general fibre factors through the bicanonical map of the fibre itself and therefore cannot be birational.

The subschemes of the moduli space corresponding to surfaces presenting the standard case are not empty for infinitely many moduli spaces, and Persson [Per81] constructed many interesting surfaces considering double covers of ruled surfaces branched on relative sextics, thereby filling a big region of the convex region represented in figure 1.

Bombieri ([Bom73]) showed that the standard case gives almost all exceptions to the birationality of the bicanonical map. More precisely, combining his results with those of Reider ([Rei88]) we know now that a minimal surface of general type with $K^2 \geq 10$ either presents the standard case, or its bicanonical map is birational. In particular, the exceptions to the birationality of the bicanonical map not presenting the standard case belong to finitely many families and many authors are trying since then to classify them (see [Cil97] for a survey updated until '96).

Du Val's Double Planes

In the same paper [Bom73] Bombieri constructed a surface of type $(K^2, p_g, q) = (9, 6, 0)$ not presenting the standard case. His example can be easily described as a hypersurface F_{14} of degree 14 in the weighted projective space $\mathbb{P}(1, 1, 2, 7)$, and from this description it follows rightaway that it bicanonical map is a double cover of the weighted projective space $\mathbb{P}(1, 1, 2)$ (isomorphic to a quadric cone in $\mathbb{P}^3$). This example is in fact a special case of a more general "geometric" situation studied first by du Val.

Let S be a minimal regular surface with $p_g \geq 2$, such that the general canonical curve is irreducible, smooth and hyperelliptic. Since the restriction of the bicanonical map φ_{2K} to a canonical curve factors through the canonical map of the curve itself, φ_{2K} cannot be birational.

Du Val [Duv52] gave a list of such surfaces obtained as double covers of rational surfaces. A generalization (see [CML00], [Bor03]) leads to the following:

Definition 6 (du Val's double planes). *A smooth surface S is* a du Val double plane *if it is birational to*

$\mathcal{D}$) *a double cover of $\mathbb{P}^2$ branched over a smooth curve of degree 8;*

$\mathcal{D}_n$) *a double cover of $\mathbb{P}^2$ branched over the union of a curve of degree $10+n$, $n \leq 6$, with n distinct lines through a point p, such that the essential singularities of the branch curve are the following:*
  - *p is a singular point of multiplicity $2n+2$,*
  - *there is a singular point of type $[5,5]$ on each line,*
  - *possibly there are some quadruple points and some points of type $[3,3]$;*

$\mathcal{B}$) *a double cover of the Hirzebruch surface $\mathbb{F}_2$ whose branch curve can be decomposed as $C_0 + G'$, $G' \in |7C_0 + 14\Gamma|$ (where $|\Gamma|$ is the ruling of $\mathbb{F}_2$ and C_0 is the section with self intersection -2), whose only essential singularities are $[3,3]$ points that are tangent to a fibre.*

Recall that a singular point of type $[d,d]$ is a singular point of multiplicity d having a further singular point of multiplicity d infinitely near to the first one. In other words, if we blow-up the singular point, the strict transform of the curve has one more singular point of multiplicity d lying on the exceptional divisor.

Remark 1. In the definition of the du Val's double planes of type $\mathcal{D}_n$ and $\mathcal{B}$ we only care about the essential singularities of the branch curve (as usual in the theory of double covers) since adding a simple singularity to the branch curve does not affect the properties of the resulting surface we are interested in.

On the contrary, in the definition of the double planes of type $\mathcal{D}$, we assume the branch curve to be smooth. In fact, if we take a double cover of $\mathbb{P}^2$ branched over a curve of degree 8 with a double point, the pull back of the pencil of lines through this point to the surface defines a pencil of curves of genus 2 (through a singular point of the surface), so the resulting surface presents the standard case.

Note that this example shows that one can degenerate surfaces presenting a nonstandard case to surfaces presenting the standard case (just take a family of smooth plane curves of degree 8 degenerating to a singular one and consider the corresponding family of double covers).

Borrelli proved that this list is "complete" in the following sense

Theorem 7 (Borrelli [Bor03]). *If S is a minimal surface of general type, not presenting the standard case, whose bicanonical map factors through a degree 2 rational map onto a rational or ruled surface: then S is the smooth minimal model of a du Val double plane. In particular, either $q = 0$ or $p_g = q = 1$.*

The "Classification"

The standard case and the du Val's double planes do not give all possible surfaces of general type with nonbirational bicanonical map, but the remaining exceptions are really few.

What is known about these is summarized in the following

Theorem 8. *Let S be a smooth minimal surface of general type whose bicanonical map φ_{2K} is not birational. Then one of the following cases occur:*

i) *S presents the standard case;*
ii) *S is the smooth minimal model of a du Val double plane;*
iii) *S is a surface of type $(1,1,0)$ (for these automatically $\deg \varphi_{2K} = 4$ and S is a complete intersection of two sextics in $\mathbb{P}(1,2,2,3,3)$);*
iv) *S is of type $(2,1,0)$ with Picard group having torsion $\mathbb{Z}/2\mathbb{Z}$ (for these automatically $\deg \varphi_{2K} = 4$ and its double cover corresponding to the torsion class is a complete intersection of two quartics in $\mathbb{P}(1,1,1,2,2)$);*
v) *φ_{2K} is $2:1$ onto a $K3$ surface and $p_g = 1$, $q = 0$, $2 \leq K^2 \leq 8$;*
vi) *S is of type $(6,3,3)$ or of type $(4,2,2)$ (for these both cases automatically φ_{2K} has degree 2 and we have a nonstandard case)*
vii) *S has $p_g = q \leq 1$.*

All these cases with the exception of $p_g = q \leq 1$ are now rather clear.

The history of this theorem is rather complicated and combines the efforts of several authors. We try to reconstruct its more important steps here, giving some more details on each class.

One of the first results in this direction is due to Xiao Gang [Xia90], giving, in the nonstandard case, and under the assumption that the degree d of the bicanonical map is at least 3, a list of the possible values of d and of the possible places in the Enriques classification of the bicanonical image Σ.

In 1997, Ciliberto, Francia and Mendes Lopes [CFML97] gave a complete classification of the case $p_g \geq 4$, essentially confirming du Val's list.

Then, Ciliberto and Mendes Lopes, with contributions of the second author and Borrelli, worked in the next years to extend the classification to $p_g \geq 2$ (see [CCML98], [CML00], [CML02a], [CML02b], [Bor02]). The case $p_g = 1$ and $q = 0$, giving cases *iii*), *iv*) and *v*) is classified in [Bor03].

In fact, cases *iii*) and *iv*) resulted already from the analysis of [Xia90] where it is proven that, if $\deg \varphi_{2K} \geq 3$, then either S is of type $(1,1,0)$ or of type $(2,1,0)$, or with $p_g = q \leq 2$.

The description given in *iii*) and *iv*) of the first two cases comes from the papers [Cat79] and [CD89], where all surfaces of respective types $(1,1,0)$ and $(2,1,0)$ are classified. In particular, it is shown that all surfaces of type $(1,1,0)$ are as in *iii*).

Remark 2. Surfaces of type $(2,1,0)$ without torsion in homology, also sometimes called Catanese-Debarre surfaces, offer the following interesting phenomenon: there is an irreducible component of the moduli space such that

1) for the general surface the bicanonical map is birational, while there are subvarieties for which the bicanonical map can respectively be

2) of degree 2 onto a K3- quartic surface,

3) of degree 2 onto a rational quartic surface,

4) of degree 4 onto a smooth quadric surface.

The surfaces in *v)* are usually called Todorov surfaces, since they were introduced in [Tod81]. The subspaces of the moduli spaces corresponding to them is described in [Mor88].

Finally, the largely open case *vii)* is very strongly related with the problem, of independent interest, of the classication of surfaces of general type with $p_g = q$.

We shall describe in the next subsection what is currently known on these surfaces, showing in particular that we have a very precise description of the two cases in *vi)*.

2.2 Surfaces with $p_g = q$

These are the surfaces corresponding to the "vertical" piece of the bold line in figure 1. In particular, $1 \leq K^2 \leq 9$.

Surfaces with $p_g = q \geq 4$

This case is clear, by corollary 1 of Beauville's theorem 4. If $p_g = q \geq 4$, then S is a product of two genus 2 curves and $p_g = q = 4$. We recall that then $K^2 = 8$ and clearly the bicanonical map has degree 4, and we have a standard case.

Surfaces with $p_g = q = 3$

These surfaces have been first studied in [CCML98], and a complete classification has been recently achieved independently by Pirola [Pir02] and Hacon-Pardini [HP02].

The result is the following

Theorem 9. *A minimal surface of general type with $p_g = q = 3$ has $K^2 = 6$ or $K^2 = 8$ and, more precisely,*

- *if $K^2 = 6$, S is the symmetric square of a genus 3 curve;*
- *otherwise $S = C_2 \times C_3/\tau$, where C_g denotes a curve of genus g and τ is an involution of product type acting on C_2 as an elliptic involution (i.e., with elliptic quotient), and on C_3 as a fixed point free involution.*

In particular, the moduli space of minimal surfaces of general type with $p_g = q = 3$ is the disjoint union of $\mathcal{M}_{6,3,3}$ and $\mathcal{M}_{8,3,3}$, which are both irreducible of respective dimension 6 and 5.

We sketch the idea of the proof.

By Debarre's inequality (in the "stronger" form: $q > 0 \Rightarrow K_S^2 \geq 2p_g$), $p_g = q = 3$ implies $K^2 \geq 6$.

As in the proof of Beauville's theorem, consider now the map

$$\wedge : \Lambda^2(H^0(\Omega^1_S)) \to H^0(\Omega^2_S).$$

Since $p_g = q = 3$, it is a linear map between two three dimensional spaces. If this is not an isomorphism, then (since every vector in $\Lambda^2\mathbb{C}^3$ is decomposable) there are two nontrivial $1-$forms ω_1 and ω_2 with $\omega_1 \wedge \omega_2 \equiv 0$. This yields then (by Castelnuovo-De Franchis) a pencil $f : S \to B$ with $b, g \geq 2$.

Then, by Beauville's inequality, $b = g = 2$ and the fibration is a holomorphic bundle (this forces $K_S^2 = 8$). Therefore f is induced by a map $\pi_1(B) \to Aut(F)$ (where F is a smooth fibre), whose kernel induces an unramified cover $\varphi : C \to B$ Galois with group G.

Since $q = 3$, the quotient of F by the group G has genus 1. By Hurwitz's formula one easily sees that, if $\phi : F \to F/G$ is branched in one point, then $|G| \leq 4$, hence G is Abelian, contradicting that ϕ is ramified. Again Hurwitz's formula shows that ϕ is branched in 2 points and $G \cong \mathbb{Z}/2$.

Otherwise $\wedge$ is an isomorphism, and therefore S does not have any pencil $f : S \to B$ with $b \geq 2$. In particular $\alpha(S)$ is a surface, a divisor Θ in $Alb(S)$. Pirola noticed that Θ must be ample, else it would have an elliptic fibration and therefore an irrational pencil with base of genus $b \geq 2$.

This implies, by Lefschetz's hyperplane theorem, that the induced map $H^1(\Omega^1_{Alb(S)}) \to H^1(\Omega^1_S)$ is injective: since, for any class $\eta \in H^1(\Omega^1_{Alb(S)})$ $\int c_1(\Theta) \wedge \eta \wedge \bar{\eta} > 0$.

In particular, $h^1(\Omega^1_S) \geq 9$ and this (since by Hodge theory $12\chi - K^2 = c_2 = 2p_g - 4q + 2 + h^1(\Omega^1)$) implies $K_S^2 \leq 7$.

The case $K_S^2 = 6$ was already settled in [CCML98], where it is first shown that the degree of the scheme of base points is $K^2 - 6$, and then that in the case $K_S^2 = 6$ α is an embedding. More precisely it is shown that its image is a theta divisor in a principally polarized abelian threefold and therefore S is the symmetric square of a genus 3 curve.

What remains to prove is $\mathcal{M}_{7,3,3} = \emptyset$, and this is done in [Pir02] by a careful study of the paracanonical system.

The fact that the bicanonical map has degree 2 is an easy consequence of the adjunction formula by which K_S is the pull back of Θ, and of the fact that the sections of $\mathcal{O}_A(2\Theta)$ are invariant, as well as Θ, for the symmetry of A sending $x \to -x$.

Surfaces with $p_g = q = 2$

This case is still far from being classified. Ciliberto and Mendes Lopes [CML02a] classified all surfaces with $p_g = q = 2$ and non-birational bicanonical map (not presenting the standard case). Their result, corresponding to the subcase $(4, 2, 2)$ of case vi) of theorem 8, is the following

Theorem 10. *If S is a minimal surface of general type with $p_g = q = 2$ and non-birational bicanonical map not presenting the standard case, then S is a double cover of a principally polarized abelian surface (A, Θ), with Θ*

irreducible. The double cover $S \to A$ is branched along a divisor $B \in |2\Theta|$, having at most double points. In particular $K_S^2 = 4$.

Note that, again by Debarre's inequality, $p_g = q = 2 \Rightarrow K^2 \geq 4$, so Ciliberto and Mendes Lopes' surfaces belong to the limit case. Their theorem solves completely the problem of the non birationality of the bicanonical map in this case, but of course a complete classification of minimal surfaces of general type with $p_g = q = 2$ would be interesting by itself.

Results in this direction have been recently obtained by F. Zucconi; to explain them we need to give the following definition.

Definition 7. *A surface S is said to be isogenous to a (higher) product if S admits an unramified finite covering which is biholomorphic to a product of two curves of respective genera at least* 2.

We have already seen surfaces isogeneous to a product in our analysis of surfaces with $p_g = q$, namely all surfaces in $\mathcal{M}_{8,4,4}$ and all surfaces in $\mathcal{M}_{8,3,3}$.

Zucconi's theorem is the following

Theorem 11 (2.9 in [Zuc03]). *There are two classes of minimal surfaces of general type with $p_g = q = 2$ whose Albanese image is a surface and having an irrational pencil, and they are both isogenous to a higher product.*

More precisely, either they have a double cover which is a product of two genus 2 *curves or they are a quotient of the product of two genus* 3 *curves by an action of $\mathbb{Z}/2\mathbb{Z}$.*

In both cases Zucconi describes precisely the group action as a diagonal action induced by actions on the two curves. The interested reader will find all details in Zucconi's paper.

Zucconi managed also to remove the hypothesis on the Albanese map, by use of a special class of surfaces isogenous to a higher product, the generalized hyperelliptic surfaces introduced in [Cat00].

Definition 8. *Let C_1 and C_2 be two smooth curves, G a finite group with two injections respectively in $Aut(C_1)$ and $Aut(C_2)$. Then the quotient surface $S = C_1 \times C_2/G$ by the diagonal action is said to be a generalized hyperelliptic surface if*

- *the projection $C_1 \to C_1/G$ is unramified;*
- *C_2/G is rational.*

Then Zucconi proved

Theorem 12. *If S has $p_g = q = 2$, and the image of the Albanese map is a curve, then S is a generalized hyperelliptic surface.*

What remains to be classified is the class of surfaces with $p_g = q = 2$ having no irrational pencils.

Chen and Hacon, in a preprint, constructed an example of surfaces with $p_g = q = 2$, $K^2 = 5$ and Albanese morphism of degree 3.

Surfaces with $p_g = q = 1$

In this case Debarre's inequality gives only $K^2 \geq 2$.

The Albanese morphism is a map onto an elliptic curve, in particular, all these surfaces have a fibration with base of genus $b = 1$. We summarize in the following statement what is known about these surfaces.

Theorem 13.

- $\mathcal{M}_{2,1,1}$ *is unirational (by this we mean: irreducible and unirational) of dimension* 7*. The Albanese map of all these surfaces is a genus* 2 *fibration.*
- $\mathcal{M}_{3,1,1}$ *has* 4 *connected components, all unirational of dimension* 5*. The Albanese map is a genus* 3 *fibration for the surfaces in one of those components, and a genus* 2 *fibration in all other cases.*
- $\mathcal{M}_{4,1,1}$*,* $\mathcal{M}_{5,1,1}$ *and* $\mathcal{M}_{8,1,1}$ *are non empty.*

Actually much more can be said, and we try to be more precise in the following.

First the most mysterious cases. It remains unsettled the existence of surfaces of general type with $p_g = q = 1$ and $K^2 = 6, 7, 9$. In a recent preprint by Rito appears the construction of a surface with $p_g = q = 1$ and $K^2 = 6$ as a double cover of a Kummer surface modifying slightly (i.e., adding a singular point to the branch curve) Todorov's construction of a surface with $p_g = 1, q = 0$ and $K^2 = 8$ in [Tod81]. Its construction makes use of the computer program MAGMA (to find a branch curve with the right singularities).

Second, the "partially understood" cases. Examples of surfaces with $p_g = q = 1$ and $K^2 = 4, 5$ were constructed by the second author as bidouble covers in [Cat99]. In both cases the Albanese map turns out to be a genus 2 fibration, so they present the standard case. The case $K^2 = 8$ was studied by Polizzi [Pol06], who considered the cases of surfaces having bicanonical map of degree 2. He could prove that all these surfaces are isogenous to a product and that they form three components of the moduli space, one of dimension 5 and two of dimension 4. All these surfaces do not contain any genus 2 pencil and they are in fact du Val double planes.

Finally, the cases $K^2 = 2, 3$ are completely classified.

The first to be settled was $K^2 = 2$, done by the second author in [Cat81], representing all those surfaces as double covers of the symmetric square of their Albanese curve.

The case $K^2 = 3$ was first studied in [CC91] where it was shown, among other things, that the Albanese map could be either a genus 2 or a genus 3 fibration. The case $g = 3$ was then classified in [CC93], showing that it gives a unirational family of dimension 5.

Note that, if there is surface with $p_g = q = 1$, $K^2 \leq 3$ and nonbirational bicanonical map not presenting the standard case, it must belong to this family.

The question whether such a surface exists is still open. Recently Polizzi [Pol05] has shown that a general surface in this component has birational bicanonical map, but this is not true for all of them, since Xiao [Xia85b] has found a subfamily of dimension 1 having a genus 2 pencil.

The classification of the case $K^2 = 3$ was completed in [CP05] classifying all those surfaces having Albanese fibres of genus 2.

The main tool for this classification is a new method for studying fibrations $f : S \to B$ of genus 2, and fibrations of genus 3 with general fibre non hyperelliptic, basically giving *generators* and *relations* of their relative canonical algebra $\mathcal{R}(f) = \oplus V_n$, seen as a sheaf of algebras over B.

Let us recall the vector bundles V_n introduced in the previous section as $V_n = f_* \omega_{S|B}^{\otimes n}$. Roughly speaking then, $\mathcal{R}(f)$ is a bundle whose fibres are the canonical rings of the fibres of f.

We state here only the theorem for genus 2 fibrations, since it is the one used in order to complete this classification.

Theorem 14. *A genus* 2 *fibration* $f : S \to B$ *is determined by the following* 5 *data*

- *the base curve* B*;*
- *the rank* 2 *vector bundle* $V_1 := f_* \omega_{S|B}$ *over* B*;*
- *an effective divisor* τ *on* B*;*
- *a class* $\xi \in Ext^1_{\mathcal{O}_B}(S^2(V_1), \mathcal{O}_\tau)/(Aut_{\mathcal{O}_B}(\mathcal{O}_\tau)$ *yielding* V_2*;*
- *letting* $\mathcal{A}$ *be the subring of the relative canonical algebra generated by* V_2*,* V_3^+ *the* $(+1)$ *eigenbundle for the hyperelliptic involution on the fibres, and defining* $\tilde{\mathcal{A}}_6 := \mathcal{H}om((V_3^+)^2, \mathcal{A}_6)$ *(where* $\mathcal{A}_6$ *is the image in* $\mathcal{A}$ *of* $S^3(V_2)$*), the last datum is an element* $w \in \mathbb{P}(H^0(\tilde{\mathcal{A}}_6))$*.*

Moreover, $\deg V_1 = \chi(\mathcal{O}_S) - (b-1)$*,* $\deg \tau = K_S^2 - 2\chi(\mathcal{O}_S) - 10(b-1)$*.*

We want to explain here the geometry behind this theorem, which at a first glance can appear slightly technical.

The vector bundles V_n yield the degree n part of the canonical ring of each fibre. So each of these vector bundles induces a rational map, the *relative n-canonical map*, from S to the corresponding projective bundle $\mathbb{P}(V_n)$, mapping each fibre via its n-canonical map.

The multiplication map of degree 1 forms give a morphism of sheaves $S^2(V_1) \to V_2$ which fits into an exact sequence

$$0 \to S^2(V_1) \to V_2 \to \mathcal{O}_\tau \to 0$$

for an effective divisor τ on B supported on the image of the "bad" fibres (those which are not 2-connected, i.e., the fibres that can be decomposed as $A + B$ with A, B effective divisors such that $A \cdot B = 1$).

ξ is the class of this extension. Therefore ξ yields V_2 and determines the relative bicanonical map. Since the bicanonical map of a genus 2 curve is a double cover of a conic (branched in 6 points), this map has degree 2 onto a conic subbundle $\mathcal{C}$ of the $\mathbb{P}^2$-bundle $\mathbb{P}(V_2)$. We have $\mathcal{C} = Proj(\mathcal{A})$.

In [CP05] it is proven that the relative bicanonical map is a morphism contracting at most some rational curves with self-intersection (-2), which implies that the branch curve has no "essential" singularities.

In fact the 5^{th} datum w determines the branch curve $div_{\mathcal{C}}(w)$. In fact, sections of $S^3(V_2)$, or of a twist of it, are "equations" of divisors in $\mathbb{P}(V_2)$ which cut a cubic curve on each fibre. Taking the quotient by the subsheaf corresponding to the equations vanishing on the conic bundle $\mathcal{C}$, one gets an equation for a divisor on the conic bundle, which cuts 6 points (intersection of a cubic and a conic) on a general fibre, and gives our branch curve.

Let us come back to the case $p_g = q = 1$, $K^2 = 3$ and $g = 2$. One needs to construct a suitable genus 2 fibration over an elliptic curve, with, in the sense of theorem 14, $\deg V_1 = \deg \tau = 1$. This is done in [CP05], by studying vector bundles on elliptic curves, and three different families are found.

Let us finally mention that $\mathbb{P}(V_1)$ is the symmetric square of B. In fact, our double cover is birational to a double cover of it. The behaviour of this double cover was described in [CC91], characterising these surfaces as double covers of the symmetric product of an elliptic curve with branch locus belonging to a certain algebraic system with prescribed singularities.

This new method shows then, rather surprisingly, that this algebraic system is not connected.

Surfaces with $p_g = q = 0$

The class of surfaces with $p_g = q = 0$ is one of the most complicated and intriguing classes of surfaces of general type. By the standard inequalities we have: $1 \leq K_S^2 \leq 9$.

We have already mentioned the case $K^2 = 1$, of the numerical Godeaux surfaces, the only case for which the bicanonical map is not finite, so let us restrict to $K_S^2 \geq 2$.

These surfaces are very far from being classified. From the point of view of the bicanonical system, this case was object of an intensive analysis by Mendes Lopes and Pardini in the last years.

What it is known on the degree of the bicanonical map can be summarized in the following

Theorem 15 ([MLP05], [MLP02]). *Let S be a surface with $p_g = q = 0$. Then*

- *if* $K^2 = 9 \Rightarrow \deg \varphi_{2K} = 1$*,*
- *if* $K^2 = 7, 8 \Rightarrow \deg \varphi_{2K} = 1$ *or* 2*,*
- *if* $K^2 = 5, 6 \Rightarrow \deg \varphi_{2K} = 1$*,* 2 *or* 4*,*
- *if* $K^2 = 3, 4 \Rightarrow \deg \varphi_{2K} \leq 5$ *and if moreover* φ_{2K} *is a morphism, then* $\deg \varphi_{2K} = 1$*,* 2 *or* 4*,*

- *if* $K^2 = 2$ *then obviously, since the image of the bicanonical map is* $\mathbb{P}^2$, *then the bicanonical map is non birational, and obviously we have:* $\deg \varphi_{2K} \le 8$, *equality holding if and only if* φ_{2K} *is a morphism.*

Let us recall that, by Reider's theorem, the bicanonical map is a morphism as soon as $K^2 \ge 5$. In fact the bicanonical map of all known examples of surfaces with $p_g = q = 0$ and $K^2 \ge 2$ is a morphism. So one could suspect[4] that the bicanonical map is always a morphism whenever $K^2 \ge 2$.

Langer ([Lan00]) has proven that the bicanonical map of a minimal surface of general type with $p_g = q = 0$ and $K_S^2 = 4$ has no fixed part.

Mendes Lopes and Pardini gave also a description of some of these surfaces having non birational bicanonical map, in particular for $K^2 \ge 6$. We summarize here some of their results

Theorem 16 ([MLP03], [MLP01], [MLP04a], [MLP04b]). *Let* S *be a minimal surface of general type with* $p_g = q = 0$ *whose bicanonical map is not birational. Then the image of the bicanonical map is a rational surface unless* $K^2 = 3$, $\deg \varphi_{2K} = 2$, *and the image is an Enriques sextic. These last surfaces form an irreducible and unirational family of dimension* 6 *of the moduli space.*

Moreover,

- *if* $K^2 = 8$, S *has an isotrivial genus* 3 *rational fibration whose general fibre is hyperelliptic with* 6 *double fibres;*
- *if* $K^2 = 7$, S *has a genus* 3 *rational fibration whose general fibre is hyperelliptic with* 5 *double fibres and a fibre with reducible support, consisting of two components;*
- *if* $K^2 = 6$ *and* $\deg \varphi_{2K} = 2$, S *has a genus* 3 *rational fibration whose general fibre is hyperelliptic with* 4 *or* 5 *double fibres;*
- *if* $K^2 = 6$ *and* $\deg \varphi_{2K} = 4$, S *is a Burniat surface.*

Remark 3. Surfaces with $p_g = q = 0$ and $K^2 = 3, 4$ were constructed by Keum ([Keu88]) and Naie ([Nai94]). For $K^2 = 3$ the degree of the bicanonical map can be equal to 2 and to 4, and it is an open question if it can be birational.

Concerning the classification of surfaces with $p_g = q = 0$ there has been recent progress.

We would like to mention that in [BCG05b] a complete classification of surfaces with $p_g = q = 0$ isogenous to a product (this forces $K^2 = 8$) is given. All the known surfaces with $K^2 = 8$, $p_g = q = 0$ have the bidisk as universal covering.

In the case $p_g = q = 0$ and $K^2 = 9$ all the surfaces in question are, by Yau's theorem ([Yau77] and [Yau78]), quotient of the complex unit ball in $\mathbb{C}^2$ by a discrete group Γ acting freely. The first effective example of such

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[4] ADDED IN PROOF: Mendes Lopes and Pardini give in math.AG/0602633 an example of surfaces with $K^2 = 2$, $p_g = q = 0$ such that $|2K_S|$ has base points.

surfaces, called *fake projective planes* since they have the same Betti numbers as the projective plane, was given by Mumford ([Mum79]) using 2-adic uniformization. Other examples were given later in [IK98], while recently Keum ([Keu05]) gave an explicit geometric construction as a cyclic cover of a particular Dolgachev surface.

In a recent preprint ([PY05]) G. Prasad and S.K. Yeung, using the arithmeticity of Γ, give twelve rather explicit lists of fake projective planes, each corresponding to an imaginary quadratic field $\mathbb{Q}(\sqrt{-a})$ and a prime p which ramifies in it.

The interesting geometric features of these examples are that:

i) all these groups Γ are indeed contained in $SU(2,1)$, hence the canonical divisor K is divisible by 3,

ii) all these surfaces have a nontrivial first homology group $H_1(S,\mathbb{Z})$.

3 Surfaces with $p_g = 4$

In Enriques' book on algebraic surfaces [Enr49] much emphasis was put on the effective construction of surfaces whose canonical map is birational, particularly for surfaces with $p_g = 4$, where the canonical image is a surface in $\mathbb{P}^3$.

Later on, in particular in the last thirty years, many authors studied surfaces with $p_g = 4$, with particular interest in the construction of surfaces with $p_g = 4$, birational canonical map and K^2 as high as possible.

If the canonical map of a minimal surface of general type S with $p_g = 4$ is birational, then the standard inequalities give $5 \leq K^2 \leq 45$.

Nowadays we know examples, by the contribution of several authors, for every value of K_S^2 in the range $5 \leq K_S^2 \leq 28$ (cf. e.g. [Cil81], [Cat99]). An example with $K_S^2 = 31$ has been recently obtained in [Lie03], although the example constructed has a big fixed part of the canonical system so that its canonical image has “only” degree 12. Moreover, the first two authors together with F. Grunewald have constructed a canonical surface in the projective 3-space with $K^2 = 45$. This surface is obtained as a Galois covering of the plane with group $(\mathbb{Z}/5\mathbb{Z})^2$, branched over a configuration of lines introduced by Hirzebruch (cf. [BCG05c]).

In this case we have a rigid surface such that its canonical system has a fixed part.

Obviously also in this case classification is the biggest challenge: for which values of K_S^2 is it possible to classify all possible minimal surfaces of general type with $p_g = 4$? And more ambitiously: for which values of K_S^2, q it is possible to completely describe the moduli space $\mathcal{M}_{K^2,4,q}$?

3.1 $K^2 = 4, 5$

The cases $K^2 = 4, 5$ were already treated by Enriques ([Enr49], section 2, chapter VIII, pp.268–271), and the corresponding moduli spaces were completely understood already in the 70's.

We briefly recall these results.

By Debarre's inequality, all these surfaces are regular (in fact, this is true for $K^2 \leq 7$). The canonical map of surfaces with $K^2 = 4$ and $p_g = 4$ is a morphism of degree 2 onto an irreducible quadric in $\mathbb{P}^3$, i.e., either a smooth quadric or a quadric cone. The general surface is a double cover of a smooth quadric branched over a smooth complete intersection with a sextic surface.

A detailed analysis of the corresponding moduli space can be found in [Hor76], where the following is proven.

Theorem 17. *$\mathcal{M}_{4,4,0}$ is irreducible, unirational of dimension 42, its singular locus is irreducible of codimension 1 and corresponds exactly to the surfaces whose canonical image is a quadric cone.*

We have two classes of minimal surfaces of general type with $K^2 = 4$: let us say surfaces of type I (double covers of a smooth quadric) and II (double covers of a quadric cone). Correspondingly we have a stratification of $\mathcal{M}_{4,4,0}$ as a union of two locally closed strata, both irreducible, which we denote simply by I and II, of respective dimension 42 and 41.

To draw a picture of this moduli space we need the following notation:

Definition 9. *Let A and B be two (locally closed) irreducible strata of a moduli space $\mathcal{M}_{K^2,p_g,q}$.*

If we write "$A \to B$", it means that there is a flat family with base a small disc $\Delta_\varepsilon \subset \mathbb{C}$, whose central fibre is of type B and whose general fibre is of type A. In other words it means that the closure of the stratum A intersects the stratum B.

With this notation a picture of $\mathcal{M}_{4,4,0}$ is the following:

$$\begin{tikzcd}
42 & \mathrm{I} \arrow[d] \\
41 & \mathrm{II}
\end{tikzcd}$$

Note that at the left of each stratum stands the dimension of the corresponding irreducible stratum.

The case $K^2 = 5$ is slightly more complicated, and completely described in [Hor75]: the canonical map is either a birational morphism to a quintic in $\mathbb{P}^3$ (type I), or a rational map of degree 2 onto an irreducible quadric, which can be as in the previous case either smooth (type II_a) or a quadric cone (type II_b).

Theorem 18. *$\mathcal{M}_{5,4,0}$ has two irreducible components, both unirational of dimension 40, intersecting in a 39 dimensional subvariety.*