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Bronius Grigelionis

Student's *t*-Distribution and Related Stochastic Processes

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To my family

Preface

Stochastic processes with heavy-tailed marginal distributions, including Student's t-distribution, are used commonly for modelling in communication networks, econometrics, insurance, logarithmic stock returns and stochastic volatility in finance, electric activity of neurons, turbulence, etc.

The aim of this short book is the survey of recent result on the Student–Lévy processes as a subclass of Thorin subordinated Gaussian–Lévy processes. Criteria of self-decomposability of such processes are discussed in detail and related Ornstein–Uhlenbeck-type processes are constructed.

The univariate Student diffusion processes are considered in the framework of the H-diffusions, i.e., stationary ergodic diffusions with the predetermined marginal distribution H. Asymptotic distributions of the normalised extreme values of these diffusions are given. Special attention is paid to the statistically tractable case of the Kolmogorov–Pearson diffusions.

Using the independently scattered random measures, defined by means of the bivariate Student–Lévy processes, strictly stationary Student processes with the arbitrary correlation function are defined. Further, via the Lamperti's transform, the self-similar Student–Lamperti processes are introduced.

As a promising direction for future work in constructing and investigating of new multivariate Student–Lévy-type processes, the notion of Lévy copulas and the related analogue of Sklar's theorem is briefly explained.

Statistical inference problems as well as general studentised statistics and self-normalised processes are not considered at all. List of references is far from to be complete.

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Vilnius, September 2011 Bronius Grigelionis

Contents

Abstract and Keywords

Abstract This brief monograph contains a deep study of infinite divisibility and self-decomposability properties of central and non-central Student's distributions, represented as variance and mean-variance mixtures of multivariate Gaussian distributions with the reciprocal gamma mixing distribution, respectively. These results permit to define and analyse Student–Lévy processes as Thorin subordinated Gaussian–Lévy processes. Analogously, Student–Ornstein–Uhlenbeck-type processes are described. A wide class of one-dimensional strictly stationary diffusions with the Student's t marginal distribution is defined as the unique weak solution for the stochastic differential equation. Extreme value theory for such diffusions is developed. A flexible and statistically tractable Kolmogorov–Pearson diffusions are also described. Using the independently scattered random measures, generated by the bivariate centered Student–Lévy process, and stochastic integration theory with respect to them, it is defined as an univariate strictly stationary process with the centered Student's t marginals and the arbitrary correlation structure. As a promising direction for future work in constructing and analysing of new multivariate Student–Lévy-type processes, the notion of Lévy copulas and the related analogue of Sklar's theorem is explained.

Keywords Bessel function · Gaussian Lévy process · *H*-diffusion · Self-decomposability · Stationary Student process · Student–Lévy process · Student's *t*-distribution · Thorin subordinator · Tweedie class

Chapter 1
Introduction

Considering a sample of independent observations $X_1, \ldots, X_n$ from the normal population with mean α and variance σ^2 for testing the null hypothesis $H_0 : \alpha = \alpha_0$ against the alternative $H_1 : \alpha = \alpha_1$, Gosset ("Student") in 1908 [1] suggested the test statistic

$$t_n = \frac{\sqrt{n}(\bar{X}_n - \alpha_0)}{s_n}, \quad n \geq 2,$$

where $\bar{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$, $s_n^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \bar{X}_n)^2$. He derived that the distribution law

$$\mathscr{L}(t_n) = T_1(n-1, 1, 0),$$

where $T_1(\nu, \sigma^2, \alpha)$ denotes the univariate Student's t-distribution with $\nu > 0$ degrees of freedom, a scaling parameter $\sigma^2 > 0$ and a location parameter $\alpha \in R^1$, defined by its probability density function (pdf for short) $f_{\nu,\sigma^2}(x - \alpha)$, where

$$f_{\nu,\sigma^2}(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi\nu}\sigma\Gamma(\frac{\nu}{2})}\left[1 + \frac{1}{\nu}\left(\frac{x}{\sigma}\right)^2\right]^{-\frac{\nu+1}{2}}, \quad x \in R^1,$$

and $\Gamma(z)$ is the Euler's gamma function (see [2]).

Having in mind that the statistics $\bar{X}_n$ and s_n^2 are independent, $\mathscr{L}(\bar{X}_n) = N(\alpha_0, \frac{\sigma^2}{n})$ and $\mathscr{L}(\sigma^{-2}s_n^2) = \Gamma_{\frac{n-1}{2},\frac{n-1}{2}}$, we easily find that

$$f_{\nu,\sigma^2}(x) = \int_0^\infty \frac{1}{\sqrt{2\pi y\sigma^2}} e^{-\frac{1}{2y}\left(\frac{x}{\sigma}\right)^2} h_\nu(y)\mathrm{d}y,$$

where $\Gamma_{\beta,\gamma}$ is the gamma distribution with the pdf

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$$p_{\beta,\gamma}(x) = \begin{cases} \dfrac{\beta^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\beta x}, & \text{if} \quad x > 0, \\ 0, \text{x} \le 0, \end{cases}$$

and

$$h_\nu(y) = \frac{(\frac{\nu}{2})^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} y^{-\frac{\nu}{2}-1} e^{-\frac{\nu}{2y}}, \quad y > 0,$$

is the pdf of the inverse (reciprocal) gamma distribution $I\Gamma_{\frac{\nu}{2},\frac{\nu}{2}}$.

In 1931 [3] Fisher introduced the univariate noncentral t-distribution with the pdf $f_{\nu,\sigma^2,a}(x-\alpha)$ as a mean–variance mixture of normal distributions with the inverse gamma mixing distribution, i.e.

$$\begin{aligned} f_{\nu,\sigma^2,a}(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi y\sigma^2}} e^{-\frac{1}{2y}\left(\frac{x-ay}{\sigma}\right)^2} h_\nu(y)\mathrm{d}y \\ &= \frac{(\frac{\nu}{2})^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} \frac{2\exp\{\frac{xa}{\sigma^2}\}}{\sqrt{2\pi\sigma^2}} \left(\frac{a^2}{\nu\sigma^2+x^2}\right)^{\frac{\nu+d}{4}} K_{\frac{\nu+1}{2}}\left(\sigma^{-2}[a^2(\nu\sigma^2+x^2)]^{\frac{1}{2}}\right), \quad x \in R^1, \end{aligned}$$

where $K_\nu(z)$ is the modified Bessel function of the third kind (see Appendix).

There are unlimited possibilities to introduce classes of multivariate extensions of Student's t-distributions with the univariate Student's marginals. An excellent survey of such useful generalizations are given by Kotz and Nadarajah in [4] (see also [5]). Further we shall mainly restrict ourselves to the cases of variance mixtures and mean–variance mixtures of multivariate Gaussian distributions with the inverse gamma mixing distribution h_ν.

Let

$$g_{a,\Sigma}(x) = \frac{1}{\sqrt{|\Sigma|}(2\pi)^{\frac{d}{2}}} \exp\left\{-\frac{1}{2}\langle(x-a)\Sigma^{-1}, x-a\rangle\right\}, \quad x \in R^d,$$

be a Gaussian pdf, where $a \in R^d$, Σ is a symmetric positive definite $d \times d$ matrix, $|\Sigma| := \det \Sigma$, $\langle\cdot,\cdot\rangle$ signs the scalar product in R^d.

Definition 1.1 We say that $T_d(\nu, \Sigma, \alpha)$ is a multivariate Student's t-distribution with $\nu > 0$ degrees of freedom, a scaling matrix Σ and a location vector $\alpha \in R^d$, if its pdf is $f_{\nu,\Sigma}(x-\alpha)$, $x \in R^d$, where

$$f_{\nu,\Sigma}(x) = \int_0^\infty g_{0,u\Sigma}(x) h_\nu(u)\mathrm{d}u$$