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Rüdiger U. Seydel

Tools for Computational Finance

Fourth Edition

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Preface to the First Edition

Basic principles underlying the transactions of financial markets are tied to probability and statistics. Accordingly it is natural that books devoted to *mathematical finance* are dominated by stochastic methods. Only in recent years, spurred by the enormous economical success of financial derivatives, a need for sophisticated computational technology has developed. For example, to price an American put, quantitative analysts have asked for the numerical solution of a free-boundary partial differential equation. Fast and accurate numerical algorithms have become essential tools to price financial derivatives and to manage portfolio risks. The required methods aggregate to the new field of *Computational Finance*. This discipline still has an aura of mysteriousness; the first specialists were sometimes called *rocket scientists*. So far, the emerging field of computational finance has hardly been discussed in the mathematical finance literature.

This book attempts to fill the gap. Basic principles of computational finance are introduced in a monograph with textbook character. The book is divided into four parts, arranged in six chapters and seven appendices. The general organization is

Part I (Chapter 1): Financial and Stochastic Background
Part II (Chapters 2, 3): Tools for Simulation
Part III (Chapters 4, 5, 6): Partial Differential Equations for Options
Part IV (Appendices A1...A7): Further Requisits and Additional Material.

The first chapter introduces fundamental concepts of financial options and of stochastic calculus. This provides the financial and stochastic background needed to follow this book. The chapter explains the terms and the functioning of standard options, and continues with a definition of the Black-Scholes market and of the principle of risk-neutral valuation. As a first computational method the simple but powerful binomial method is derived. The following parts of Chapter 1 are devoted to basic elements of stochastic analysis, including Brownian motion, stochastic integrals and Itô processes. The material is discussed only to an extent such that the remaining parts of the book can be understood. Neither a comprehensive coverage of derivative products nor an explanation of martingale concepts are provided. For such in-depth coverage of financial and stochastic topics ample references to special literature are given as hints for further study. The focus of this book is on numerical methods.

Chapter 2 addresses the computation of random numbers on digital computers. By means of congruential generators and Fibonacci generators, uniform deviates are obtained as first step. Thereupon the calculation of normally distributed numbers is explained. The chapter ends with an introduction into low-discrepancy numbers. The random numbers are the basic input to integrate stochastic differential equations, which is briefly developed in Chapter 3. From the stochastic Taylor expansion, prototypes of numerical methods are derived. The final part of Chapter 3 is concerned with Monte Carlo simulation and with an introduction into variance reduction.

The largest part of the book is devoted to the numerical solution of those partial differential equations that are derived from the Black-Scholes analysis. Chapter 4 starts from a simple partial differential equation that is obtained by applying a suitable transformation, and applies the finite-difference approach. Elementary concepts such as stability and convergence order are derived. The free boundary of American options —the optimal exercise boundary— leads to variational inequalities. Finally it is shown how options are priced with a formulation as linear complimentarity problem. Chapter 5 shows how a finite-element approach can be used instead of finite differences. Based on linear elements and a Galerkin method a formulation equivalent to that of Chapter 4 is found. Chapters 4 and 5 concentrate on standard options.

Whereas the transformation applied in Chapters 4 and 5 helps avoiding spurious phenomena, such artificial oscillations become a major issue when the transformation does not apply. This is frequently the situation with the non-standard *exotic* options. Basic computational aspects of exotic options are the topic of Chapter 6. After a short introduction into exotic options, Asian options are considered in some more detail. The discussion of numerical methods concludes with the treatment of the advanced total variation diminishing methods. Since exotic options and their computations are under rapid development, this chapter can only serve as stimulation to study a field with high future potential.

In the final part of the book, seven appendices provide material that may be known to some readers. For example, basic knowledge on stochastics and numerics is summarized in the appendices A2, A4, and A5. Other appendices include additional material that is slightly tangential to the main focus of the book. This holds for the derivation of the Black-Scholes formula (in A3) and the introduction into function spaces (A6).

Every chapter is supplied with a set of exercises, and hints on further study and relevant literature. Many examples and 52 figures illustrate phenomena and methods. The book ends with an extensive list of references.

This book is written from the perspectives of an applied mathematician. The level of mathematics in this book is tailored to readers of the advanced undergraduate level of science and engineering majors. Apart from this basic knowledge, the book is self-contained. It can be used for a course on the subject. The intended readership is interdisciplinary. The audience of this book includes professionals in financial engineering, mathematicians, and scientists of many fields.

An expository style may attract a readership ranging from graduate students to practitioners. Methods are introduced as tools for immediate application. Formulated and summarized as algorithms, a straightforward implementation in computer programs should be possible. In this way, the reader may learn by computational experiment. *Learning by calculating* will be a possible way to explore several aspects of the financial world. In some parts, this book provides an algorithmic introduction into computational finance. To keep the text readable for a wide range of readers, some of the proofs and derivations are exported to the exercises, for which frequently hints are given.

This book is based on courses I have given on computational finance since 1997, and on my earlier German textbook *Einführung in die numerische Berechnung von Finanz-Derivaten*, which Springer published in 2000. For the present English version the contents have been revised and extended significantly.

The work on this book has profited from cooperations and discussions with Alexander Kempf, Peter Kloeden, Rainer Int-Veen, Karl Riedel and Roland Seydel. I wish to express my gratitude to them and to Anita Rother, who `TEX`ed the text. The figures were either drawn with `xfig` or plotted and designed with `gnuplot`, after extensive numerical calculations.

Additional material to this book, such as hints on exercises and colored figures and photographs, is available at the website address

`www.mi.uni-koeln.de/numerik/compfin/`

It is my hope that this book may motivate readers to perform own computational experiments, thereby exploring into a fascinating field.

Köln *Rüdiger Seydel*
February 2002

Preface to the Second Edition

This edition contains more material. The largest addition is a new section on jump processes (Section 1.9). The derivation of a related partial integro-differential equation is included in Appendix A3. More material is devoted to Monte Carlo simulation. An algorithm for the standard workhorse of inverting the normal distribution is added to Appendix A7. New figures and more exercises are intended to improve the clarity at some places. Several further references give hints on more advanced material and on important developments.

Many small changes are hoped to improve the readability of this book. Further I have made an effort to correct misprints and errors that I knew about.

A new domain is being prepared to serve the needs of the computational finance community, and to provide complementary material to this book. The address of the domain is

`www.compfin.de`

The domain is under construction; it replaces the website address `www.mi.uni-koeln.de/numerik/compfin/`.

Suggestions and remarks both on this book and on the domain are most welcome.

Köln | *Rüdiger Seydel*
July 2003

Preface to the Third Edition

The rapidly developing field of financial engineering has suggested extensions to the previous editions. Encouraged by the success and the friendly reception of this text, the author has thoroughly revised and updated the entire book, and has added significantly more material. The appendices were organized in a different way, and extended. In this way, more background material, more jargon and terminology are provided in an attempt to make this book more self-contained. New figures, more exercises, and better explanations improve the clarity of the book, and help bridging the gap to finance and stochastics.

The largest addition is a new section on analytic methods (Section 4.8). Here we concentrate on the interpolation approach and on the quadratic approximation. In this context, the analytic method of lines is outlined. In Chapter 4, more emphasis is placed on extrapolation and the estimation of the accuracy. New sections and subsections are devoted to risk-neutrality. This includes some introducing material on topics such as the theorem of Girsanov, state-price processes, and the idea of complete markets. The analysis and geometry of early-exercise curves is discussed in more detail. In the appendix, the derivations of the Black-Scholes equation, and of a partial integro-differential equation related to jump diffusion are rewritten. An extra section introduces multidimensional Black-Scholes models. Hints on testing the quality of random-number generators are given. And again more material is devoted to Monte Carlo simulation. The integral representation of options is included as a link to quadrature methods. Finally, the references are updated and expanded.

It is my pleasure to acknowledge that the work on this edition has benefited from helpful remarks of Rainer Int-Veen, Alexander Kempf, Sebastian Quecke, Roland Seydel, and Karsten Urban.

The material of this Third Edition has been tested in courses the author gave recently in Cologne and in Singapore. Parallel to this new edition, the website `www.compfin.de` is supplied by an option calculator.

Köln *Rüdiger Seydel*
October 2005

Preface to the Fourth Edition

Financial engineering is evolving at a fast pace; new methods are being developed and efficient algorithms are being demanded. This fourth edition of *Tools for Computational Finance* carefully integrates new directions set forth by recent research. Insight from conferences and workshops has been validated by us and tested in the class room. In this fourth edition the main focus is still largely, albeit not exclusively, on the Black–Scholes world, which is considered a bench mark and the central point within a slightly more general setting.

New topics of this fourth edition include a section on calibration, with background material on minimization in the Appendix. Heston's model is also included. Two examples of exotic options have been added, namely: a two-dimensional barrier option and a two-dimensional binary option. And the exposition on Monte Carlo methods for American options has been extended by regression methods, including the Longstaff–Schwartz algorithm. Furthermore, the tradeoff bias versus variance is discussed. Bermudan-based algorithms play a larger role in this edition, with more emphasis on the dynamic programming principle based on continuation values. Section 4.6 on finite-difference methods has been reorganized, now stressing the efficiency of direct methods. — A few minor topics of the previous edition have become obsolete and have been removed.

Every endeavor has been made to further improve the clarity of this exposition. Amendments have been made throughout. And numerous additional references provide hints for further study.

It is my pleasure to acknowledge that this edition has benefited from inspiring discussions with several people, including Marco Avellaneda, Peter Carr, Peter Forsyth, Tat Fung, Jonathan Goodman, Pascal Heider, Christian Jonen, Jan Kallsen, Sebastian Quecke, and Roland Seydel.

Köln, August 2008 *Rüdiger Seydel*

Contents

Notations

elements of options:

t	time
T	maturity date, time to expiration
S	price of underlying asset
	S_j, S_{ji} specific values of the price S
S_t	price of the asset at time t
K	strike price, exercise price
Ψ	payoff function
V	value of an option (V_C value of a call, V_P value of a put, ^{Am} American, ^{Eur} European)
σ	volatility
r	interest rate (Appendix A1)

general mathematical symbols:

$\mathbb{R}$	set of real numbers
$\mathbb{N}$	set of integers > 0
$\in$	element in
$\subseteq$	subset of, $\subset$ strict subset
$[a, b]$	closed interval $\{x \in \mathbb{R} : a \leq x \leq b\}$
$[a, b)$	half-open interval $a \leq x < b$ (analogously $(a, b], (a, b)$)
P	probability
E	expectation (Appendix B1)
Var	variance
Cov	covariance
log	natural logarithm
$:=$	defined to be
$\doteq$	equal except for rounding errors
$\equiv$	identical
$\Longrightarrow$	implication
$\Longleftrightarrow$	equivalence
$O(h^k)$	Landau-symbol: for $h \to 0$
	$f(h) = O(h^k) \iff \frac{f(h)}{h^k}$ is bounded
$\sim \mathcal{N}(\mu, \sigma^2)$	normal distributed with expectation μ and variance σ^2
$\sim \mathcal{U}[0, 1]$	uniformly distributed on $[0, 1]$

Δt	small increment in t
^{tr}	transposed; A^{tr} is the matrix where the rows and columns of A are exchanged.
$\mathcal{C}^0[a,b]$	set of functions that are continuous on $[a,b]$
$\in \mathcal{C}^k[a,b]$	k-times continuously differentiable
$\mathcal{D}$	set in $\mathbb{R}^n$ or in the complex plane, $\bar{\mathcal{D}}$ closure of $\mathcal{D}$, $\mathcal{D}^\circ$ interior of $\mathcal{D}$
$\partial\mathcal{D}$	boundary of $\mathcal{D}$
$\mathcal{L}^2$	set of square-integrable functions
$\mathcal{H}$	Hilbert space, Sobolev space (Appendix C3)
$[0,1]^2$	unit square
Ω	sample space (in Appendix B1)
$f^+ := \max\{f, 0\}$	
d	symbol for differentiation
$\dot{u}$	time derivative $\frac{\mathrm{d}u}{\mathrm{d}t}$ of a function $u(t)$
f'	derivative of a function f
i	symbol for imaginary unit
e	symbol for the basis of the exponential function exp
∂	symbol for partial differentiation
$\mathbf{1}_{\mathcal{M}}$	=1 on $\mathcal{M}$, =0 elsewhere (indicator function)

integers:

$i, j, k, l, m, n, M, N, \nu$

various variables:

$X_t, X, X(t)$	random variable
W_t	Wiener process, Brownian motion (Definition 1.7)
$y(x,\tau)$	solution of a partial differential equation for (x,τ)
w	approximation of y
h	discretization grid size
φ	basis function (Chapter 5)
ψ	test function (Chapter 5)

abbreviations:

BDF	Backward Difference Formula, see Section 4.2.1
CIR	Cox Ingersoll Ross model, see Section 1.7.4
CFL	Courant-Friedrichs-Lewy, see Section 6.5.1
Dow	Dow Jones Industrial Average
FE	Finite Element
FFT	Fast Fourier Transformation
FTBS	Forward Time Backward Space, see Section 6.5.1
FTCS	Forward Time Centered Space, see Section 6.4.2
GBM	Geometric Brownian Motion, see (1.33)
LCP	Linear Complementary Problem

MC	Monte Carlo
ODE	Ordinary Differential Equation
OTC	Over the Counter
OU	Ornstein Uhlenbeck
PDE	Partial Differential Equation
PIDE	Partial Integro-Differential Equation
PSOR	Projected Successive Overrelaxation
QMC	Quasi Monte Carlo
SDE	Stochastic Differential Equation
SOR	Successive Overrelaxation
TVD	Total Variation Diminishing
i.i.d.	independent and identical distributed
inf	infimum, largest lower bound of a set of numbers
sup	supremum, least upper bound of a set of numbers
supp(f)	support of a function f: $\{x \in \mathcal{D} : f(x) \neq 0\}$
t.h.o.	terms of higher order

hints on the organization:

(2.6)	number of equation (2.6)
	(The first digit in all numberings refers to the chapter.)
(A4.10)	equation in Appendix A; similarly B, C, D
$\longrightarrow$	hint (for instance to an exercise)

Chapter 1 Modeling Tools for Financial Options

1.1 Options

What do we mean by option? An option is the right (but not the obligation) to buy or sell a risky asset at a prespecified fixed price within a specified period. An option is a financial instrument that allows —amongst other things— to make a bet on rising or falling values of an underlying asset. The **underlying** asset typically is a stock, or a parcel of shares of a company. Other examples of underlyings include stock indices (as the Dow Jones Industrial Average), currencies, or commodities. Since the value of an option depends on the value of the underlying asset, options and other related financial instruments are called *derivatives* ($\longrightarrow$ Appendix A2). An option is a contract between two parties about trading the asset at a certain future time. One party is the *writer*, often a bank, who fixes the terms of the option contract and sells the option. The other party is the *holder*, who purchases the option, paying the market price, which is called *premium*. How to calculate a fair value of the premium is a central theme of this book. The holder of the option must decide what to do with the rights the option contract grants. The decision will depend on the market situation, and on the type of option. There are numerous different types of options, which are not all of interest to this book. In Chapter 1 we concentrate on standard options, also known as *vanilla options*. This Section 1.1 introduces important terms.

Options have a limited life time. The *maturity date* T fixes the time horizon. At this date the rights of the holder expire, and for later times ($t > T$) the option is worthless. There are two basic types of option: The **call** option gives the holder the right to *buy* the underlying for an agreed price K by the date T. The **put** option gives the holder the right to *sell* the underlying for the price K by the date T. The previously agreed price K of the contract is called **strike** or **exercise price**[1]. It is important to note that the holder is not obligated to *exercise* —that is, to buy or sell the underlying according to the terms of the contract. The holder may wish to close his position by selling the option. In summary, at time t the holder of the option can choose to

[1] The price K as well as other prices are meant as the price of one unit of an asset, say, in $.

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- sell the option at its current market price on some options exchange (at $t < T$),
- retain the option and do nothing,
- exercise the option ($t \leq T$), or
- let the option expire worthless ($t \geq T$).

In contrast, the writer of the option has the obligation to deliver or buy the underlying for the price K, in case the holder chooses to exercise. The risk situation of the writer differs strongly from that of the holder. The writer receives the premium when he issues the option and somebody buys it. This up-front premium payment compensates for the writer's potential liabilities in the future. The asymmetry between writing and owning options is evident. This book mostly takes the standpoint of the holder (long position in the option).

Not every option can be exercised at any time $t \leq T$. For **European options**, exercise is only permitted at expiration T. **American options** can be exercised at any time up to and including the expiration date. For options the labels American or European have no geographical meaning. Both types are traded in each continent. Options on stocks are mostly American style.

The value of the option will be denoted by V. The value V depends on the price per share of the underlying, which is denoted S. This letter S symbolizes stocks, which are the most prominent examples of underlying assets. The variation of the asset price S with time t is expressed by S_t or $S(t)$. The value of the option also depends on the remaining time to expiry $T - t$. That is, V depends on time t. The dependence of V on S and t is written $V(S, t)$. As we shall see later, it is not easy to define and to calculate the fair value V of an option for $t < T$. But it is an easy task to determine the terminal value of V at expiration time $t = T$. In what follows, we shall discuss this topic, and start with European options as seen with the eyes of the holder.

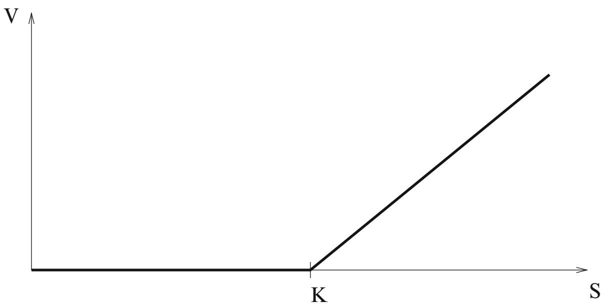

Fig. 1.1. Intrinsic value of a call with exercise price K (payoff function)

The Payoff Function

At time $t = T$, the holder of a European call option will check the current price $S = S_T$ of the underlying asset. The holder has two alternatives to acquire the underlying asset: either buying the asset on the spot market (costs S), or buying the asset by exercising the call option (costs K). The decision is easy: the costs are to be minimal. The holder will exercise the call only when $S > K$. For then the holder can immediately sell the asset for the spot price S and makes a gain of $S - K$ per share. In this situation the value of the option is $V = S - K$. (This reasoning ignores transaction costs.) In case $S < K$ the holder will not exercise, since then the asset can be purchased on the market for the cheaper price S. In this case the option is worthless, $V = 0$. In summary, the value $V(S, T)$ of a call option at expiration date T is given by

$$V(S_T, T) = \begin{cases} 0 & \text{in case } S_T \le K \text{ (option expires worthless)} \\ S_T - K & \text{in case } S_T > K \text{ (option is exercised)} \end{cases}$$

Hence

$$V(S_T, T) = \max\{S_T - K, 0\} .$$

Considered for all possible prices $S_t > 0$, $\max\{S_t - K, 0\}$ is a function of S_t, in general for $0 \le t \le T$.[2] This **payoff function** is shown in Figure 1.1. Using the notation $f^+ := \max\{f, 0\}$, this payoff can be written in the compact form $(S_t - K)^+$. Accordingly, the value $V(S_T, T)$ of a call at maturity date T is

$$V(S_T, T) = (S_T - K)^+ . \tag{1.1C}$$

For a European put, exercising only makes sense in case $S < K$. The payoff $V(S, T)$ of a put at expiration time T is

$$V(S_T, T) = \begin{cases} K - S_T & \text{in case } S_T < K \text{ (option is exercised)} \\ 0 & \text{in case } S_T \ge K \text{ (option is worthless)} \end{cases}$$

Hence

$$V(S_T, T) = \max\{K - S_T, 0\} ,$$

or

$$V(S_T, T) = (K - S_T)^+ , \tag{1.1P}$$

compare Figure 1.2.

[2] In this chapter, the payoff evaluated at t only depends on the current value S_t. Payoffs that depend on the *entire path* S_t for all $0 \le t \le T$ occur for exotic options, see Chapter 6.

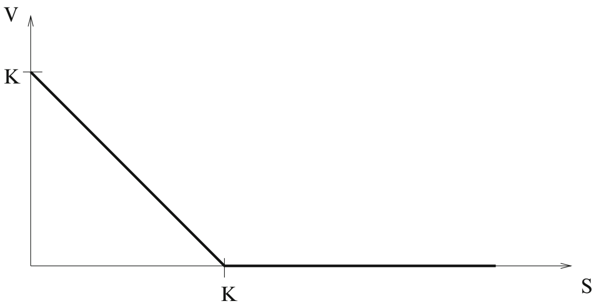

Fig. 1.2. Intrinsic value of a put with exercise price K (payoff function)

The curves in the payoff diagrams of Figures 1.1 and 1.2 show the option values from the perspective of the holder. The profit is not shown. For an illustration of the profit, the initial costs for buying the option at $t = t_0$ must be subtracted. The initial costs basically consist of the premium and the transaction costs. Since both are paid upfront, they are multiplied by $e^{r(T-t_0)}$ to take account of the time value; r is the continuously compounded interest rate. Subtracting the costs leads to shifting down the curves in Figures 1.1 and 1.2. The resulting *profit diagram* shows a negative profit for some range of S-values, which of course means a loss (see Figure 1.3).

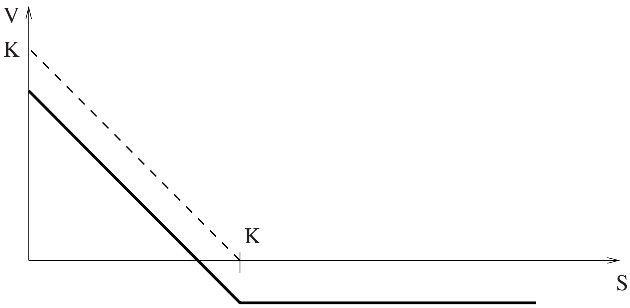

Fig. 1.3. Profit diagram of a put

The payoff function for an American call is $(S_t - K)^+$ and for an American put $(K - S_t)^+$ for any $t \le T$. The Figures 1.1 and 1.2 as well as the equations (1.1C), (1.1P) remain valid for American type options.

The payoff diagrams of Figures 1.1, 1.2 and the corresponding profit diagrams show that a potential loss for the purchaser of an option (long position) is limited by the initial costs, no matter how bad things get. The situation for the writer (short position) is reverse. For him the payoff curves of Figures 1.1, 1.2 as well as the profit curves must be reflected on the S-axis. The writer's profit or loss is the reverse of that of the holder. Multiplying the payoff of a call in Figure 1.1 by (-1) illustrates the potentially unlimited risk of a short call. Hence the writer of a call must carefully design a strategy to compensate for his risks. We will come back to this issue in Section 1.5.

A Priori Bounds

No matter what the terms of a specific option are and no matter how the market behaves, the values V of the options satisfy certain bounds. These bounds are known a priori. For example, the value $V(S,t)$ of an American option can never fall below the payoff, for all S and all t. These bounds follow from the *no-arbitrage principle* ($\longrightarrow$ Appendices A2, A3).

To illustrate the strength of no-arbitrage arguments, we assume for an American put that its value is below the payoff. $V < 0$ contradicts the definition of the option. Hence $V \geq 0$, and S and V would be in the triangle seen in Figure 1.2. That is, $S < K$ and $0 \leq V < K - S$. This scenario would allow arbitrage. The strategy would be as follows: Borrow the cash amount of $S + V$, and buy both the underlying and the put. Then immediately exercise the put, selling the underlying for the strike price K. The profit of this arbitrage strategy is $K - S - V > 0$. This is in conflict with the no-arbitrage principle. Hence the assumption that the value of an American put is below the payoff must be wrong. We conclude for the put

$$V_{\mathrm{P}}^{\mathrm{Am}}(S,t) \geq (K-S)^+ \quad \text{for all } S,t\ .$$

Similarly, for the call

$$V_{\mathrm{C}}^{\mathrm{Am}}(S,t) \geq (S-K)^+ \quad \text{for all } S,t\ .$$

(The meaning of the notations $V_{\mathrm{C}}^{\mathrm{Am}}$, $V_{\mathrm{P}}^{\mathrm{Am}}$, $V_{\mathrm{C}}^{\mathrm{Eur}}$, $V_{\mathrm{P}}^{\mathrm{Eur}}$ is evident.)

Other bounds are listed in Appendix D1. For example, a European put on an asset that pays no dividends until T may also take values below the payoff, but is always above the lower bound $K\mathrm{e}^{-r(T-t)} - S$. The value of an American option should never be smaller than that of a European option because the American type includes the European type exercise at $t = T$ and in addition *early exercise* for $t < T$. That is

$$V^{\mathrm{Am}} \geq V^{\mathrm{Eur}}$$

as long as all other terms of the contract are identical. When no dividends are paid until T, the values of put and call for European options are related by the *put-call parity*

$$S + V_{\mathrm{P}}^{\mathrm{Eur}} - V_{\mathrm{C}}^{\mathrm{Eur}} = K\mathrm{e}^{-r(T-t)}\ ,$$

which can be shown by applying arguments of arbitrage ($\longrightarrow$ Exercise 1.1).

Options in the Market

The features of the options imply that an investor purchases puts when the price of the underlying is expected to fall, and buys calls when the prices are about to rise. This mechanism inspires speculators. An important application of options is hedging ($\longrightarrow$ Appendix A2).

The value of $V(S,t)$ also depends on other factors. Dependence on the strike K and the maturity T is evident. Market parameters affecting the price are the interest rate r, the **volatility** σ of the price S_t, and dividends in case of a dividend-paying asset. The interest rate r is the risk-free rate, which applies to zero bonds or to other investments that are considered free of risks ($\longrightarrow$ Appendices A1, A2). The important volatility parameter σ can be defined as standard deviation of the fluctuations in S_t, for scaling divided by the square root of the observed time period. The larger the fluctuations, respresented by large values of σ, the harder is to predict a future value of the asset. Hence the volatility is a standard measure of risk. The dependence of V on σ is highly sensitive. On occasion we write $V(S,t;T,K,r,\sigma)$ when the focus is on the dependence of V on market parameters.

Time is measured in years. The units of r and σ^2 are per year. Writing $\sigma = 0.2$ means a volatility of 20%, and $r = 0.05$ represents an interest rate of 5%. Table 1.1 summarizes the key notations of option pricing. The notation is standard except for the strike price K, which is sometimes denoted X, or E.

The time period of interest is $t_0 \le t \le T$. One might think of t_0 denoting the date when the option is issued and t as a symbol for "today." But this book mostly sets $t_0 = 0$ in the role of "today," without loss of generality. Then the interval $0 \le t \le T$ represents the remaining life time of the option. The price S_t is a stochastic process, compare Section 1.6. In real markets, the interest rate r and the volatility σ vary with time. To keep the models and the analysis simple, we mostly assume r and σ to be constant on $0 \le t \le T$. Further we suppose that all variables are arbitrarily divisible and consequently can vary continuously —that is, all variables vary in the set $\mathbb{R}$ of real numbers.

Table 1.1. List of important variables

t	current time, $0 \le t \le T$
T	expiration time, maturity
$r > 0$	risk-free interest rate, continuously compounded
$S,\ S_t$	spot price, current price per share of stock/asset/underlying
σ	annual volatility
K	strike, exercise price per share
$V(S,t)$	value of an option at time t and underlying price S

The Geometry of Options

As mentioned, our aim is to calculate $V(S,t)$ for fixed values of K, T, r, σ. The values $V(S,t)$ can be interpreted as a piece of surface over the subset

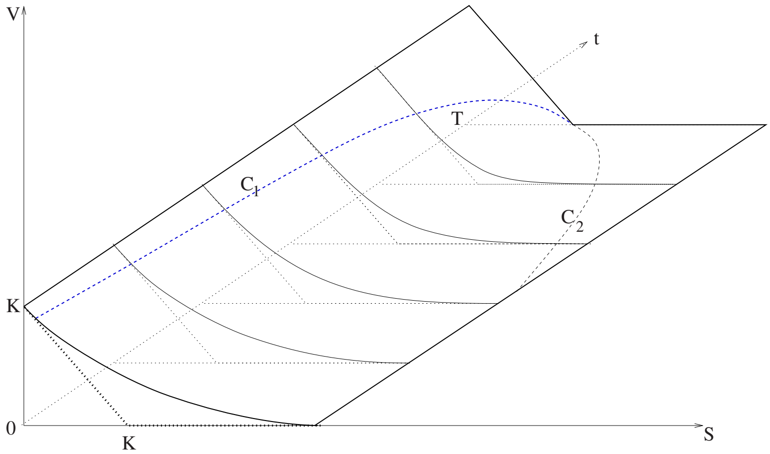

Fig. 1.4. Value $V(S,t)$ of an American put (schematically)

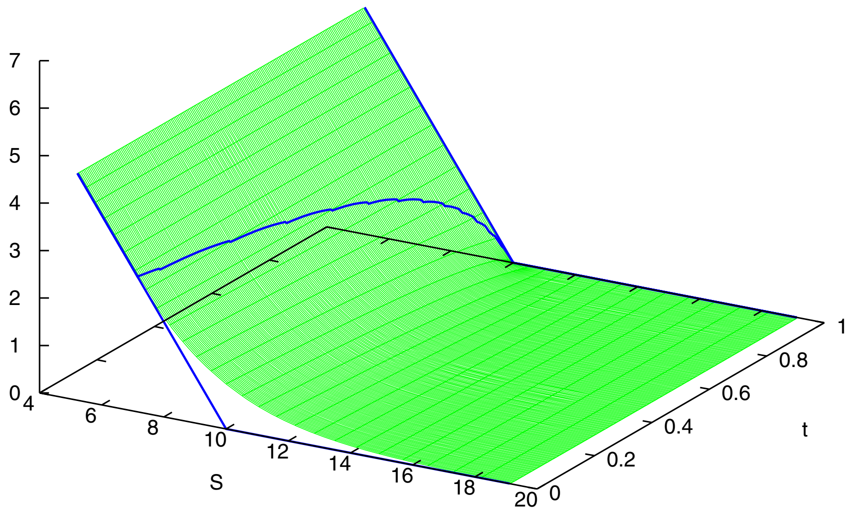

Fig. 1.5. Value $V(S,t)$ of an American put with $r = 0.06$, $\sigma = 0.30$, $K = 10$, $T = 1$

$$S > 0 \ , \ 0 \leq t \leq T$$

of the (S,t)-plane. Figure 1.4 illustrates the character of such a surface for the case of an American put. For the illustration assume $T = 1$. The figure depicts six curves obtained by cutting the *option surface* with the planes $t = 0, 0.2, \ldots, 1.0$. For $t = T$ the payoff function $(K - S)^+$ of Figure 1.2 is clearly visible.

Shifting this payoff parallel for all $0 \le t < T$ creates another surface, which consists of the two planar pieces $V = 0$ (for $S \ge K$) and $V = K - S$ (for $S < K$). This *payoff surface* $(K - S)^+$ is a lower bound to the option surface, $V(S,t) \ge (K - S)^+$. Figure 1.4 shows two curves C_1 and C_2 on the option surface. The curve C_1 is the *early-exercise curve*, because on the planar part with $V(S,t) = K - S$ holding the option is not optimal. (This will be explained in Section 4.5.) The curve C_2 has a technical meaning explained below. Within the area limited by these two curves the option surface is clearly above the payoff surface, $V(S,t) > (K - S)^+$. Outside that area, both surfaces coincide. This is strict "above" C_1, where $V(S,t) = K - S$, and holds approximately for S beyond C_2, where $V(S,t) \approx 0$ or $V(S,t) < \varepsilon$ for a small value of $\varepsilon > 0$. The location of C_1 and C_2 is not known, these curves are calculated along with the calculation of $V(S,t)$. Of special interest is $V(S,0)$, the value of the option "today." This curve is seen in Figure 1.4 for $t = 0$ as the front edge of the option surface. This front curve may be seen as smoothing the corner in the payoff function. The schematic illustration of Figure 1.4 is completed by a concrete example of a calculated put surface in Figure 1.5. An approximation of the curve C_1 is shown.

The above was explained for an American put. For other options the bounds are different ($\longrightarrow$ Appendix D1). As mentioned before, a European put takes values above the lower bound $K\mathrm{e}^{-r(T-t)} - S$, compare Figure 1.6 and Exercise 1.1b.

In summary, this Section 1.1 has introduced an option with the following features: it depends on *one* underlying, and its payoff is $(K - S)^+$ or $(S - K)^+$, with S evaluated at the current time instant. This is the standard option called vanilla option. All other options are called *exotic*. To clarify the distinction between vanilla options and exotic options, we hint at ways how an option can be "exotic." For example, an option may depend on a basket of several underlying assets, or the payoff may be different, or the option may be *path-dependent* in that V no longer depends solely on the current (S_t, t) but on the entire path S_t for $0 \le t \le T$. To give an example of the latter, we mention an *Asian option*, where the payoff depends on the average value of the asset for all times until expiry. Or for a *barrier option* the value also depends on whether the price S_t hits a prescribed barrier during its life time. We come back to exotic options later in the book.

1.2 Model of the Financial Market

Ultimately it is the market which decides on the value of an option. If we try to *calculate* a reasonable value of the option, we need a mathematical model of the market. Mathematical models can serve as approximations and idealizations of the complex reality of the financial world. For modeling financial options, the models named after the pioneers Black, Merton and Scholes have

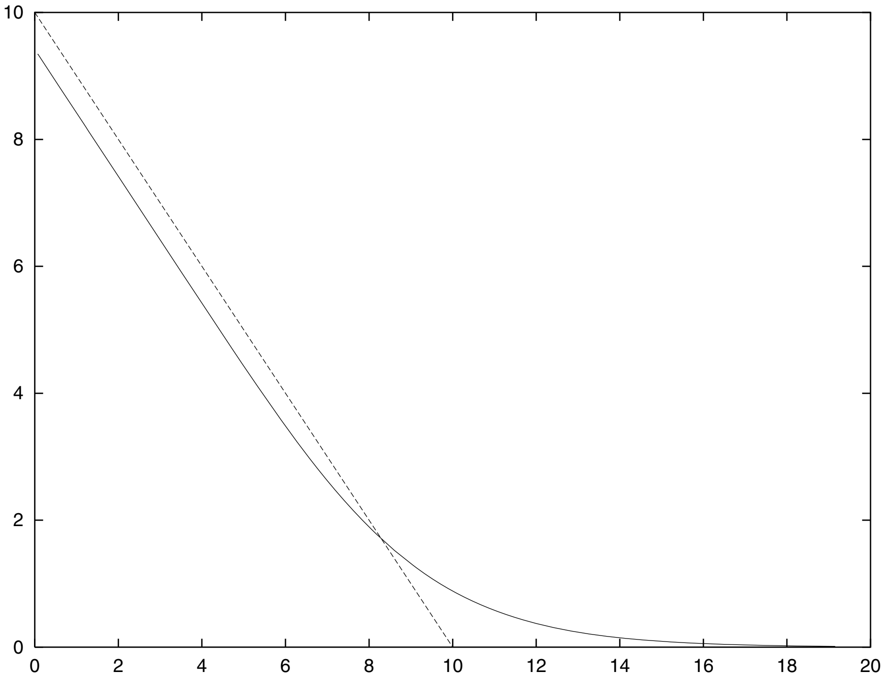

Fig. 1.6. Value of a European put $V(S, 0)$ for $T = 1$, $K = 10$, $r = 0.06$, $\sigma = 0.3$. The payoff $V(S, T)$ is drawn with a dashed line. For small values of S the value V approaches its lower bound, here $9.4 - S$.

been both successful and widely accepted. This Section 1.2 introduces some key elements of market models.

The ultimate aim is to value the option —that is, to calculate $V(S, t)$. It is attractive to define the option surfaces $V(S, t)$ on the *half strip* $S > 0$, $0 \leq t \leq T$ as solutions of suitable equations. Then calculating V amounts to solving the equations. In fact, a series of assumptions allows to characterize the *value functions* $V(S, t)$ as solutions of certain partial differential equations or partial differential inequalities. The model is represented by the famous Black–Scholes equation, which was suggested in 1973.

Definition 1.1 (Black–Scholes equation)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1.2}$$

Equation (1.2) is a partial differential equation for the value function $V(S, t)$ of options. This equation may serve as symbol of the classical market model. But what are the assumptions leading to the Black–Scholes equation?

Assumptions 1.2 (Black–Merton–Scholes model of the market)

(a) *There are no arbitrage opportunities.*

(b) *The market is frictionless.*
This means that there are no transaction costs (fees or taxes), the interest rates for borrowing and lending money are equal, all parties have immediate access to any information, and all securities and credits are available at any time and in any size. Consequently, all variables are perfectly divisible —that is, may take any real number. Further, individual trading will not influence the price.

(c) *The asset price follows a geometric Brownian motion.*
(This stochastic motion will be discussed in Sections 1.6–1.8.)

(d) r and σ are constant for $0 \le t \le T$. No dividends are paid in that time period. The option is European.

These are the assumptions that lead to the Black–Scholes equation (1.2). Some of the assumptions (c), (d) are rather strong, in particular, the volatility σ being constant. Some of the assumptions can be weakened. We come to more complex models later in the text. A derivation of the Black–Scholes partial differential equation (1.2) is given in Appendix A4. Admitting all real numbers t within the interval $0 \le t \le T$ leads to characterize the model as *continuous-time model*. In view of allowing also arbitrary $S > 0$, $V > 0$, we speak of a continuous model.

A value function $V(S,t)$ is not fully defined by merely requesting that it solves (1.2) for all S and t out of the half strip. In addition to solving this partial differential equation, the function $V(S,t)$ must satisfiy a terminal condition and boundary conditions. The **terminal condition** for $t = T$ is

$$V(S,T) = \Psi(S) \; ,$$

where Ψ denotes the payoff function (1.1C) or (1.1P), depending on the type of option. The boundaries of the half strip $0 < S$, $0 \le t \le T$ are defined by $S = 0$ and $S \to \infty$. At these boundaries the function $V(S,t)$ must satisfy **boundary conditions**. For example, a European call must obey

$$V(0,t) = 0; \quad V(S,t) \to S - K\mathrm{e}^{-r(T-t)} \text{ for } S \to \infty \; . \tag{1.3C}$$

This completes one possibility of defining a value function $V(S,t)$. In Chapter 4 we will come back to the Black–Scholes equation and to boundary conditions. For (1.2) an analytic solution is known [equation (A4.10) in Appendix A4]. Note that the partial differential equation (1.2) is linear in the value function V. The nonlinearity of the Black–Scholes problem comes from the payoff; the functions $\Psi(S) = (K - S)^+$ or $\Psi(S) = (S - K)^+$ are convex. The partial differential equation (PDE) is no longer linear when Assumptions 1.2(b) are relaxed. For example, for considering trading intervals Δt and transaction costs as k per unit, one could add the nonlinear term

$$-\sqrt{\frac{2}{\pi}}\,\frac{k\sigma S^2}{\sqrt{\Delta t}}\,\left|\frac{\partial^2 V}{\partial S^2}\right|$$

to (1.2), see [WDH96], [Kwok98]. Also finite liquidity (feedback of trading to the price of the underlying) leads to nonlinear terms in the PDE. In the general case, closed-form solutions do not exist, and a solution is calculated numerically, especially for American options. For the latter a further nonlinearity stems from the early-exercise feature (⟶ Chapter 4). For solving (1.2) numerically, a variant with dimensionless variables can be used (⟶ Exercise 1.2).

Of course, the calculated value V of an option depends on the chosen market model. Writing $V(S,t;\,T,K,r,\sigma)$ suggests a focus on the Black–Scholes equation. This could be made definite by writing V^{BS}, for example. Other market models may involve more parameters. Then, in general, the corresponding value of the value function V is different from V^{BS}. Since we mostly stick to the market model of Assumptions 1.2, we drop the superscript. All our prices V are model prices, not market prices. They depend on the underlying choice of assumptions. For the relation of our model prices V to market prices V^{mar}, see Section 1.10.

At this point, a word on the notation is appropriate. The symbol S for the asset price is used in different roles: First it comes without subscript in the role of an independent real variable $S > 0$ on which the value function $V(S,t)$ depends, say as solution of the partial differential equation (1.2). Second it is used as S_t with subscript t to emphasize its random character as stochastic process. When the subscript t is omitted, the current role of S becomes clear from the context.

1.3 Numerical Methods

Applying numerical methods is inevitable in all fields of technology including financial engineering. Often the important role of numerical algorithms is not noticed. For example, an analytical formula at hand [such as the Black–Scholes formula (A4.10)] might suggest that no numerical procedure is needed. But closed-form solutions may include evaluating the logarithm or the computation of the distribution function of the normal distribution. Such elementary tasks are performed using sophisticated numerical algorithms. In pocket calculators one merely presses a button without being aware of the numerics. The robustness of those elementary numerical methods is so dependable and the efficiency so large that they almost appear not to exist. Even for apparently simple tasks the methods are quite demanding (⟶ Exercise 1.3). The methods must be carefully designed because inadequate strategies might produce inaccurate results (⟶ Exercise 1.4).

Spoilt by generally available black-box software and graphics packages we take the support and the success of numerical workhorses for granted. We make use of the numerical tools with great respect but without further comments, and we just assume an elementary education in numerical methods. An introduction into important methods and hints on the literature are given in Appendix C1.

Since financial markets undergo apparently stochastic fluctuations, stochastic approaches provide natural tools to simulate prices. These methods are based on formulating and simulating stochastic differential equations. This leads to Monte Carlo methods ($\longrightarrow$ Chapter 3). In computers, related simulations of options are performed in a deterministic manner. It will be decisive how to simulate randomness ($\longrightarrow$ Chapter 2). Chapters 2 and 3 are devoted to tools for simulation. These methods can be applied even in case the Assumptions 1.2 are not satisfied.

More efficient methods will be preferred provided their use can be justified by the validity of the underlying models. For example it may be advisable to solve the partial differential equations of the Black–Scholes type. Then one has to choose among several methods. The most elementary ones are finite-difference methods ($\longrightarrow$ Chapter 4). A somewhat higher flexibility concerning error control is possible with finite-element methods ($\longrightarrow$ Chapter 5). The numerical treatment of exotic options requires a more careful consideration of stability issues ($\longrightarrow$ Chapter 6). The methods based on differential equations will be described in the larger part of this book.

The various methods are discussed in terms of accuracy and speed. Ultimately the methods must give quick and accurate answers to real-time problems posed in financial markets. Efficiency and reliability are key demands. Internally the numerical methods must deal with diverse problems such as convergence order or stability. So the numerical analyst is concerned in error estimates and error bounds. Technical criteria such as complexity or storage requirements are relevant for the implementation.

The mathematical formulation benefits from the assumption that all variables take values in the continuum $\mathbb{R}$. This idealization is practical since it avoids initial restrictions of technical nature, and it gives us freedom to impose *artificial* discretizations convenient for the numerical methods. The hypothesis of a continuum applies to the (S,t)-domain of the half strip $0 \leq t \leq T$, $S > 0$, and to the differential equations. In contrast to the hypothesis of a continuum, the financial reality is rather discrete: Neither the price S nor the trading times t can take any real value. The artificial discretization introduced by numerical methods is at least twofold:

1.) The (S,t)-domain is replaced by a **grid** of a finite number of (S,t)-points, compare Figure 1.7.
2.) The differential equations are adapted to the grid and replaced by a finite number of algebraic equations.

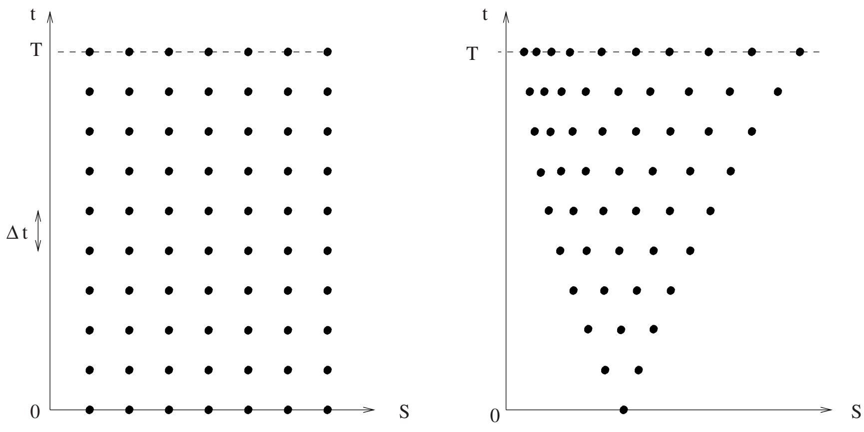

Fig. 1.7. Grid points in the (S, t)-domain

Another kind of discretization is that computers replace the real numbers by a finite number of rational numbers, namely, the floating-point numbers. The resulting rounding error will not be relevant for much of our analysis, except for investigations of stability.

The restriction of the differential equations to the grid causes **discretization errors**. The errors depend on the coarsity of the grid. In Figure 1.7, the distance between two consecutive t-values of the grid is denoted Δt.[3] So the errors will depend on Δt and on ΔS. It is one of the aims of numerical algorithms to control the errors. The left-hand figure in Figure 1.7 shows a simple rectangle grid, whereas the right-hand figure shows a tree-type grid as used in Section 1.4. The type of the grid matches the kind of underlying equations. The values of $V(S, t)$ are primarily approximated at the grid points. Intermediate values can be obtained by interpolation.

The continuous model is an idealization of the discrete reality. But the numerical discretization does not reproduce the original discretization. For example, it would be a rare coincidence when Δt represents a day. The derivations that go along with the twofold transition

$$\text{discrete} \longrightarrow \text{continuous} \longrightarrow \text{discrete}$$

do not compensate.

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[3] The symbol Δt denotes a small increment in t (analogously $\Delta S, \Delta W$). In case Δ would be a number, the product with u would be denoted $\Delta \cdot u$ or $u\Delta$.