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Lorentzian Geometry and Related Topics

GeLoMa 2016, Málaga, Spain, September 20–23

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Preface

Lorentzian Geometry was born as the geometric theory on which General Relativity could be mathematically stated. Nowadays, it constitutes a very active area of research with its proper specific weight in Differential Geometry and Mathematical Relativity. Many mathematical techniques are involved in this field (geometric analysis, functional analysis, partial differential equations, Lie groups and Lie algebras…), making any text focused on it of great interest to a broad audience.

Fifteen years ago, several Spanish researchers interested in Lorentzian Geometry and related mathematical topics, launched the biennial meetings on Lorentzian Geometry in Benalmádena 2001 (Málaga). This first meeting was developed in a very friendly atmosphere and had a vocation to be the first of a following suit on the same topic. As a consequence, a fruitful series of international meetings on Lorentzian Geometry started since that moment: Murcia 2003 (Spain), Castelldefels 2005 (Spain), Santiago de Compostela 2007 (Spain), Martina Franca 2009 (Italy), Granada 2011 (Spain), and Sao Paulo 2013 (Brazil). A special edition on "Lorentzian and conformal Geometry" was held in Greifswald (Germany) in 2014 in honour of Prof. Helga Baum. Along the years, the international character of these meetings has increased spectacularly. The excellent ambiance in which this series of meetings had been originated, recalling the first edition, led us to speak about the "Benalmádena's spirit".

The most recent edition was held at the University of Málaga, Spain, in September, 2016. This volume contains a picture of the trends in Lorentzian Geometry exposed in this VIII International Meeting on Lorentzian Geometry. Among others, it contains topics such as notable (maximal, trapped, null, spacelike, constant mean curvature, umbilical, isoparametric) submanifolds, causal completion of spacetimes, stationary regions and horizons in spacetimes, solitons in semi-Riemannian manifolds, relation between Lorentzian and Finslerian Geometries and the oscillator spacetime.

Let us provide a more specific summary about the contributions included here.

The Euclidean space and the Lorentz-Minkowski spacetime share the same underlying manifold. Therefore a natural question on spacelike hypersurfaces in the Lorentz-Minkowski spacetime arises. When a spacelike hypersurface has the same mean curvature as hypersurface of the Lorentz-Minkowski spacetime and as hypersurface in the Euclidean space? **Eva M. Alarcón, Alma L. Albujer** and **Magdalena Caballero** summarize results for the case when the two mean curvature functions are equal and constant. For the case when the mean curvature functions are equal but not necessarily constant, the authors generalize, to arbitrary dimension, some previous results on spacelike surfaces.

In their contribution, **Stephanie B. Alexander** and **William A. Karr** show that sectional curvature bounds of the form $\mathcal{R} \leq K$ are closely tied to space-time convex functions. Several consequences are obtained: a natural construction of such functions as well as an analogue of a theorem by Alías, Bessa and de Lira ruling out trapped submanifolds in new domains. In addition, they point several connections between totally independent researches by different authors.

Motivated by the very special role played by the Lorentzian oscillator group, **Giovanni Calvaruso** describes an explicit system of global coordinates for this Lie group, and uses it to compute its symmetries and solutions to the Ricci soliton equation.

Nastassja Cipriani and **José M.M. Senovilla** provide necessary and sufficient conditions for a spacelike submanifold of *arbitrary* co-dimension to be umbilical along normal directions. To this aim, they use the so-called *total shear tensor,* i.e. the trace-free part of the second fundamental form. They also show that the sum of the dimensions of the spaces generated by the total shear tensor and by the umbilical vector fields equals the co-dimension.

Ivan P. Costa e Silva presents a personal review of a number of results on the global geometric properties of stationary regions of spacetimes, both new and well-known (many of the latter with new proofs). He also discusses the general structure and regularity of the horizons associated with these regions. The analysis is largely carried out without assuming any field equations, asymptotic flatness/ hyperbolicity or dimensional restrictions, thereby emphasizing their independence of such extra, often physically motivated hypotheses.

J. Carlos Díaz-Ramos, Miguel Domínguez-Vázquez and **Víctor Sanmartín-López** are interested in isoparametric hypersurfaces in complex hyperbolic spaces, whose classification has recently been obtained by the authors. Concretely, they prove that given an isoparametric hypersurface in $\mathbb{C}H^n$, the principal curvatures of this hypersurface are pointwise the same, as the principal curvatures of a homogeneous hypersurface of $\mathbb{C}H^n$. Although this result can be obtained following the classification mentioned above, they prove it by a more direct approach, by lifting hypersurfaces in complex hyperbolic spaces to anti-de Sitter spacetimes.

The causal completion of spacetimes has succeeded as a tool to study different global properties of spacetimes. However, it is not always easy to extend the structure of the spacetime to the causal boundary. **Stacey (Steven) Harris** examines the extent to which various causal constructions and properties for spacetimes can be applied to the future completion of a strongly causal spacetime, considered as a topological space using the future chronological topology.

A very useful link between Lorentzian and Finslerian Geometries has been developed in the last years. In this volume, **Miguel Ángel Javaloyes** and **Miguel Sánchez** develop these relationships and several applications to spacetimes. In their contribution the purpose is twofold. On one hand, the authors provide a reference summary on the subject for Lorentzian geometers. On the other hand, they consider a big class of spacetimes admitting a time function t and characterize when the slices t = constant are Cauchy hypersurfaces.

Erdem Kocakuşaklı and **Miguel Ortega** study some general properties of translating solitons in the semi-Riemannian setting. In particular, they focus on the study of translating solitons which are invariant under the action of a Lie group of isometries of the ambient space, and they examine the behaviour near the singular orbit, whenever there exits, and at infinity. They also include several examples.

Wai Yeung Lam and **Masashi Yasumoto** investigate discretizations of surfaces with vanishing mean curvature (i.e., maximal surfaces) in the three-dimensional Lorentz-Minkowski space. In particular, the case of trivalent maximal surfaces is analysed. The authors derive a Weierstrass-type representation using discrete holomorphic quadratic differentials for these surfaces. Their contribution also includes a deep analysis of singularities of trivalent maximal surfaces.

The hypersurfaces with constant mean curvature in spacetimes are convenient initial data for the Cauchy problem of the Einstein equations. **Rafael López** provides a survey on this kind of hypersurfaces on the steady-state space. Three different models for this spacetime appear in his contribution. Each one is conveniently employed depending on the problems. The author focuses on Bernstein-type theorems by using as key tools the tangency principle and the Omori-Yau maximum principle.

Continuing with the celebrated Calabi–Bernstein theorem, **Rafael Rubio** presents a useful review about some of the classical and recent proofs of this theorem in the Lorentz-Minkowski spacetime for the two-dimensional case, as well as several extensions for Lorentzian-warped products and other relevant spacetimes. He also analyzes the problem of uniqueness of complete maximal hypersurfaces under the perspective of some new results.

Benjamín Olea focuses on the geometry of null hypersurfaces in Lorentzian manifolds. To this aim he uses the *rigging techniques,* one of the approaches to overcome the difficulties derived from the degeneracy of these objects. Concretely, he studies under which conditions the *rigged connection,* i.e. the Levi-Civita connection associated to the Riemannian metric induced by the rigging, coincides with the connection directly induced from the rigging, and gives some examples.

Finally, by means of the application of the Cauchy–Kovalevski theorem for partial differential equations, **Masaaki Umehara** and **Kotaro Yamada** construct all real analytic germs of zero mean curvature surfaces by using several concrete examples of such surfaces in $\mathbb{R}^3_1$ containing a certain light-like line. Besides they use a new approach to the subject, they obtain, as a consequence, new examples of zero mean curvature surfaces such that the causal type of one-side of the line is space-like and the other-side is time-like. Moreover, they provide several applications of these results.

Summing up, this volume constitutes a representative picture of the last progresses in the field of Lorentzian Geometry and related topics. Moreover, we think that the topics here included provide a nice approach to several very active research problems.

We would like to thank the careful work of the contributors, as well as the exhaustive revisions by the anonymous referees. We would also like to acknowledge Springer for its interest on this branch of knowledge and its staff for their friendly assistance.

Last but not least, we warmly thank all participants of the Lorentzian meeting for the excellent scientific level and pleasant atmosphere of this congress http://gigda.ugr.es/geloma/, as well as the support of the sponsors: the University of Malaga, the Vice-Rectorate for Research and Knowledge Transfer (UMA), the Department of Algebra, Geometry and Topology (UMA) and the Department of Applied Mathematics (UMA); the Spanish projects MTM2013-47828-C2-1-P, MTM2013-47828-C2-2-P and MTM2013-41768-P; the Academia Malagueña de Ciencias, the Sociedad Malagueña de Astronomía and the Real Sociedad Española de Física; and the Metro de Málaga.

Málaga, Spain
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Contents

Spacelike Hypersurfaces in the Lorentz-Minkowski Space with the Same Riemannian and Lorentzian Mean Curvature

Eva M. Alarcón, Alma L. Albujer and Magdalena Caballero

Abstract Spacelike hypersurfaces in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ can be endowed with two different Riemannian metrics, the metric inherited from $\mathbb{L}^{n+1}$ and the one induced by the Euclidean metric of $\mathbb{R}^{n+1}$. Consequently, we can consider two mean curvature functions naturally attached to any spacelike hypersurface, H_R and H_L. In this manuscript, we revise some known results for the case where $H_R = H_L$ is constant, and generalize to arbitrary dimension some recent results for spacelike surfaces with $H_R = H_L$ not necessarily constant obtained by Albujer and Caballero. Specifically, we prove that spacelike hypersurfaces with $H_R = H_L$ do not have any elliptic points. As an application of this result, jointly with a classical argument on the existence of elliptic points due to Osserman, we present several geometric consequences for the hypersurfaces we are considering. Finally, as any spacelike hypersurface in $\mathbb{L}^{n+1}$ is locally a graph over the hyperplane $x_{n+1} = 0$, our hypersurfaces are locally determined by the solutions to a certain partial differential equation called the $H_R = H_L$ hypersurface equation. The character of this equation is studied, and some uniqueness results for its related Dirichlet problem are given.

Keywords Mean curvature · Spacelike hypersurfaces · Elliptic points · Dirichlet problem

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1 Introduction and Background

A hypersurface in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ is said to be spacelike if its induced metric is a Riemannian one. We can endow a spacelike hypersurface in $\mathbb{L}^{n+1}$ with another Riemannian metric, the one inherited from the Euclidean space $\mathbb{R}^{n+1}$. Therefore, we can consider two different mean curvature functions on a spacelike hypersurface, the mean curvature function related to the metric induced by $\mathbb{R}^{n+1}$, that we will denote by H_R, and the one related to the metric inherited from $\mathbb{L}^{n+1}$, H_L.

A hypersurface in $\mathbb{R}^{n+1}$ is said to be minimal if its mean curvature function vanishes identically, that is $H_R \equiv 0$. Analogously, a spacelike hypersurface in $\mathbb{L}^{n+1}$ is said to be maximal if $H_L \equiv 0$. The study of minimal and maximal hypersurfaces is a topic of wide interest. One of the main results about the global geometry of minimal surfaces is the well-known Bernstein theorem, proved by Bernstein [5] in 1915, which states that the only entire minimal graphs in $\mathbb{R}^3$ are the planes. Some decades later, in 1970, Calabi [7] proved its analogous version for spacelike surfaces in the Lorentz-Minkowski space, the Calabi-Bernstein theorem, which states that the only entire maximal graphs in $\mathbb{L}^3$ are the spacelike planes. An important difference between both results is that the Bernstein theorem can be extended to minimal graphs in $\mathbb{R}^{n+1}$ up to dimension $n = 7$, as it was proved by Bombieri et al. [6], but it is no longer true for higher dimensions. However, the Calabi-Bernstein theorem holds true for any dimension as it was proved by Calabi [7] for dimension $n \leq 4$, and by Cheng and Yau [8] for arbitrary dimension.

It is interesting to note that any complete spacelike hypersurface in $\mathbb{L}^{n+1}$ is necessarily an entire graph over any spacelike hyperplane, see [4, Proposition 3.3]. Consequently, the Calabi-Bernstein theorem can also be expressed in a parametric way by asserting that the only complete maximal hypersurfaces in $\mathbb{L}^{n+1}$ are the spacelike hyperplanes. Its Riemannian analogue is not true since there exists a wide family of nonplanar complete minimal hypersurfaces in $\mathbb{R}^{n+1}$, even in the 2-dimensional case ($n = 2$).

As an immediate consequence of the above results, we conclude that the only complete hypersurfaces that are simultaneously minimal in $\mathbb{R}^{n+1}$ and maximal in $\mathbb{L}^{n+1}$ are the spacelike hyperplanes.

Going a step further, we can consider spacelike hypersurfaces with the same constant mean curvature functions H_R and H_L. In 1955, as a direct consequence of the classical divergence theorem, Heinz [12] proved that given a graph in $\mathbb{R}^3$ defined over a disk of radius R in $\mathbb{R}^2$ centered at the origin, $B_0(R)$, if $|H_R| \geq c > 0$ for a certain constant c, then $R \leq \frac{1}{c}$ necessarily. Some years later, Chern [9] and Flanders [10] simultaneously and independently extended this result to general dimension. Therefore, the only entire graphs with constant mean curvature H_R in $\mathbb{R}^{n+1}$ are the minimal ones. The Lorentzian version of this result is not true, there are examples of entire spacelike graphs with constant mean curvature H_L in $\mathbb{L}^{n+1}$ which are not maximal, for instance the hyperbolic spaces. However, taking into account the Calabi-Bernstein theorem, we conclude again that the only complete spacelike hypersurfaces in $\mathbb{L}^{n+1}$ with the same constant mean curvature functions H_R and H_L are the spacelike hyperplanes.

Without assuming any completeness hypothesis, Kobayashi [14] studied the problem for $H_R = H_L = 0$ in the 2-dimensional case. After presenting a classification of maximal ruled surfaces in $\mathbb{L}^3$, he showed that the only surfaces that are simultaneously minimal and maximal are necessarily ruled. And consequently, they are open pieces of a spacelike plane or of a helicoid in the region where the helicoid is spacelike. Recently, Albujer et al. [1, 2] have continued with the study of spacelike surfaces with the same mean curvature in $\mathbb{R}^3$ and in $\mathbb{L}^3$, not necessarily constant. Specifically, they have shown that those surfaces have non-positive Gaussian curvature with respect to the metric induced from $\mathbb{R}^3$ in all their points, and have obtained several interesting consequences about the geometry of such surfaces.

In general dimension, Lee and Lee [15] have recently presented nonplanar examples of simultaneously minimal and maximal spacelike graphs in the Lorentz-Minkowski space. Their examples can be seen as generalized ruled hypersurfaces, in fact they are a natural generalization of helicoids. However, there is no known classification of such hypersurfaces similar to Kobayashi's result.

Our main purpose in this manuscript is to generalize some of the results in [1, 2], providing some geometric properties of spacelike hypersurfaces in $\mathbb{L}^{n+1}$ with $H_R = H_L$. In Sect. 2 we present some basic preliminaries on spacelike hypersurfaces in $\mathbb{L}^{n+1}$ and their mean curvature functions with respect to the metrics inherited from $\mathbb{R}^{n+1}$ and $\mathbb{L}^{n+1}$. It is well known that any spacelike hypersurface can be locally seen as a graph over an open subset of a spacelike hyperplane, which without loss of generality can be supposed to be the hyperplane $x_{n+1} = 0$, see [16] for the proof in the two-dimensional case. Therefore, we also describe the normal vector fields and the mean curvature functions with respect to both metrics in terms of the differential operators of the function which locally describes the hypersurface. Finally, we recall a characterization result by Osserman [17] for hypersurfaces in $\mathbb{R}^{n+1}$ without elliptic points and we present its Lorentzian version for spacelike hypersurfaces in $\mathbb{L}^{n+1}$, see Theorem 2, which extends [1, Theorem 3].

In Sects. 3 and 4 we consider spacelike hypersurfaces in $\mathbb{L}^{n+1}$ such that $H_R = H_L$. Specifically, in Sect. 3 we prove that at any point of those hypersurfaces the principal curvatures cannot have all of them the same sign. From this theorem, as well as from Osserman's result, we get some geometric consequences to which the rest of the section is devoted.

In Sect. 4 we present the $H_R = H_L$ hypersurface equation. Any spacelike hypersurface is locally determined by a solution of this equation satisfying $|Du| < 1$, where D and $|\cdot|$ stand for the Euclidean gradient and Euclidean norm in $\mathbb{R}^n$, respectively. We prove the uniqueness of the Dirichlet problem associated to this partial differential equation under some appropriate boundary conditions. This is not trivial, since the equation is not always elliptic. We also consider rotationally invariant spacelike graphs with $H_R = H_L$, obtaining a uniqueness result for them.

Some of the proofs are analogous to the two-dimensional case, still we will include them for the sake of completeness.

2 Preliminaries

Let $\mathbb{L}^{n+1}$ be the $(n+1)$-dimensional Lorentz-Minkowski space, that is, $\mathbb{R}^{n+1}$ endowed with the metric

$$\langle , \rangle_L = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2,$$

where $(x_1, \ldots, x_{n+1})$ are the canonical coordinates in $\mathbb{R}^{n+1}$, and let $|\cdot|_L$ denote its norm. It is easy to see that the Levi-Civita connections of the Euclidean space $\mathbb{R}^{n+1}$ and the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ coincide, so we will just denote them by $\overline{\nabla}$.

A (connected) hypersurface Σ^n in $\mathbb{L}^{n+1}$ is said to be a spacelike hypersurface if $\mathbb{L}^{n+1}$ induces a Riemannian metric on Σ, which is also denoted by $\langle , \rangle_L$. Given a spacelike hypersurface Σ, we can choose a unique future-directed unit normal vector field N_L on Σ. Let ∇^L denote the Levi-Civita connection in Σ with respect to $\langle , \rangle_L$. Then the Gauss and Weingarten formulae for the spacelike hypersurface Σ become

$$\overline{\nabla}_X Y = \nabla^L_X Y - \langle A_L X, Y \rangle_L N_L$$

and

$$A_L X = -\overline{\nabla}_X N_L,$$

respectively, for any tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$, where $A_L : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ stands for the shape operator of Σ with respect to N_L. The mean curvature function of Σ with respect to N_L is defined by

$$H_L = -\frac{1}{n}\mathrm{tr}\, A_L = -\frac{1}{n}(k_1^L + \cdots + k_n^L),$$

where $k_i^L,\ i = 1, \ldots, n$, stand for the principal curvatures of $(\Sigma, \langle , \rangle_L)$.

It is well known that there exists no closed (compact and without boundary) spacelike hypersurface in $\mathbb{L}^{n+1}$ [3, 4]. Therefore, every compact spacelike hypersurface Σ in the Lorentz-Minkowski space necessarily has nonempty boundary.

The same topological hypersurface can be also considered as a hypersurface of the Euclidean space, that is $\mathbb{R}^{n+1}$ with its usual Euclidean metric. For simplicity, we will just denote the Euclidean space by $\mathbb{R}^{n+1}$, the Euclidean metric and the induced metric on Σ by $\langle , \rangle_R$, and its norm by $|\cdot|_R$. In such a case, Σ admits a unique upward directed unit normal vector field, N_R. In an analogous way as in the Lorentzian case, let ∇^R denote the Levi-Civita connection in Σ with respect to $\langle , \rangle_R$. The Gauss and Weingarten formulae read now

$$\overline{\nabla}_X Y = \nabla^R_X Y + \langle A_R X, Y \rangle_R N_R$$

and

$$A_R X = -\overline{\nabla}_X N_R,$$

respectively, $A_R : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ being the shape operator of Σ with respect to N_R. The mean curvature function of Σ with respect to N_R is defined by

$$H_R = \frac{1}{n}\mathrm{tr}\, A_R = \frac{1}{n}(k_1^R + \cdots + k_n^R),$$

where $k_i^R,\ i = 1, \ldots, n$, stand for the principal curvatures of $(\Sigma, \langle , \rangle_R)$. Let us recall that a point $p \in \Sigma$ is said to be elliptic if all the principal curvatures of Σ at p have the same sign.

It is interesting to observe that the mean curvature functions have an expression in terms of the normal curvatures of any set of orthogonal directions. Specifically,

$$H_R = \frac{1}{n}(\kappa_{v_1}^R + \cdots + \kappa_{v_n}^R) \quad \text{and} \quad H_L = -\frac{1}{n}(\kappa_{w_1}^L + \cdots + \kappa_{w_n}^L), \tag{2.1}$$

where $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are orthonormal basis of $T_p\Sigma$ with respect to $\langle , \rangle_R$ and $\langle , \rangle_L$, respectively.

A spacelike hypersurface is locally a graph over an open subset of the hyperplane $x_{n+1} = 0$, which can be identified with $\mathbb{R}^n$. Therefore, for each $p \in \Sigma$ there exists an open neighborhood of p, $\Omega \subseteq \mathbb{R}^n$, and a smooth function $u \in C^\infty(\Omega)$ such that $\Sigma = \Sigma_u$ on this neighborhood, where

$$\Sigma_u = \{(x_1, \ldots, x_n, u(x_1, \ldots, x_n)) : (x_1, \ldots, x_n) \in \Omega\}.$$

It is easy to check that Σ_u is a spacelike hypersurface if and only if $|Du| < 1$, where D and $|\cdot|$ stand for the gradient operator and the norm in the Euclidean space $\mathbb{R}^n$, respectively. In this case, it is possible to get expressions for the normal vector fields N_L and N_R, as well as for the mean curvature functions H_L and H_R, in terms of u. Specifically, with a straightforward computation we get

$$N_L = \frac{(Du, 1)}{\sqrt{1 - |Du|^2}} \quad \text{and} \quad N_R = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}. \tag{2.2}$$

And for the mean curvature functions we have

$$H_L = \frac{1}{n}\,\mathrm{div}\left(\frac{Du}{\sqrt{1 - |Du|^2}}\right) \quad \text{and} \quad H_R = \frac{1}{n}\,\mathrm{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right), \tag{2.3}$$

where div denotes the divergence operator in $\mathbb{R}^n$. Let us observe that

$$\cos\theta = \frac{1}{\sqrt{1 + |Du|^2}} \quad \text{and} \quad \cosh\psi = \frac{1}{\sqrt{1 - |Du|^2}},$$

where θ and ψ denote the angle between N_R and $e_{n+1} = (0, \ldots, 0, 1)$ and the hyperbolic angle between N_L and e_{n+1}, respectively. Moreover, from (2.2) it is immediate to get

$$\frac{\langle X, N_L \rangle_L}{\cosh \psi} = -\frac{\langle X, N_R \rangle_R}{\cos \theta}, \tag{2.4}$$

for any $X \in \mathfrak{X}(\Sigma)$, which is a global equality since it does not depend on u. Let us observe that, in the previous expressions, we are writing Du instead of $Du \circ \pi$, where π is the canonical projection of Σ_u onto Ω. On behalf of simplicity, we will continue using this identification along the manuscript.

According to Osserman [17], a hypersurface Σ in the Euclidean space $\mathbb{R}^{n+1}$ satisfies the convex hull property if every compact subset $D \subseteq \Sigma$ lies in the convex hull of its boundary. In [17, Theorem], he gave the following simple geometric condition characterizing those hypersurfaces.

Theorem 1 ([17, Theorem]) *A hypersurface Σ in $\mathbb{L}^{n+1}$ has the convex hull property if and only if there is no elliptic point in Σ.*

Theorem 1 also holds for spacelike hypersurfaces in $\mathbb{L}^{n+1}$. This yields from the following lemma contained in [1]. We expose the proof of the lemma for the sake of completeness.

Lemma 1 ([1, Lemma 2]) *Let Σ be a spacelike hypersurface in $\mathbb{L}^{n+1}$. Given $p \in \Sigma$ and $v \in T_p\Sigma$, let $\kappa_v^L(p)$ and $\kappa_v^R(p)$ denote the normal curvatures at p in the direction of v with respect to $\langle , \rangle_L$ and $\langle , \rangle_R$, respectively. Then,*

$$\frac{|v|_R^2}{\cos \theta(p)} \kappa_v^R(p) = -\frac{|v|_L^2}{\cosh \psi(p)} \kappa_v^L(p).$$

Proof Given $p \in \Sigma$ and $v \in T_p\Sigma$, let α be a smooth curve on Σ such that $\alpha(0) = p$ and $\alpha'(0) = v$. We will work at p, but for simplicity we will omit it. Then, by definition,

$$\kappa_v^R = \langle \overline{\nabla}_{t_R} t_R, N_R \rangle_R \qquad \text{and} \qquad \kappa_v^L = \langle \overline{\nabla}_{t_L} t_L, N_L \rangle_L, \tag{2.5}$$

where $t_R = \frac{\alpha'}{|\alpha'|_R}$ and $t_L = \frac{\alpha'}{|\alpha'|_L}$. We combine (2.4) and (2.5) to finish the proof. □

Consequently, κ_v^L and κ_v^R always have opposite signs. And so, all the principal curvatures of Σ with respect to $\langle , \rangle_R$ are positive if and only if all its principal curvatures with respect to $\langle , \rangle_L$ are negative, and vice versa. Equivalently, a point in Σ is elliptic with respect to the metric $\langle , \rangle_L$ if and only if it is elliptic with respect to $\langle , \rangle_R$. Hence, we have proved the Lorentzian version of Theorem 1.

Theorem 2 *A spacelike hypersurface Σ in $\mathbb{R}^{n+1}$ has the convex hull property if and only if there is no elliptic point in Σ.*

The above result is a generalization of [1, Theorem 3], which states that if Σ is a compact spacelike hypersurface in $\mathbb{L}^{n+1}$ not contained in the convex hull of its boundary, then it necessarily has an elliptic point.

3 Spacelike Hypersurfaces with $H_R = H_L$

We can now state and prove our first main result.

Theorem 3 *Let Σ be a spacelike hypersurface in $\mathbb{L}^{n+1}$ such that $H_R = H_L$. Then not all the principal curvatures have the same sign. That is, there is no elliptic point in Σ.*

Proof We are going to work locally, so we can assume that there exists an open subset $\Omega \subseteq \mathbb{R}^n$ and a smooth function $u \in \mathcal{C}^\infty(\Omega)$ such that $\Sigma = \Sigma_u$. We define Σ^* as the graph of u over the following open set

$$\Omega^* = \{(x_1, \ldots, x_n) \in \Omega : Du(x_1, \ldots, x_n) \neq 0\}.$$

Given $p \in \Sigma^*$, we consider its corresponding level hypersurface contained in $\mathbb{R}^n$ and we call its lifting to Σ, S_c. We are working in a neighborhood of p, hence we can assume that S_c lies on Σ^*. Since $Du \neq 0$ in Ω^*, its distribution is integrable, so we can consider the integral curve through $\pi(p)$. We denote by α its lifting to Σ^*. We observe that a vector field tangent to α is $\alpha' = (Du, |Du|^2) \circ \pi$. Therefore, we have two submanifolds defined on a neighborhood of p which are orthogonal at p for both $\langle , \rangle_R$ and $\langle , \rangle_L$. Now, let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal basis of $T_p S_c$ in $\mathbb{R}^{n+1}$. These vectors are also orthonormal in $\mathbb{L}^{n+1}$, and orthogonal to α' with respect to both metrics. Then, Lemma 1 gives us the following relationships, where we have omitted the point p on behalf of simplicity

$$\kappa^R_{e_i} = -\frac{|e_i|^2_L}{|e_i|^2_R}\frac{\cos\theta}{\cosh\psi}\,\kappa^L_{e_i} = -\sqrt{\frac{1-|Du|^2}{1+|Du|^2}}\,\kappa^L_{e_i}, \quad i = 1, \ldots, n-1 \quad \text{and}$$

$$\kappa^R_{\alpha'} = -\frac{|\alpha'|^2_L}{|\alpha'|^2_R}\frac{\cos\theta}{\cosh\psi}\,\kappa^L_{\alpha'} = -\left(\frac{1-|Du|^2}{1+|Du|^2}\right)^{\frac{3}{2}}\kappa^L_{\alpha'}.$$

By denoting $A = \sqrt{\frac{1-|Du|^2}{1+|Du|^2}}$, we rewrite the previous expressions as

$$\kappa^R_{e_i} = -A\,\kappa^L_{e_i}, \qquad i = 1, \ldots, n-1 \qquad \text{and} \qquad \kappa^R_{\alpha'} = -A^3\,\kappa^L_{\alpha'}. \tag{3.1}$$

As we are dealing with orthogonal directions at p for both $\langle , \rangle_R$ and $\langle , \rangle_L$, and we are assuming $H_R = H_L$, from (2.1) we get

$$-\kappa^L_{e_1} - \cdots - \kappa^L_{e_{n-1}} - \kappa^L_{\alpha'} = \kappa^R_{e_1} + \cdots + \kappa^R_{e_{n-1}} + \kappa^R_{\alpha'},$$

which jointly with (3.1) implies

$$\kappa^L_{e_1} + \cdots + \kappa^L_{e_{n-1}} = -(A^2 + A + 1)\kappa^L_{\alpha'},$$

and so

$$\left(\kappa^L_{e_1} + \cdots + \kappa^L_{e_{n-1}}\right) \kappa^L_{\alpha'} \leq 0.$$

On the other hand, we can express the normal curvature given by a unitary vector $v = \sum_{i=1}^n a_i e_i^* \in T_p\Sigma$, where e_i^*, $i = 1, \ldots, n$, stand for the principal directions of Σ at p, in the next way

$$\kappa^L_v = a_1^2 k_1^L + \cdots + a_n^2 k_n^L.$$

Then, if we suppose that all the principal curvatures have the same sign, we get a contradiction.

Consider now $p \in \Sigma \setminus \Sigma^*$. If $p \in \text{int}\,(\Sigma \setminus \Sigma^*)$, then Σ is locally a horizontal hyperplane around p, and so $\kappa_i^L = \kappa_i^R = 0$ for all $i = 1, \ldots, n$. Otherwise $p \in \partial\Sigma^*$, and the result follows from a continuity argument. □

We have just proved that a hypersurface Σ such that $H_R = H_L$ does not have any elliptic point, which jointly which Theorem 1 leads to some interesting geometric consequences, to which the rest of the manuscript is devoted. The first of them is immediate from both results.

Theorem 4 *Let Σ be a compact spacelike hypersurface with (necessarily) nonempty boundary such that $H_R = H_L$. Then Σ is contained in the convex hull of its boundary.*

Let us recall that any spacelike hypersurface is locally a graph Σ_u over an open subset $\Omega \subseteq \mathbb{R}^n$. From now on, we will focus on spacelike graphs.

First, we present a uniqueness result for graphs which are asymptotic to a spacelike hyperplane, where the term asymptotic is defined as follows. We say that two entire graphs Σ_u and Σ_v over $\mathbb{R}^n$ are *asymptotic* if for every $\varepsilon > 0$ there exists a compact set $K \subset \mathbb{R}^n$ such that $|u(x_1, \ldots, x_n) - v(x_1, \ldots, x_n)| < \varepsilon$ for every $(x_1, \ldots, x_n) \in \mathbb{R}^n \setminus K$. Observe that, without loss of generality, we can consider that those compact sets are Euclidean balls of a certain radius. If we define the *width* of a set in $\mathbb{R}^n$ as the supremum of the diameter of the closed balls contained in it, the concept of asymptotic graphs is not only well defined in the case of entire graphs, but also in the case of graphs over a domain of infinite width, that is, a domain which contains closed balls of any radius. Notice that this definition is a generalization of the classical concept of width for a convex body, see [18].

Theorem 5 *The only spacelike graphs Σ_u in $\mathbb{L}^{n+1}$ defined over an open subset $\Omega \subseteq \mathbb{R}^n$ of infinite width, with $H_R = H_L$, and asymptotic to a spacelike hyperplane, are (pieces of) spacelike hyperplanes.*

Proof Let us notice that Σ_u is a graph over any spacelike hyperplane, and in particular, over the hyperplane to which it is asymptotic. To prove it, it is enough to observe that if some timelike line intersects Σ_u twice, the plane with director vector $e_{n+1} = (0, \ldots, 0, 1)$ and containing that line cuts Σ_u in a curve that is timelike at some point, which is a contradiction.

Let us denote by Π the hyperplane to which Σ_u is asymptotic and let $v \in \mathcal{C}^\infty(\Omega')$ be the function such that $\Sigma_u = \Sigma_v$, $\Omega' \subseteq \Pi$ being the domain of definition of v. Notice that the width of Ω' is also infinite.

For any $\varepsilon > 0$ there exists $(y_1, \ldots, y_n) \in \Omega'$ and $R > 0$ such that $|v(x_1, \ldots, x_n)| < \varepsilon$ for every $(x_1, \ldots, x_n) \in \Omega' \setminus \bar{B}_{(y_1,\ldots,y_n)}(R)$. By Theorem 4, we know that the graph of the restriction of v to $\bar{B}_{(y_1,\ldots,y_n)}(R)$ is contained in the convex hull of its boundary. Therefore, $|v(x_1, \ldots, x_n)| \leq \varepsilon$ for all $(x_1, \ldots, x_n) \in \bar{B}_{(y_1,\ldots,y_n)}(R)$, so this inequality holds globally on Ω'. Taking limits when ε approaches 0, we conclude that $\Sigma_u = \Omega'$. □

4 A Quasi-linear PDE Related to Spacelike Hypersurfaces with $H_R = H_L$

As we have already mentioned, any spacelike hypersurface is locally a graph Σ_u over an open subset $\Omega \subseteq \mathbb{R}^n$. Thanks to (2.3), if we consider the differential operator given by

$$Q(u) = \operatorname{div}\left(\left(\frac{1}{\sqrt{1-|Du|^2}} - \frac{1}{\sqrt{1+|Du|^2}}\right) Du\right),$$

those graphs are the solutions to the equation

$$Q(u) = 0, \tag{4.1}$$

satisfying $|Du| < 1$. We will refer to the above equation as *the $H_R = H_L$ hypersurface equation*.

Let us observe firstly that (4.1) is an elliptic quasi-linear partial differential equation, everywhere except at those points where $Du = 0$, at which it is parabolic. In fact, it is a tedious but straightforward computation to show that (4.1) can be expressed as

$$\langle B(Du), D^2u \rangle = 0,$$

D^2u being the Hessian matrix of u with respect to the Euclidean metric of $\mathbb{R}^n$ and $B(Du)$ a symmetric and positive definite matrix on Du, everywhere except at those points where $Du = 0$, where it vanishes.

It is well known that the solutions to a second order elliptic quasi-linear partial differential equation for an analytic operator Q are always analytic, see [13] for a proof of this fact. Therefore, if u is a solution of (4.1) with $0 < |Du| < 1$, it is necessarily analytic. However, in general the analyticity of the solutions of (4.1) cannot be guaranteed.

Along this section, let Ω be a domain of $\mathbb{R}^n$, that is an open and bounded subset of $\mathbb{R}^n$. Given a domain $\Omega \subset \mathbb{R}^n$ and $\psi \in \mathcal{C}^0(\partial\Omega)$, the Dirichlet problem related to the $H_R = H_L$ hypersurface equation consists in finding a solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ to the boundary value problem

$$\left.\begin{array}{r} Q(u) = 0 \ \text{in} \quad \Omega \\ |Du| < 1 \ \text{in} \quad \Omega \\ u = \psi \quad \text{on } \partial\Omega \end{array}\right\}. \tag{4.2}$$

As a consequence of a uniqueness theorem for the Dirichlet problem associated to quasilinear elliptic operators [11, Theorem 9.3], we get our next result.

Theorem 6 *Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary and $\psi \in \mathcal{C}^0(\partial\Omega)$ such that the Dirichlet problem* (4.2) *admits a solution u without critical points. Then, the solution is unique.*

Remark 1 It is interesting to observe that [11, Theorem 9.3] holds under four assumptions on the operator defining the equation, one of which does not hold in our case. Since we are assuming the spatially condition $|Du| < 1$, the coefficients of Q are not well defined on the whole $\Omega \times \mathbb{R} \times \mathbb{R}^n$, as it is required in the last hypothesis, but just on $\Omega \times \mathbb{R} \times B_0(1)$. However, studying in detail the proof of the cited theorem, we can realize that it is sufficient to consider the coefficients defined on $\Omega \times \mathbb{R} \times B_0(1)$.

It is still more interesting to put emphasis on the fact that the proof does not work if the ellipticity fails somewhere. Therefore, we can not omit the hypothesis on the gradient of u. However, as a consequence of Theorem 4, we get the following result on the uniqueness of the Dirichlet problem under appropriate boundary values.

Theorem 7 *The only solutions to the Dirichlet problem* (4.2) *with affine boundary value are the affine functions.*

Proof Let u be a solution of (4.2) and Σ_u its associated graph. From Theorem 4, Σ_u is contained in the convex hull of its boundary, which is contained in a hyperplane. Therefore, the spacelike graph Σ_u must be also contained in the same hyperplane, and consequently u is affine.

In the previous reasoning it is crucial to observe that Theorem 4 also works for $\mathcal{C}^2$-hypersurfaces. □

4.1 On Rotationally Invariant Spacelike Graphs

From now on, let us consider rotationally invariant spacelike graphs with respect to a vertical axis. Therefore, we can assume without loss of generality that the graph Σ_u is determined by a function

$$u(x_1, \ldots, x_n) = f(r), \quad r = x_1^2 + \cdots + x_n^2, \tag{4.3}$$

where $f \in \mathcal{C}^\infty(I)$ for certain $I \subseteq [0, +\infty)$. As an immediate consequence of Theorem 7 we get the following uniqueness result for entire rotationally invariant spacelike graphs with $H_R = H_L$.

Theorem 8 *The only entire spacelike graphs Σ_u determined by a function u given by (4.3) such that $H_R = H_L$ are the horizontal hyperplanes.*

Proof Given a positive constant R, any entire solution to the $H_R = H_L$ hypersurface equation of the form (4.3) is a solution of the Dirichlet problem (4.2) over $B_0(R)$ with constant boundary value. By Theorem 7, the function u must also be constant in $\overline{B_0}(R)$. The result is proven taking limits when R approaches infinity. □

Let us observe that Theorem 8 works not only for entire graphs, but for graphs defined over a ball centered at the origin of $\mathbb{R}^n$.

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Space-Time Convex Functions and Sectional Curvature

Stephanie B. Alexander and William A. Karr

Abstract We show that in Lorentzian manifolds, sectional curvature bounds of the form $\mathcal{R} \leq K$, as defined by Andersson and Howard, are closely tied to space-time convex and λ-convex ($\lambda > 0$) functions, as defined by Gibbons and Ishibashi. Among the consequences are a natural construction of such functions, and an analog, that applies to domains of a new type, of a theorem of Alías, Bessa and deLira ruling out trapped submanifolds.

Keywords Space-time · Convex function · Distance · Hessian · Trapped submanifold

1 Introduction

A study of the possible uses of convex functions in General Relativity was initiated by Gibbons and Ishibashi, according to whom: "Convexity and convex functions play an important role in theoretical physics ... [and] also have important applications to geometry, including Riemannian geometry ... It is surprising therefore that, to our knowledge, that techniques making use of convexity and convex functions have played no great role in General Relativity" [8].

Gibbons and Ishibashi introduce and mainly consider "space-time convex" functions on Lorentzian manifolds (M, g), or more generally, functions f satisfying

$$\overline{\nabla}^2 f \geq \lambda g, \quad \lambda > 0.$$

They find examples and nonexamples of such functions on regions in cosmological space-times and black-hole space-times. They show, for example, that such functions

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rule out closed marginally inner and outer trapped surfaces. Curvature bounds do not arise in their considerations.

The purpose of this note is to show that sectional curvature bounds of the form $\mathcal{R} \leq K$ are closely tied to space-time convex functions. Among the consequences:

- A natural construction of such functions.
- New domains that cannot support trapped submanifolds, namely a full neighborhood of a point q, rather than a neighborhood of q in the chronological future of q as has been considered previously, in particular by Alías et al. [2].

The bound $\mathcal{R} \leq K$, introduced by Andersson and Howard [4], extends Sec $\leq K$ from the Riemannian to the semi-Riemannian setting by requiring spacelike sectional curvatures to be $\leq K$ and timelike ones to be $\geq K$. Equivalently, the curvature tensor is required to satisfy

$$g(R(v, w)v, w) \;\leq\; K\big(g(v, v)\, g(w, w) - g(v, w)^2\big).$$

For $\mathcal{R} \geq K$, reverse the inequalities.

In addition, we indicate connections between investigations that have been pursued independently by various authors, including:

- Comparison theorems for Lorentzian distance on domains in the chronological future of a source point or hypersurface on which the source has no Lorentzian cut points, given timelike sectional curvature controls (see, for example, [2, 3, 6, 12]).
- Hessian comparisons on level hypersurfaces in exponentially embedded neighborhoods of a point or hypersurface, given a sectional curvature bound of the form $\mathcal{R} \leq K$ or $\mathcal{R} \geq K$ [1, 4].
- Space-time convex functions [8].

1.1 Outline of the Paper

Section 2 is an introduction to space-time convex and λ-convex functions, as defined in [8].

Section 3 summarizes certain theorems about Hessian and Laplacian comparisons on the Lorentzian distance function from a point or achronal spacelike hypersurface, under comparisons on timelike sectional curvature [2, 3, 6, 12].

Section 4 describes results from [1, 4] concerning the conditions $\mathcal{R} \geq K$ and $\mathcal{R} \leq K$ in semi-Riemannian manifolds. In particular, in [4] Andersson and Howard prove a comparison theorem for matrix Ricatti equations which applies to the second fundamental forms of parallel families of hypersurfaces under curvature comparisons. In [1], this theorem is adapted to tubes around points; as an application, the geometric meaning of the bounds $\mathcal{R} \geq K$ and $\mathcal{R} \leq K$ is found by introducing signed lengths of geodesics.

In Sect. 5, we use this framework to rule out trapped submanifolds in an exponentially embedded neighborhood of a point in a space-time satisfying $\mathcal{R} \leq K$.

2 Space-Time Convex Functions

Definition 2.1 Given smooth functions $f : M \to \mathbf{R}$ and $\lambda : M \to \mathbf{R}$ on a semi-Riemannian manifold (M, g), f will be called λ-*convex* if the Hessian $\overline{\nabla}^2 f$ satisfies

$$\overline{\nabla}^2 f \geq \lambda\, g, \tag{2.1}$$

or equivalently,

$$(f \circ \gamma)'' \geq (\lambda \circ \gamma)\, g(\gamma', \gamma') \tag{2.2}$$

for every geodesic γ.

Suppose M is Lorentzian. We say f is *space-time* λ-*convex* if f is λ-convex for some *positive* function λ, and $\overline{\nabla}^2 f$ has Lorentzian signature.

Note that this definition differs from the classical definition of convexity in that the right-hand sides of (2.1) and (2.2) need not be positive when $\lambda > 0$. Rather, controlled concavity is allowed along timelike geodesics, and is imposed in the definition of space-time convexity.

One of the simplest examples of a space-time λ-convex function is

$$f(\mathbf{x}, t) = \frac{1}{2}(\mathbf{x} \cdot \mathbf{x} - \lambda t^2), \quad (\mathbf{x}, t) \in \mathbf{E}_1^{n+1}, \tag{2.3}$$

on Minkowski space for some constant $0 < \lambda \leq 1$.

As pointed out in [8], the geometric meaning of space-time convexity is that at each point, the forward light cone defined by the Hessian $\overline{\nabla}^2 f$ lies inside the light cone defined by the space-time metric.

Definition 2.1 is consistent with current Riemannian/Alexandrov usage of "λ-convex" (see [14]); and also with the definition of "space-time convex" in [8] except that our λ is a positive function and Gibbons and Ishibashi take λ to be a positive constant. (However, Definition 2.1 differs from the usage in [1].)

In [8], Gibbons and Ishibashi begin an investigation of the geometric implications of space-time convex functions. For example, they show that a space-time with a closed marginally inner and outer trapped surface cannot support a space-time convex function.

Here a *marginally inner and outer trapped surface* Σ is a spacelike submanifold of codimension 2 whose mean curvature vanishes.

Seeking examples of space-time convex functions, Gibbons and Ishibashi consider *Robertson-Walker spaces*

$$M = -I \times_f F,$$

that is, M is the product manifold $I \times F$ carrying the warped product metric

$$-d\tau^2 + f^2 ds_F^2$$

where $I = (a, b)$, $a \in [-\infty, \infty)$, $b \in (-\infty, \infty]$, $f : I \to \mathbf{R}_+$, and F has constant sectional curvature. They ask when the function

$$-f^2/2 \tag{2.4}$$

is space-time convex (here we use f to denote both the warping function and its lift to M). For instance, various cosmological charts are considered on de-Sitter space $\mathbf{dS}^{n+1}$ and anti-de-Sitter space $\mathbf{adS}^{n+1}$. One of these yields an affirmative answer: namely, the function (2.4) is space-time convex on the region

$$(0, \pi/2) \times_{\sin} \mathbf{H}^n$$

in $\mathbf{adS}^{n+1}$.

Gibbons and Ishibashi do not consider curvature bounds when seeking examples. The perspective of space-times with curvature bounds of the form $\mathcal{R} \leq K$ suggests an alternative, namely analogs of the "square norm" ((2.3) with $\lambda = 1$). For instance, these analogs yield space-time convex functions adapted to some of the domains in de-Sitter and anti-de-Sitter space considered in [8].

Our theorems show that space-time convex functions arise naturally in all Lorentzian manifolds satisfying $\mathcal{R} \leq K$.

3 Comparisons for Lorentzian Distance

Let us mention some related works concerning the Lorentian distance functions from a point or spacelike hypersurface. All these investigations are restricted to domains containing no Lorentzian cut points of the source point or hypersurface.

(1) In [6], Erkekoglu, Garcia-Rio and Kupeli prove Hessian and Laplacian comparison theorems for level sets of the Lorentzian distance function from points or from achronal spacelike hypersurfaces, in two space-times M and $\tilde{M}$. They consider corresponding timelike, distance-realizing unit geodesics in M and $\tilde{M}$, where sectional curvatures of two-planes tangent to the geodesics at corresponding values of the time parameter are no greater in M than in $\tilde{M}$. Some space-time singularity theorems are given.

(2) In [3], Alías, Hurtado and Palmer study the restriction of Lorentzian distance from a point or achronal spacelike hypersurface to a spacelike hypersurface satisfying the Omori-Yau maximum principle. Under constant bounds either above or below on timelike sectional (or Ricci) curvatures, they obtain sharp estimates on the mean curvature of such hypersurfaces.

(3) In [12], Impera studies Hessian and Laplacian comparisons for Lorentzian distance from a point, assuming timelike sectional curvatures are bounded above or below by a function of the Lorentzian distance. Estimates are obtained on the higher order mean curvatures of spacelike hypersurfaces satisfying the Omori-Yau maximum principle.
(4) In [2], Alías, Bessa and deLira prove nonexistence results and sharp mean curvature estimates for trapped submanifolds (of arbitrary codimension), based on comparison inequalities for the Laplacian of the restriction to a spacelike submanifold of the Lorentzian distance function from a point or achronal spacelike hypersurface. They use a weak Omori-Yau maximum principle equivalent to stochastic completeness.

4 Curvature Bounds $\mathcal{R} \leq K, \mathcal{R} \geq K$

Recall that $\mathcal{R} \leq K$ means that spacelike sectional curvatures are $\leq K$ and timelike ones are $\geq K$. For $\mathcal{R} \geq K$, reverse the inequalities. (Note that $\mathcal{R} \leq K \leq K'$ does not imply $\mathcal{R} \leq K'$!)

4.1 Geometric Meaning

Briefly, $\mathcal{R} \leq K$ means, as in the Riemannian case, that unit geodesics radiating from a point "repel" each other at least as much as in a space of constant curvature K, assuming the same initial conditions. However, repulsion here is meant in the *signed* sense. In particular, in the Lorentzian case, if the initial direction of variation of the geodesics is timelike, we see *negative repulsion*, that is, at least as much *attraction* as in a Lorentzian space of constant curvature K. This is explained below in Sects. 4.3 and 4.4.

4.2 GRW Spaces

Space-times satisfying $\mathcal{R} \leq K$ and $\mathcal{R} \geq K$ are abundant. We mention as examples, *generalized Robertson-Walker (GRW) spaces*, namely warped products $M = (-I) \times_f F$ for arbitrary Riemannian manifolds F.

Lemma 4.1 *[[1], Corollary 7.2] A GRW space $M = -I \times_f F$ satisfies $\mathcal{R} \leq K$ if and only if $f : I \to \mathbf{R}_+$ satisfies*

$$f'' \geq Kf,$$

and F either is one-dimensional or has sectional curvature $\leq C$ where