Springer Proceedings in Mathematics & Statistics

Ayman Badawi
Mohammad Reza Vedadi
Siamak Yassemi
Ahmad Yousefian Darani *Editors*

Homological and Combinatorial Methods in Algebra

SAA 4, Ardabil, Iran, August 2016

Springer

Springer Proceedings in Mathematics & Statistics

Volume 228

Springer Proceedings in Mathematics & Statistics

This book series features volumes composed of selected contributions from workshops and conferences in all areas of current research in mathematics and statistics, including operation research and optimization. In addition to an overall evaluation of the interest, scientific quality, and timeliness of each proposal at the hands of the publisher, individual contributions are all refereed to the high quality standards of leading journals in the field. Thus, this series provides the research community with well-edited, authoritative reports on developments in the most exciting areas of mathematical and statistical research today.

More information about this series at http://www.springer.com/series/10533

Ayman Badawi · Mohammad Reza Vedadi
Siamak Yassemi · Ahmad Yousefian Darani
Editors

Homological and Combinatorial Methods in Algebra

SAA 4, Ardabil, Iran, August 2016

Springer

Editors
Ayman Badawi
Department of Mathematics
American University of Sharjah
Sharjah
United Arab Emirates

Mohammad Reza Vedadi
Faculty of Mathematical Science
Isfahan University of Technology
Isfahan
Iran

Siamak Yassemi
School of Mathematics, Statistics, and Computer Science
University of Tehran
Tehran
Iran

Ahmad Yousefian Darani
Department of Mathematics and Applications
University of Mohaghegh Ardabili
Ardabil
Iran

ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-3-319-74194-9 ISBN 978-3-319-74195-6 (eBook)
https://doi.org/10.1007/978-3-319-74195-6

Library of Congress Control Number: 2017963530

Mathematics Subject Classification (2010): 13A15, 12F15, 13B05, 13C10, 16E05, 13F05

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

The 4th SAA 2016, fourth in the series of international seminars on algebra and its applications, was held at the Department of Mathematics and Applications, University of Mohaghegh Ardabili, Iran, during 9–11 August, 2016. Following the tradition of its predecessors, this meeting gathered researchers around topics in present recent progress and new trends in algebra and its applications. A total number of 105 participants from several countries have attended the conference in the University of Mohaghegh Ardabili (UMA). The main goal of the seminar is to introduce Iranian faculty and graduate students to important ideas in the mainstream of algebra. A secondary goal is for Iranian mathematicians to open channels of communication with algebraists from around the globe and eventually begin collaborative research projects. The audience was multidisciplinary allowing the participants to exchange diversified ideas and to show the wide attraction of algebra and its applications. There were two kinds of lectures: invited talks of one hour presented by distinguished experts and half an hour contributions.

The Scientific Committee consisted of Kamran Divaani-Azar (Alzahra University), Hossien Abdolzadeh, Jafar Azami—Chair, Kamal Bahmanpour, Adel P. Kazemi, Ahmad Khojali, Majid Rahro-Zargar, Ahmad Yousefian Darani, and Naser Zamani all from UMA.

The Organizing Committee was constituted by Goudarz Sadeghi, Mohammad Narimani, Yousef Abbaspour, Daioush Latifi, Kazem Haghnejad, Hossein Abdolzadeh, Abbas Najati, and Ahmad Yousefian Darani (Chair) all from UMA.

We are particularly indebted to our plenary speakers: Moharam Aghapour (Arak University), Fariborz Azar Panah (Shahid Chamran University of Ahvaz), Ayman Badawi (American University of Sharjah), Reza Naghipour (University of Tabriz), Peyman Nasehpour (University of Tehran), Mohammad Reza Vedadi (Isfahan University of Technology), Roger Wiegand (University of Nebraska-Lincoln), Sylvia Wiegand (University of Nebraska Lincoln), Siamak Yassemi (University of Tehran), and Rahim Zaare-Nahandi (University of Tehran). Thanks are also due to the presenters of contributed papers, as well as everyone who attended for making the seminar a success. According to the evaluations of the scientific committee, there were several excellent talks presented by invited speakers.

The 4th SAA 2016 was sponsored by the UMA, and organized by the Faculty of Mathematics and Department of Mathematics and Applications, UMA. We would like to publicly acknowledge the financial support of the Vice-Chancellorship for Research and Technology of UMA, as well as the hospitality of the Faculty of Mathematics and Department of Mathematics and Applications of UMA. We are also very grateful for the secretarial help of Negin Karimi. Selected papers of 4th SAA 2016 are presented in the volume *Homological and Combinatorial Methods in Algebra* in the series Springer Proceedings of Mathematics & Statistics published by Springer. With the publication of this proceeding, we hope that a wider mathematical audience will benefit from the seminar research achievements and new contributions to the field of algebra and its applications. More details of the event can be found at http://uma.ac.ir//links/4saa.

Sharjah, United Arab Emirates — Ayman Badawi
Isfahan, Iran — Mohammad Reza Vedadi
Tehran, Iran — Siamak Yassemi
Ardabil, Iran — Ahmad Yousefian Darani

Contents

b-Symbol Distance Distribution of Repeated-Root Cyclic Codes

Hojjat Mostafanasab and Esra Sengelen Sevim

Abstract Symbol-pair codes, introduced by Cassuto and Blaum (Proc IEEE Int Symp Inf Theory, 988–992, 2010 [1]), have been raised for symbol-pair read channels. This new idea is motivated by the limitations of the reading process in high-density data storage technologies. Yaakobi et al. (IEEE Trans Inf Theory **62**(4):1541–1551, 2016 [8]) introduced codes for b-symbol read channels, where the read operation is performed as a consecutive sequence of $b > 2$ symbols. In this paper, we come up with a method to compute the b-symbol-pair distance of two n-tuples, where n is a positive integer. Also, we deal with the b-symbol-pair distances of some kind of cyclic codes of length p^e over F_{p^m}.

Keywords b-Symbol pair · Distance distribution · Cyclic codes

1 Introduction

Recently, it is possible to write information on storage devices with high resolution using advances in data storage systems. However, it causes a problem of the gap between write resolution and read resolution. Cassuto and Blaum [1, 2] laid out a framework for combating pair-errors, relating pair-error correction capability to a new metric called pair-distance. They proposed the model of symbol-pair read channels. Such channels are mainly motivated by magnetic-storage channels with high write resolution, due to physical limitations, each channel contains contributions from two adjacent symbols. Cassuto and Listsyn [3] studied algebraic construction of

H. Mostafanasab (✉)
Department of Mathematics and Applications, University of Mohaghegh Ardabili,
P.O. Box 179, Ardabil, Iran
e-mail: h.mostafanasab@gmail.com

E. S. Sevim
Eski Silahtarağa Elektrik Santrali, Kazim Karabekir, Istanbul Bilgi University,
Cad. No: 2/1334060, Eyüp, Istanbul, Turkey
e-mail: esra.sengelen@bilgi.edu.tr

A. Badawi et al. (eds.), *Homological and Combinatorial Methods in Algebra*, Springer Proceedings in Mathematics & Statistics 228,
https://doi.org/10.1007/978-3-319-74195-6_1

cyclic symbol-pair codes. Yaakobi et al. [9] proposed efficient decoding algorithms for the cyclic symbol-pair codes. Chee et al. [4, 5] established a Singleton-type bound for symbol-pair codes and constructed codes that meet the Singleton-type bound. Hirotomo et al. [7] proposed the decoding algorithm for symbol-pair codes based on the newly defined parity-check matrix and syndromes.

For this new channels, the codes defined as usual over some discrete symbol alphabet, but whose reading from the channel is performed as overlapping pairs of symbols. Let Ξ be the alphabet consisting of q elements. Each element in Ξ is called a symbol. We use Ξ^n to denote the set of all n-tuples, where n is a positive integer. In the symbol-pair read channel, there are in fact two channels. If the stored information is $x = (x_0, x_1, \ldots, x_{n-1}) \in \Xi^n$, then the symbol-pair read vector of x is

$$\pi(x) = [(x_0, x_1), (x_1, x_2), \ldots, (x_{n-2}, x_{n-1}), (x_{n-1}, x_0)],$$

and the goal is to correct a large number of the so called symbol-pair errors. The pair distance, $d_p(x, y)$, between two pair-read vectors x and y is the Hamming distance over the symbol-pair alphabet $(\Xi \times \Xi)$ between their respective pair-read vectors, that is, $d_p(x, y) = d_H(\pi(x), \pi(y))$. The minimum pair distance of a code $\mathscr{C}$ is defined as $d_p(\mathscr{C}) = \min\{d_p(x, y) | x, y \in \mathscr{C} \text{ and } x \neq y\}$. Accordingly, the pair weight of x is $\omega_p(x) = \omega_H(\pi(x))$. If $\mathscr{C}$ is a linear code, then the minimum pair-distance of $\mathscr{C}$ is the smallest pair-weight of nonzero codewords of $\mathscr{C}$. The minimum pair-distance is one of the important parameters of symbol-pair codes. This distance distribution is very difficult to compute in general, however, for the class of cyclic codes of length p^e over F_{p^m}, their Hamming distance has been completely determined in [6]. In [10], Zhu et al. investigated the symbol-pair distances of cyclic codes of length p^e over F_{p^m}.

For $b \geq 3$, the b-symbol read vector corresponding to the vector

$$x = (x_0, x_1, \ldots, x_{n-1}) \in \Xi^n$$

is defined as

$$\pi_b(x) = [(x_0, x_1, \ldots, x_{b-1}), (x_1, x_2, \ldots, x_b), \ldots, (x_{n-1}, x_0, \ldots, x_{b-2})] \in (\Xi^b)^n.$$

We refer to the elements of $\pi_b(x)$ as b-symbols. The b-symbol distance between x and y, denoted by $d_b(x, y)$, is defined as $d_b(x, y) = d_H(\pi_b(x), \pi_b(y))$. Similarly, we define the b-weight of the vector x as $\omega_H(\pi_b(x))$. In the analogy of the definition of symbol-pair codes, the minimum b-symbol distance of $\mathscr{C}$, $d_b(\mathscr{C})$, is given by $d_b(\mathscr{C}) = \min\{d_b(x, y) | x, y \in \mathscr{C} \text{ and } x \neq y\}$. For more information on these notions see [8].

We can rewrite [8, Proposition 9] for any arbitrary alphabet Ξ.

Proposition 1. *Let $x \in \Xi^n$ be such that $0 < \omega_H(x) \leq n - (b - 1)$. Then*

$$\omega_H(\mathscr{C}) + b - 1 \leq \omega_b(\mathscr{C}) \leq b \cdot \omega_H(\mathscr{C}).$$

Referring to Proposition 1, we see that:

Corollary 1. *Let* $\mathscr{C}$ *be a code. If* $0 < d_H(\mathscr{C}) \leq n - (b-1)$*, then*

$$d_H(\mathscr{C}) + b - 1 \leq d_b(\mathscr{C}) \leq b \cdot d_H(\mathscr{C}).$$

In the next section we give a method to calculate the b-symbol distance of two n-tuples. We know that all cyclic codes of length p^e over a finite field of characteristic p are generated by a single "monomial" of the form $(x-1)^i$, where $0 \leq i \leq p^e$ (see [6]). Determining the b-symbol-pair distances of some kind of these cyclic codes is the main purpose of the next section.

2 Main Results

In the following theorem we give a formula to calculate the b-symbol distance of two n-tuples.

Theorem 1. *Let* $x = (x_1, x_2, \ldots, x_n)$ *and* $y = (y_1, y_2, \ldots, y_n)$ *be two vectors in* Ξ^n *with* $0 < d_H(x, y) \leq n - (b-1)$. *Suppose that*

$$A = \{1, 2, \ldots, n\} \backslash \{r, r+1, r+2, \ldots, s \mid r, s \text{ are such that } s - r \geq b-2 \text{ and } x_i = y_i \\ \text{for each } r \leq i \leq s \text{ and indices may wrap around modulo} n\},$$

and $A = \cup_{l=1}^{L} B_l$ *is a minimal partition of the set A to subsets of consecutive indices (every subset* $B_l = [s_l, e_l]$ *is the sequence of all indices between* s_l *and* e_l*, inclusive, and is the smallest integer that achieves such partition, also indices may wrap around modulo n). Then*

$$d_b(x, y) = d_H(x, y) + e + L(b-1),$$

where $e = |\{i \mid i \in B_l \text{ for some } 1 \leq l \leq L \text{ such that } x_i = y_i\}|$.

Proof. Since the partition is minimal, there are no two indices $i, i+j$, where $j \in \{1, \ldots, b-1\}$, that belong to different subsets $B_l, B_{l'}$. The b-symbol distance between x and y is equal to the sum of the sizes of the b-tuple subsets

$$\{(s_l - b + 1, s_l - b + 2, \ldots, s_l), (s_l - b + 2, s_l - b + 3, \ldots, s_l, s_l + 1), \ldots, (s_l, s_l + 1, \ldots, s_l + b - 1), \\ (s_l + 1, s_l + 2, \ldots, s_l + b), \ldots, (e_l, e_l + 1, \ldots, e_l + b - 1)\}.$$

The number of b-tuples in each b-tuple subset equals $|B_l| + b - 1$, whence $d_b(x, y) = \sum_{l=1}^{L} B_l + L(b-1)$. Furthermore, it is easy to see that $\sum_{l=1}^{L} B_l = d_H(x, y) + e$ where $e = |\{i \mid i \in B_l$ for some $1 \leq l \leq L$ such that $x_i = y_i\}|$.

Corollary 2. *Let $x = (x_1, x_2, \ldots, x_n) \in \Xi^n$ with $0 < \omega_H(x) \leq n - (b-1)$. Suppose that*

$$A = \{1, 2, \ldots, n\} \backslash \{r, r+1, r+2, \ldots, s \mid r, s \text{ are such that } s - r \geq b-2 \text{ and } x_i = 0 \text{ for each } r \leq i \leq s \text{ and indices may wrap around modulo } n\},$$

and $A = \cup_{l=1}^{L} B_l$ is a minimal partition of the set A to subsets of consecutive indices (every subset $B_l = [s_l, e_l]$ is the sequence of all indices between s_l and e_l, inclusive, and is the smallest integer that achieves such partition, also indices may wrap around modulo n). Then $\omega_b(x) = \omega_H(x) + e + L(b-1)$, where

$$e = |\{i \mid i \in B_l \text{ for some } 1 \leq l \leq L \text{ such that } x_i = 0\}|.$$

Example 1. Let $n = 15$, $b = 4$ and $x = (0, 0, 1, 3, 0, 5, 0, 0, 0, 2, 0, 7, 0, 0, 0) \in Z^{15}$. We list all of the 4-tuples as follows:

$$(0, 0, 1, 3), (0, 1, 3, 0), (1, 3, 0, 5), (3, 0, 5, 0), (0, 5, 0, 0), (5, 0, 0, 0), (0, 0, 0, 2),$$
$$(0, 0, 2, 0), (0, 2, 0, 7), (2, 0, 7, 0), (0, 7, 0, 0), (7, 0, 0, 0), (0, 0, 0, 0), (0, 0, 0, 1).$$

Hence $\omega_4(x) = 13$. On the other hand, $\omega_H(x) = 5$, $e = 2$ and $L = 2$. Therefore, the equation $\omega_b(x) = \omega_H(x) + e + L(b-1)$ holds.

Theorem 2. ([6, Theorem 6.4]) Let $\mathscr{C}$ be a cyclic code of length p^e over F_{p^m}. Then $\mathscr{C} = \langle (x-1)^i \rangle \subseteq \frac{F_{p^m}[x]}{\langle x^{p^e} - 1 \rangle}$, for $i \in \{0, 1, \ldots, p^e\}$. Also,

(1) $d_H(\mathscr{C}) = 1$ if $i = 0$.
(2) $d_H(\mathscr{C}) = \beta + 2$ if $\beta p^{e-1} + 1 \leq i \leq (\beta+1)p^{e-1}$ where $0 \leq \beta \leq \mathrm{p} - 2$.
(3) $d_H(\mathscr{C}) = (t+1)p^k$ if $p^e - p^{e-k} + (t-1)p^{e-k-1} + 1 \leq i \leq p^e - p^{e-k} + tp^{e-k-1}$, where $1 \leq t \leq p-1$, *and* $1 \leq k \leq e-1$.
(4) $d_H(\mathscr{C}) = 0$ if $i = p^e$.

From now on, in order to simplify the notation, for $i \in \{0, 1, \ldots, p^e\}$, we denote each code $\langle (x-1)^i \rangle$ by $\mathscr{C}_i$.

Proposition 2. *If $b \leq p^e$, then $d_b(\mathscr{C}_0) = b$.*

Proof. By Theorem 2, we have that $d_H(\mathscr{C}_0) = 1$. So, by Corollary 1, $b \geq d_b(\mathscr{C}_0) \geq d_H(\mathscr{C}_0) + b - 1 = b$. Hence $d_b(\mathscr{C}_0) = b$.

Proposition 3. *Let $b < p^e$. Then $b + 1 \leq d_b(\mathscr{C}_i) \leq 2b$ for every $1 \leq i \leq p^{e-1}$.*

Proof. By Theorem 2, $d_H(\mathscr{C}_i) = 2$ for every $1 \leq i \leq p^{e-1}$. Hence, $2b \geq d_b(\mathscr{C}_i) \geq 2 + (b-1) = b + 1$, by Corollary 1.

Notice that, for two codes $\mathscr{C}, \mathscr{C}' \subseteq F_{p^m}^{p^e}$ with $\mathscr{C} \subseteq \mathscr{C}'$, we have $d_b(\mathscr{C}) \geq d_b(\mathscr{C}')$. We define $d_b(\mathscr{C}_{p^e}) = 0$.

Proposition 4. *Let $b \leq p$ and $e = 1$. Then $d_b(\mathscr{C}_i) = i + b$ for each $0 \leq i \leq p - b$.*

Proof. By Theorem 2, $d_H(\mathscr{C}_i) = i + 1$ for $0 \le i \le p - 1$. Assume that $0 \le i \le p - b$. Hence, by Corollary 1, $d_b(\mathscr{C}_i) \ge i + 1 + b - 1 = i + b$. Moreover $\omega_b((x - 1)^i) = i + 1 + (b - 1) = i + b$. Then $d_b(\mathscr{C}_i) = i + b$.

Theorem 3. *Let $e \ge 2$ and $1 \le i \le p^{e-1}$ such that $i + b \le p^e$ and $i \le b$. Then $d_b(\mathscr{C}_i) = i + b$.*

Proof. Since $i + b \le p^e$, then by Corollary 2, $\omega_b((x - 1)^i) = i + 1 + (b - 1) = i + b$. So, $d_b(\mathscr{C}_i) \le i + b$. By Proposition 3, $d_b(\mathscr{C}_i) \ge b + 1$. Let $c(x)$ be a polynomial in $F_{p^m}[x]$. If $\omega_b(c(x)) = j + b$ for some $1 \le j \le i - 1$, then $i \le b$ implies that $c(x) = x^t(a_0 + a_1x + \cdots + a_jx^j)$ where a_l's are in F_{p^m}, $a_0, a_j \ne 0$ and t is a non-negative integer. However $c(x) \notin \mathscr{C}_i$. So $d_b(\mathscr{C}_i) = i + b$.

Lemma 1. *Let e and k be two integers such that $e \ge 2$ and $1 \le k \le e - 1$. Suppose that $c(x) = (x - 1)^{p^e - p^{e-k}} g(x)$ where $g(x)$ is a nonzero polynomial in $F_{p^m}[x]$ with $d := deg(g(x)) < p^{e-k}$ and $b \le p^e - d$. Then*

(1) If $d \le p^{e-k} - b$ or $g_k = 0$ for every $0 \le k \le b - p^{e-k} + d - 1$, then $\omega_b(c(x)) = p^k\omega_b(g(x))$.
(2) If $d > p^{e-k} - b$ and $g_k \ne 0$ for some $0 \le k \le b - p^{e-k} + d - 1$, then $\omega_b(c(x)) = p^k\big(\omega_b(g(x)) - (b - 1) + \zeta\big)$ where $\zeta = p^{e-k} - d - 1$.

Proof. Assume that $g(x) = \sum_{j=0}^{d} g_jx^j$. Thus

$$c(x) = \sum_{i=0}^{p^k-1} x^{ip^{e-k}} g(x) = \sum_{i=0}^{p^k-1}\sum_{j=0}^{d} g_jx^{ip^{e-k}+j}.$$

As usual, we identify the polynomial $h(x) = h_0 + h_1x + \cdots + h_nx^n$ with the vector $h = (h_0, h_1, \ldots, h_n)$. Therefore, we have $c = (\overbrace{\widehat{g}, \ldots, \widehat{g}}^{p^k-\text{time}})$ where

$$\widehat{g} = (g_0, \ldots, g_d, \overbrace{0, \ldots, 0}^{(p^{e-k}-d-1)-\text{time}}).$$

We denote $\omega_b(\widehat{g}(x)) := \omega_b(\widehat{g})$. Since $\pi_b(c) = [\overbrace{\pi_b(\widehat{g}), \ldots, \pi_b(\widehat{g})}^{p^k-\text{time}}]$, then $\omega_b(c(x)) = p^k\omega_b(\widehat{g}(x))$. On the other hand, $\omega_b(g(x)) = \omega_b(g)$, where

$$g = (g_0, g_1, \ldots, g_d, \overbrace{0, \ldots, 0}^{(p^e-d-1)-\text{time}}).$$

We can check that:

(1) If $d \leq p^{e-k} - b$ or $g_k = 0$ for every $0 \leq k \leq b - p^{e-k} + d - 1$, then $\omega_b(g) = \omega_b(\widehat{g})$, i.e., $\omega_b(g(x)) = \omega_b(\widehat{g}(x))$. Hence $\omega_b(c(x)) = p^k \omega_b(g(x))$.

(2) If $d > p^{e-k} - b$ and $g_k \neq 0$ for some $0 \leq k \leq b - p^{e-k} + d - 1$, then $\omega_b(g) = \omega_b(\widehat{g}) + (b-1) - \zeta$ where $\zeta = p^{e-k} - d - 1$, i.e., $\omega_b(g(x)) = \omega_b(\widehat{g}(x)) + (b-1) - \zeta$. So, $\omega_b(c(x)) = p^k\big(\omega_b(g(x)) - (b-1) + \zeta\big)$.

Theorem 4. *Let e and k be two integers such that $e \geq 2$ and $1 \leq k \leq e-1$. If $0 \leq i \leq p^{e-k-1}$ such that $b + i \leq p^{e-k}$ and $i \leq b$, then $d_b(\mathscr{C}_{p^e - p^{e-k} + i}) = p^k(b+i)$.*

Proof. Fix $0 \leq i \leq p^{e-k-1}$ such that $b + i \leq p^{e-k}$ and $i \leq b$. Let $0 \neq c(x) \in \mathscr{C}_{p^e - p^{e-k} + i}$. Then, there exists $0 \neq f(x) \in F_{p^m}[x]$ such that $c(x) = (x-1)^{p^e - p^{e-k}} (x-1)^i f(x)$. Set $g(x) := (x-1)^i f(x)$ and $d := deg(g(x))$. Without loss of the generality we may assume that $d < p^{e-k}$. Notice that by Theorem 2, $\omega_H(g(x)) \geq 2$, and by Theorem 3, $\omega_b(g(x)) \geq b + i$. Regarding Lemma 1, we consider the following cases:

Case 1. If $d \leq p^{e-k} - b$ or $g_k = 0$ for every $0 \leq k \leq b - p^{e-k} + d - 1$, then $\omega_b(c(x)) = p^k \omega_b(g(x)) \geq p^k(b+i)$.

Case 2. If $d > p^{e-k} - b$ and $g_k \neq 0$ for some $0 \leq k \leq b - p^{e-k} + d - 1$, then $\omega_b(c(x)) = p^k\big(\omega_b(g(x)) - (b-1) + \zeta\big)$ where $\zeta = p^{e-k} - d - 1$. If $\omega_H(g(x)) \geq b + i$, then Corollary 1 implies that $\omega_b(g(x)) \geq b + i + b - 1$. Hence $\omega_b(c(x)) \geq p^k\big(b + i + (b-1) - (b-1)\big) = p^k(b+i)$. Assume that $\omega_H(g(x)) = i + j$ for some $2 - i \leq j \leq b - 1$. It is easy to see that $\omega_H(g(x)) + z = d + 1$ where $z = |\{l \mid 0 \leq l \leq d \text{ and } g_l = 0\}|$. We claim that, $z \geq b - j - \zeta$. Otherwise $d + 1 < \omega_H(g(x)) + b - j - \zeta = i + j + b - j - (p^{e-k} - d - 1) = i + b - p^{e-k} + d + 1$. But $b + i \leq p^{e-k}$ leads us to a contradiction. Therefore the claim holds. So, $\omega_b(g(x)) \geq i + j + b - j - \zeta + (b-1)$. Thus $\omega_b(c(x)) \geq p^k\big(\omega_b(g(x)) - (b-1) + \zeta\big) = p^k(i+b)$. Hence $d_b(\mathscr{C}_{p^e - p^{e-k} + i}) \geq p^k(i+b)$. Moreover, by part (1) of Lemma 1, $\omega_b((x-1)^{p^e - p^{e-k} + i}) = p^k \omega_b((x-1)^i) = p^k(b+i)$. Consequently, $d_b(\mathscr{C}_{p^e - p^{e-k} + i}) = p^k(b+i)$.

References

1. Cassuto, Y., Blaum. M.: Codes for symbol-pair read channels. In: Proceeding of the IEEE International Symposium on Information Theory, Austin, TX, USA, June, pp. 988–992. (2010)
2. Cassuto, Y., Blaum, M.: Codes for symbol-pair read channels. IEEE Trans. Inf. Theory **57**(12), 8011–8020 (2011)
3. Cassuto, Y., Litsyn, Y.: Symbol-pair codes: algebraic constructions and asymptotic bounds. In: Proceeding of the IEEE International Symposium on Information Theory, St. Petersburg, Russia, July/Aug, pp. 2348–2352. (2011)
4. Chee, Y.M., Ji, L., Kiah, H.M., Wang, C., Yin, J.: Maximum distance separable codes for symbol-pair read channels. IEEE Trans. Inf. Theory **59**(11), 7259–7267 (2013)

5. Chee, Y.M., Kiah, H.M., Wang, C.: Maximum distance separable symbol-pair codes. In: Proceeding of the International Symposium on Information Theory, Cambridge, MA, USA, July, pp. 2886–2890. (2012)
6. Dinh, H.Q.: On the linear ordering of some classes of negacyclic and cyclic codes and their distance distributions. Finite Fields Appl. **14**(1), 22–40 (2008)
7. Hirotomo, M., Takita, M., Morii, M.: Syndrome decoding of symbol-pair codes. In: Proceeding of the IEEE Information Theory Workshop, Hobart, TAS, Australia, pp. 162–166. (2014)
8. Yaakobi, E., Bruck, J., Siegel, P.H.: Constructions and decoding of cyclic codes over *b*-symbol read channels. IEEE Trans. Inf. Theory **62**(4), 1541–1551 (2016)
9. Yaakobi, E., Bruck, J., Siegel, P.H.: Decoding of cyclic codes over symbol-pair read channels. In: Proceeding of the International Symposium on Information Theory, Cambridge, MA, USA, pp. 2891–2895. (2012)
10. Zhu, S., Sun, Z., Wang, L.: The symbol-pair distance distribution of repeated-root cyclic codes over F_{p^m}. arXiv:1607.01887v1, (2016)