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Analysis in Banach Spaces

Volume II: Probabilistic Methods and Operator Theory

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Analysis in Banach Spaces

Volume II: Probabilistic Methods and Operator Theory

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Dedicated to the memory of

Nigel J. Kalton (1946–2010)
&
Alan G. R. McIntosh (1942–2016)

Preface

In Volume I of 'Analysis in Banach spaces' we presented essential techniques for the analysis of Banach space-valued functions, from integration theory and martingale inequalities to the extension of classical singular integral operators, such as the Hilbert transform and Mihlin Fourier multiplier operators, from L^p-spaces of scalar-valued functions to L^p-spaces of functions taking values in UMD Banach spaces.

In the present volume we concentrate on a second, closely related question central to the theory of evolution equations, namely how to extend various classical L^2-estimates and related Hilbert space techniques to the Banach space setting, in particular to the L^p-scale.

Already in the mid-1980s, motivated by the square root problem for sectorial operators, Alan McIntosh forged the classical theory of square functions in Fourier Analysis, pioneered by Paley, Marcinkiewicz–Zygmund, and Stein, into a powerful tool for the study of general sectorial operators on Hilbert spaces. Just as one can view Harmonic Analysis as the 'spectral theory of the Laplacian' (Strichartz), McIntosh's square function techniques for sectorial operators capture essential singular integral estimates still available in this more general setting. Extension of these estimates to the L^p-setting requires a substitute for the basic Hilbertian orthogonality techniques on which they rely. The theory of random sums, in particular Rademacher sums and Gaussian sums, originally developed in the context of Probability Theory in Banach spaces and the Geometry of Banach spaces, provides just that. The fine properties of Banach space-valued random sums are intimately connected with various probabilistic notions such as type, cotype, and K-convexity which often take on the role of geometrical properties of the classical L^p-spaces that are explicit or implicit in the treatment of classical inequalities. The first two chapters of this volume present those aspects that are relevant to our purpose. For a fuller treatment of this fascinating topic the reader is referred to the rich literature on the Geometry of Banach spaces and Probability in Banach spaces.

Volume I already provided a first glance into the programme outlined above when we proved an operator-valued version of the Mihlin multiplier theorem by replacing the uniform boundedness condition on certain operator families appearing in the conditions of the Mihlin theorem by the stricter requirement of R-boundedness. This magic wand can be applied to a surprising number of operator theoretic Hilbert space results. This volume presents a wealth of analytical methods that allow one to verify the R-boundedness of many sets of classical operators relevant in applications to Harmonic Analysis and Stochastic Analysis.

A second tool to extend Hilbert space techniques to a Banach space setting consists of replacing L^2-spaces by generalised square function estimates which, in an abstract Banach space setting, can be alternatively described in an operator-theoretic way through the theory of radonifying operators. This class of operators connects the theory of Banach space-valued Gaussian random sums to methods from operator theory in a rather direct way, thus paving the way to substantial applications in vector-valued Harmonic Analysis and Stochastic Analysis. On an 'operational level', they display the same function space properties (such as versions of Hölder's inequality, Fatou's lemma, and Fubini's theorem) as their classical counterparts do.

With these tools at hand we present a far reaching extension of the theory of the H^∞-functional calculus on Hilbert spaces to the L^p-setting, including characterisations of its boundedness in terms of square function estimates, R-boundedness and dilations. From these flow the results which made the H^∞-functional calculus so useful in the theory of evolution equations in L^p-spaces: the operator sum method, an operator-valued calculus and a variety of techniques to verify the boundedness of the H^∞-functional calculus for most differential operators of importance in applications.

The randomisation techniques and their operator theoretic counterparts worked out in the present book will also set the stage for Volume III. There we will present vector-valued function spaces, complete our treatment of vector-valued harmonic analysis and discuss the theory of operator-valued Itô integrals in UMD Banach spaces and their application to maximal regularity estimates for stochastic evolution equations with Gaussian noise. It is here that generalised square functions display their full power as they furnish a close link between stochastic estimates such as the vector-valued Burkholder–Davis–Gundy inequalities and harmonic analytic properties of the underlying partial differential operator, encoded in its H^∞-calculus.

*

It is perhaps interesting to notice a change of generation in the contents of this volume compared to Volume I. With important exceptions mostly on the scale of subsections, the main body of the material presented in Volume I may be considered 'classical' by now. In fact, the following subjective definition of 'classical' has been has proposed by David Cruz-Uribe (private communication): "Anything that was proved before I started graduate school." By a three-quarter majority within the present authorship, this definition would render all results obtained by mid-1980's 'classical'.

The main results of the first two chapters on random sums and their connections to Banach space theory are still largely classical in this sense. However, an important turning point occurs in the beginning of Chapter 8, dedicated to the notion of R-boundedness. Although the deep roots of this theory are older, its systematic development only begins in the 1990's and reaches its full bloom around and after the turn of the millennium; some basic questions related to the comparison of R-boundedness with related notions were settled as recently as 2016. Likewise, while the foundations of the theory of radonifying operators are certainly classical, their interpretation and systematic exploitation as generalised square functions in Chapter 9 is a successful creation of the 2000's. As for the theory of the H^∞-calculus developed in the last chapter, only the groundwork in a Hilbert space context is classical. Its extension to Banach spaces is more recent, and especially its fundamental connections with the generalised square functions, a key theme of our treatment, have only been revealed during this century.

*

Two stylistic conventions of Volume I will stay in force in the present volume as well: Most of the time, we are quite explicit with the constants appearing in our estimates, and we especially try to keep track of the dependence on the main parameters involved. Some of these explicit quantitative formulations appear here for the first time. We also pay more attention than many texts to the impact of the underlying scalar field (real or complex) on the results under consideration. A careful distinction between linear and conjugate-linear duality is particularly critical to the correct formulation of some key results concerning the generalised square functions, which are among the main characters of the present volume.

*

This project was initiated in Delft and Karlsruhe in 2008. Critical to its eventual progress was the possibility of intensive joint working periods in the serenity provided by the Banach Center in Będlewo (2012), Mathematisches Forschungsinstitut Oberwolfach (2013), Stiftsgut Keysermühle in Klingenmünster (2014 and 2015), Hotel 't Paviljoen in Rhenen (2015), and Buitengoed de Uylenburg in Delfgauw (2017). All four of us also met three times in Helsinki (2014, 2016 and 2017), and a number of additional working sessions were held by subgroups of the author team. One of us (J.v.N.) wishes to thank Marta Sanz-Solé for her hospitality during a sabbatical leave at the University of Barcelona in 2013.

Preliminary versions of parts of the material were presented in advanced courses and lecture series at various international venues and in seminars at our departments, and we would like to thank the students and colleagues who attended these events for feedback that shaped and improved the final form of the text. Special thanks go to Alex Amenta, Markus Antoni, Sonja Cox, Chiara Gallarati, Fabian Hornung, Luca Hornung, Nick Lindemulder, Emiel Lorist, Bas Nieraeth, Jan Rozendaal, Emil Vuorinen, and Ivan Yaroslavtsev who did detailed readings of portions of this book. Needless to say, we take full responsibility for any remaining errors. A list with errata will be maintained on our personal websites. We wish to thank Klaas Pieter Hart for LaTeX support.

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Delft, Helsinki and Karlsruhe,
September 15, 2017.

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Symbols and notations

Sets

$\mathbb{N} = \{0, 1, 2, \ldots\}$ - non-negative integers
$\mathbb{Z}$ - integers
$\mathbb{Q}$ - rational numbers
$\mathbb{R}$ - real numbers
$\mathbb{C}$ - complex numbers
$\mathbb{K}$ - scalar field ($\mathbb{R}$ or $\mathbb{C}$)
$\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, \infty\}$ - extended integers
$\mathbb{R}_+ = (0, \infty)$ - positive real line
B_X - open unit ball
S_X - unit sphere
$B(x, r)$ - open ball centred at x with radius r
$\mathbb{D}$ - open unit disc
$\mathbb{S} = \{z \in \mathbb{C} : 0 < \Im z < 1\}$ - unit strip
Σ_ω = open sector of angle ω
$\Sigma_\omega^{\mathrm{bi}}$ = open bisector of angle ω
$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ - unit circle

Vector spaces

c_0 - space of null sequences
C - space of continuous functions
C_0 - space of continuous functions vanishing at infinity
C^α - space of Hölder continuous functions
C_{b} - space of bounded continuous functions
C_{c} - space of continuous functions with compact support
C_{c}^∞ - space of test functions with compact support
$\mathscr{C}^p$ - Schatten class
$\varepsilon_N(X)$, $\varepsilon(X)$ - Rademacher sequence spaces
$\gamma_N(X), \gamma(X)$ - Gaussian sequence spaces

$\gamma(H, X)$ - space of γ-radonifying operators
$\gamma(S; X)$ - shorthand for $\gamma(L^2(S), X)$
$\gamma_\infty(H, X)$ - space of almost summing operators
$\gamma_\infty(S; X)$ - shorthand for $\gamma_\infty(L^2(S), X)$
H - Hilbert space
$H^{s,p}$ - Bessel potential space
H^p - Hardy space
$\mathscr{H}(X_0, X_1)$ - space of holomorphic functions on the strip
ℓ^p - space of p-summable sequences
ℓ^p_N - space of p-summable sequences of lenght N
L^p - Lebesgue space
$L^{p,q}$ - Lorentz space
$L^{p,\infty}$ - weak-L^p
$\mathscr{L}(X, Y)$ - space of bounded linear operators
$\mathfrak{M}L^p(\mathbb{R}^d; X, Y)$ - space of Fourier multipliers
$\mathfrak{M}(\mathbb{R}^d; X, Y)$ - Mihlin class
$\mathscr{S}$ - space of Schwartz functions
$\mathscr{S}'$ - space of tempered distributions
$W^{k,p}$ - Sobolev space
$W^{s,p}$ - Sobolev-Slobodetskii space
X, Y, $\ldots$ - Banach spaces
$X_{\mathbb{C}}$ - complexification
$X^{\gamma,p}_{\mathbb{C}}$ - Gaussian complexification
X^*, Y^*, $\ldots$ - dual Banach spaces
$X^{\odot}$, $Y^{\odot}$, $\ldots$ - strongly continuous semigroup dual spaces
$X \otimes Y$ - tensor product
$[X_0, X_1]_\theta$ - complex interpolation space
$(X_0, X_1)_{\theta,p}$, $(X_0, X_1)_{\theta,p_0,p_1}$ - real interpolation spaces

Measure theory and probability

$\mathscr{A}$ - σ-algebra
$df_n = f_n - f_{n-1}$ - nth martingale difference
ϵ, ϵ_n - signs in $\mathbb{K}$, i.e., scalars in $\mathbb{K}$ of modulus one
ε, ε_n - Rademacher variables with values in $\mathbb{K}$
$\mathbb{E}$ - expectation
$\mathscr{F}$, $\mathscr{G}$, $\ldots$ - σ-algebras
$\mathscr{F}_f$ - collection of sets in $\mathscr{F}$ on which f is integrable
$\mathbb{E}(\cdot|\cdot)$ - conditional expectation
γ, γ_n - Gaussian variables
h_I - Haar function
μ - measure
$\|\mu\|$ - variation of a measure
$(\Omega, \mathscr{A}, \mathbb{P})$ - probability space
$\mathbb{P}$ - probability measure

r, r_n - real Rademacher variables
$(S, \mathscr{A}, \mu)$ - measure space
$\sigma(f, g, \dots)$ - σ-algebra generated by the functions $f, g, \dots$
$\sigma(\mathscr{C})$ - σ-algebra generated by the collection $\mathscr{C}$
τ - stopping time
w_α - Walsh functions

Norms and pairings

$|\cdot|$ - modulus, Euclidean norm
$\|\cdot\| = \|\cdot\|_X$ - norm in a Banach space X
$\|\cdot\|_p = \|\cdot\|_{L^p}$ - L^p-norm
$\langle\cdot,\cdot\rangle$ - duality
$(\cdot|\cdot)$ - inner product in a Hilbert space
$a \cdot b$ - inner product of $a, b \in \mathbb{R}^d$

Operators

A - closed linear operator
A^* - adjoint operator
$A^\odot$ - part of A^* in $X^\odot$
$\mathsf{D}(A)$ - domain of A
D_j - pre-decomposition
Δ - Laplace operator
$\gamma(\mathscr{T})$ - γ-bound of the operator family $\mathscr{T}$
$\gamma_p(\mathscr{T})$ - γ-bound of $\mathscr{T}$ with respect to the L^p-norm
$\mathscr{D}$ - dyadic system
$\partial_j = \partial/\partial x_j$ - partial derivative with respect to x_j
∂^α - partial derivative with multi-index α
$\mathbb{E}(\cdot|\cdot)$ - conditional expectation
$\mathscr{F} f$ - Fourier transform
$\mathscr{F}^{-1} f$ - inverse Fourier transform
H - Hilbert transform
$\widetilde{H}$ - periodic Hilbert transform
J_s - Bessel potential operator
$\ell^2(\mathscr{T})$ - ℓ^2-bound of the operator family $\mathscr{T}$
$\mathscr{L}(X, Y)$ - space of bounded operators from X to Y
$\mathscr{L}_{\rm so}(X, Y)$ - idem, endowed with the strong operator topology
$\mathsf{N}(A)$ - null space of A
$\mathscr{R}(\mathscr{T})$ - R-bound of the operator family $\mathscr{T}$
$\mathscr{R}_p(\mathscr{T})$ - R-bound of $\mathscr{T}$ with respect to the L^p-norm
$\mathsf{R}(A)$ - range of A
R_j - jth Riesz transform
S, T, $\dots$- bounded linear operators
$S(t)$, $T(t)$, $\dots$- semigroup operators
$S^*(t)$, $T^*(t)$, $\dots$- adjoint semigroup operators on the dual space X^*

$S^{\odot}(t)$, $T^{\odot}(t)$, ...- their parts in the strongly continuous dual $X^{\odot}$
T^* - adjoint of the operator T
$T^\star$ - Hilbert space (hermitian) adjoint of Hilbert space operator T
T_m - Fourier multiplier operator associated with multiplier m
$T \otimes I_X$ - tensor extension of T

Constants and inequalities

$\alpha_{p,X}$ - Pisier contraction property constant
$\alpha^{\pm}_{p,X}$ - upper and lower Pisier contraction property constant
$\beta_{p,X}$ - UMD constant
$\beta^{\mathbb{R}}_{p,X}$ - UMD constant with signs ± 1
$\beta^{\pm}_{p,X}$ - upper and lower randomised UMD constant
$c_{q,X}$ - cotype q constant
$c^{\gamma}_{q,X}$ - Gaussian cotype q constant
$\varDelta_{p,X}$ - triangular contraction property constant
$\hbar_{p,X}$ - norm of the Hilbert transform on $L^p(\mathbb{R};X)$
$K_{p,X}$ - K-convexity constant
$K^{\gamma}_{p,X}$ - Gaussian K-convexity constant
$\kappa_{p,q}$ - Kahane–Khintchine constant
$\kappa^{\mathbb{R}}_{p,q}$ - idem, for real Rademacher variables
$\kappa^{\gamma}_{p,q}$ - idem, for Gaussian sums
$\kappa_{p,q,X}$ - idem, for a fixed Banach space X
$\tau_{p,X}$ - type p constant
$\tau^{\gamma}_{p,X}$ - Gaussian type p constant
$\varphi_{p,X}(\mathbb{R}^d)$ - norm of the Fourier transform $\mathscr{F} : L^p(\mathbb{R}^d;X) \to L^{p'}(\mathbb{R}^d;X)$.

Miscellaneous

$\hookrightarrow$ - continuous embedding
$\mathbf{1}_A$ - indicator function
$a \lesssim b$ - $\exists C$ such that $a \leqslant Cb$
$a \lesssim_{p,P} b$ - $\exists C$, depending on p and P, such that $a \leqslant Cb$
C - generic constant
$\complement$ - complement
$d(x,y)$ - distance
$f^\star$ - maximal function
$\widetilde{f}$ - reflected function
$\widehat{f}$ - Fourier transform
$\check{f}$ - inverse Fourier transform
$f * g$ - convolution
$\Im$ - imaginary part
Mf - Hardy–Littlewood maximal function
$p' = p/(p-1)$ - conjugate exponent
$p^* = \max\{p, p'\}$

$\Re$ - real part
$s \wedge t = \min\{s, t\}$
$s \vee t = \max\{s, t\}$
x - generic element of X
x^* - generic element of X^*
$x \otimes y$ - elementary tensor
x^+, x^-, $|x|$ - positive part, negative part, and modulus of x

Standing assumptions

Throughout this book, two conventions will be in force.

1. Unless stated otherwise, the scalar field $\mathbb{K}$ can be real or complex. Results which do not explicitly specify the scalar field to be real or complex are true over both the real and complex scalars.

2. In the context of randomisation, a *Rademacher variable* is a uniformly distributed random variable taking values in the set $\{z \in \mathbb{K} : |z| = 1\}$. Such variables are denoted by the letter ε. Thus, whenever we work over $\mathbb{R}$ it is understood that ε is a real Rademacher variable, i.e.,

$$\mathbb{P}(\varepsilon = 1) = \mathbb{P}(\varepsilon = -1) = \frac{1}{2},$$

   and whenever we work over $\mathbb{C}$ it is understood that ε is a complex Rademacher variable (also called a *Steinhaus variable*), i.e.,

$$\mathbb{P}(a < \arg(\varepsilon) < b) = \frac{1}{2\pi}(b - a).$$

   Occasionally we need to use real Rademacher variables when working over the complex scalars. In those instances we will always denote these with the letter r. Similar conventions are in force with respect to Gaussian random variables: a *Gaussian random variable* is a standard normal real-valued variable when working over $\mathbb{R}$ and a standard normal complex-valued variable when working over $\mathbb{C}$.

6

Random sums

One of the main themes in these volumes is the use of probabilistic techniques in general, and random sums in particular, in Banach space-valued Analysis. A first glimpse of their usefulness was already offered by the classical Theorem 2.1.9 of Paley, Marcinkiewicz and Zygmund on the extendability of bounded operators on $L^p(S)$ to bounded operators on $L^p(S;H)$, the proof of which involved estimates on Gaussian random sums. On the other hand, Rademacher random sums played a key role both in the formulation and in the proofs of the Littlewood–Paley theory in $L^p(\mathbb{R}^d;X)$ developed in Chapter 5.

In the chapter at hand, we will make a systematic investigation of the properties of the mentioned the two aforementioned species of random sums, Rademacher and Gaussian sums. The first three Sections 6.1, 6.2, and 6.3 are concerned with their basic relations and estimates in the context of finite sums, whereas Section 6.4 is devoted to two fundamental convergence results, the Itô–Nisio theorem and the Hoffmann-Jørgensen–Kwapień theorem, for infinite random series. In the final Section 6.5, we present Pisier's theorem on the comparison of Rademacher sums and trigonometric sums. It will be applied in the further development of the Littlewood–Paley theory, to which we return in Chapter 8.

With one exception, the present chapter deals with general aspects of random sums that remain valid in arbitrary Banach spaces. The rich interplay between more sophisticated estimates for these sums on the one hand, and the properties of the underlying Banach spaces on the other hand, will be taken up in the following Chapter 7. The exception is the Hoffmann-Jørgensen–Kwapień theorem, which relates the convergence of X-valued random sums to the containment of c_0 as an isomorphic subspace in X. It will play an important role in Chapter 9, where we extend the present considerations of random sums over finite or countable sequences $x_n = f(n)$ in order to deal with certain 'randomised norms' of functions f on general measure spaces. This, in turn, provides a powerful tool for the study of the H^∞-calculus in the last chapter.

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6.1 Basic notions and estimates

We begin with a brief discussion of the notion of *random variable* in the Banach space-valued context. We refer the reader to Appendix E for an introduction to some standard notions of probability theory.

Let X be a (real or complex) Banach space.

Definition 6.1.1. *An X-valued* random variable *is an X-valued strongly measurable function ξ defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$.*

The underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ will always be considered given and fixed, and when several random variables are considered simultaneously we will assume them to be defined on the same probability space, unless the contrary is stated (and there are also cases when this is useful).

The notion of strong measurability has been studied in great detail in Chapter 1. We recall that a function is said to be *strongly measurable* if it is the pointwise limit of a sequence of simple functions. By the Pettis measurability theorem (Theorem 1.1.6) a function ξ is strongly measurable if and only if it is separably-valued and weakly measurable, where the latter means that the scalar-valued functions $\langle \xi, x^* \rangle$ are measurable in the usual sense for all functionals $x \in X^*$. Moreover, strongly measurable functions are measurable (Corollary 1.1.2), i.e., the pre-images of open sets are measurable, and in separable Banach spaces the notions of strong measurability and measurability coincide (Corollary 1.1.10). As a consequence of these facts, standard definitions and results for measurable random variables taking values in separable metric spaces, such as those collected in Appendix E, apply in the present setting.

Remark 6.1.2 (Strong measurability versus strong $\mathbb{P}$-measurability). A further subtlety concerns the distinction between strong measurability and strong μ-measurability (cf. Definitions 1.1.4 and 1.1.14). Recall that a function is said to be *strongly $\mathbb{P}$-measurable* if it is $\mathbb{P}$-almost everywhere (in contrast to everywhere) the pointwise limit of a sequence of simple functions; if we work over a general measure space it is furthermore required that the approximating simple functions be supported on sets of finite measure. If ξ is a strongly $\mathbb{P}$-measurable, then ξ can be redefined on a $\mathbb{P}$-null set to become a strongly measurable random variable $\widetilde{\xi}$ (Proposition 1.1.16). We may then define

$$\mathbb{P}(\xi \in B) := \mathbb{P}(\widetilde{\xi} \in B)$$

for any set B in $\mathscr{B}(X)$, the Borel σ-algebra of X. This definition is independent of the choice of the pointwise defined representative $\widetilde{\xi}$ and allows one to treat strongly $\mathbb{P}$-measurable functions as random variables. As long as we are dealing with properties of random variables that only depend on their (joint) distributions, we may thus use the notions of 'strongly measurable' and 'strongly $\mathbb{P}$-measurable' interchangeably, and we will indeed do so. Only when the risk of confusion arises we will be more precise in this respect, for instance when dealing with the σ-algebra generated by a random variable or a family of random variables.

In the same vein, an *integrable* random variable will always mean a random variable that is Bochner integrable with respect to $\mathbb{P}$.

The Bochner integral of an integrable X-valued random variable ξ is called its *mean (value)* or *expectation* and is denoted by $\mathbb{E}(\xi)$ or just $\mathbb{E}\xi$. Thus,

$$\mathbb{E}\xi := \int_\Omega \xi \,\mathrm{d}\mathbb{P}.$$

The *distribution* of ξ is the Borel probability measure μ_ξ on X defined by

$$\mu_\xi(B) := \mathbb{P}(\xi \in B), \quad \xi \in \mathscr{B}(X).$$

As a consequence of the substitution rule (Theorem 1.2.6), the expectation of an integrable random variable is given in terms of its distribution by

$$\mathbb{E}\xi = \int_X x \,\mathrm{d}\mu_\xi(x).$$

Simple criteria for random variables to have the same distribution can be given in terms of the so-called characteristic function, which is discussed in Section E.1.c.

A family of X-valued random variables $(\xi_i)_{i\in I}$ is said to be *independent* if for all choices of distinct indices $i_1, \dots, i_N \in I$ and all $B_1, \dots, B_N \in \mathscr{B}(X)$ we have

$$\mathbb{P}(\xi_{i_1} \in B_1, \dots, \xi_{i_N} \in B_N) = \prod_{n=1}^N \mathbb{P}(\xi_{i_n} \in B_n),$$

or equivalently (by Dynkin's lemma, Lemma A.1.3) the distribution of the X^N-valued random variable $(\xi_{i_1}, \dots, \xi_{i_N})$ equals the product of the distributions of the ξ_{i_n}. For further details the reader is referred to Section E.1.b.

The identity

$$\mathbb{E}(\xi_1\xi_2) = \mathbb{E}\xi_1 \cdot \mathbb{E}\xi_2$$

for independent scalar-valued integrable random variables ξ_1 and ξ_2 admits the following extension to the vector-valued case. For a discussion of independence of random variables with values in a metric space we refer to Appendix E.

Proposition 6.1.3. *Let X_1, X_2, Y be Banach spaces and $\beta : X_1 \times X_2 \to Y$ be a bounded bilinear mapping. If the random variables ξ_1 and ξ_2 are independent and integrable, with values in X_1 and X_2 respectively, then $\beta(\xi_1, \xi_2)$ is integrable and*

$$\mathbb{E}\beta(\xi_1, \xi_2) = \beta(\mathbb{E}\xi_1, \mathbb{E}\xi_2).$$

Proof. The integrability of $\beta(\xi_1, \xi_2)$ follows from

$$\mathbb{E}\|\beta(\xi_1, \xi_2)\| \leqslant \|\beta\|\mathbb{E}(\|\xi_1\|\|\xi_2\|) = \|\beta\|\,\mathbb{E}\|\xi_1\|\,\mathbb{E}\|\xi_2\| < \infty,$$

where $\|\beta\| := \sup\{\|\beta(x_1, x_2)\| : \ \|x_1\|, \|x_2\| \leqslant 1\}$ is finite by assumption.

To prove the identity for the expectation we proceed as follows. Let $A_1 \in \sigma(\xi_1)$ and $A_2 \in \sigma(\xi_2)$ (we use the notation $\sigma(\xi)$ to denote the σ-algebra generated by ξ). Then, by bilinearity,

$$\begin{aligned}\mathbb{E}\beta(\mathbf{1}_{A_1} \otimes x_1, \mathbf{1}_{A_2} \otimes x_2) &= \mathbb{E}(\mathbf{1}_{A_1}\mathbf{1}_{A_2}\beta(x_1, x_2)) \\ &= \mathbb{E}\mathbf{1}_{A_1}\mathbb{E}\mathbf{1}_{A_2}\beta(x_1, x_2) \\ &= \beta(\mathbb{E}(\mathbf{1}_{A_1} \otimes x_1), \mathbb{E}(\mathbf{1}_{A_2} \otimes x_2)).\end{aligned}$$

Once again by bilinearity, this proves the identity $\mathbb{E}\beta(\phi_1, \phi_2) = \beta(\mathbb{E}\phi_1, \mathbb{E}\phi_2)$ for all simple $\phi_k : \Omega \to X_k$ that are $\sigma(\xi_k)$-measurable, $k = 1, 2$. By dominated convergence, this implies the same identity for arbitrary $\phi_k \in L^1(\Omega, \sigma(\xi_k); X_k)$; in particular, this proves the identity for ξ_1 and ξ_2. $\square$

It is evident how to extend this result to the multilinear case, and to the sesquilinear case when $\mathbb{K} = \mathbb{C}$. The prime examples of interest include duality and scalar multiplication

$$\begin{aligned}\beta(x, x^*) &:= \langle x, x^*\rangle \qquad && X_1 = X,\ X_2 = X^*,\ Y = \mathbb{K}, \\ \beta(\lambda, x) &:= \lambda x && X_1 = \mathbb{K},\ X_2 = Y = X.\end{aligned}$$

In such cases, we will apply Proposition 6.1.3 casually, without an explicit reference to either the proposition or a particular "bilinear form".

6.1.a Symmetric random variables and randomisation

A distinguished role in the subsequent developments is played by random variables with a simple additional property:

Definition 6.1.4 (Symmetric random variables). *An X-valued random variable is called:*

(1) symmetric*, if ξ and $\epsilon\xi$ are identically distributed for all $\epsilon \in \mathbb{K}$ with $|\epsilon| = 1$;*
(2) real-symmetric*, if ξ and $-\xi$ are identically distributed.*

Clearly, symmetry implies real-symmetry, and in real Banach spaces the two notions coincide. In complex Banach spaces a random variable ξ is symmetric if and only if ξ and $e^{i\theta}\xi$ are identically distributed for all $\theta \in [0, 2\pi]$. We will sometimes refer to the latter property as *complex-symmetry.*

Symmetric random variables have a useful monotonicity property with respect to taking L^p-norms:

Proposition 6.1.5. *Let ξ and η be X-valued random variables. If η is real-symmetric and independent of ξ, then for all $1 \leqslant p \leqslant \infty$ we have*

$$\|\xi\|_{L^p(\Omega;X)} \leqslant \|\xi + \eta\|_{L^p(\Omega;X)}.$$

Somewhat informal statements such as this one are to be interpreted in the obvious way; for instance, here we are saying that $\xi + \eta \in L^p(\Omega; X)$ implies $\xi \in L^p(\Omega; X)$ along with the stated inequality.

Proof. The real-symmetry of η and the independence of ξ and η imply that $\xi + \eta$ and $\xi - \eta$ are identically distributed, and therefore

$$\begin{aligned}\big(\mathbb{E}\|\xi\|^p\big)^{1/p} &= \frac{1}{2}\big(\mathbb{E}\|(\xi+\eta) + (\xi-\eta)\|^p\big)^{1/p} \\ &\leqslant \frac{1}{2}\big(\mathbb{E}\|\xi+\eta\|^p\big)^{1/p} + \frac{1}{2}\big(\mathbb{E}\|\xi-\eta\|^p\big)^{1/p} = \big(\mathbb{E}\|\xi+\eta\|^p\big)^{1/p}.\end{aligned}$$

Here we took $1 \leqslant p < \infty$; the modification needed for $p = \infty$ is obvious. $\square$

As a typical application, suppose $(\xi_n)_{n=1}^N$ is a sequence of independent real-symmetric scalar-valued random variables and $(x_n)_{n=1}^N$ is a sequence in X. Then, for all $I \subseteq \{1, \dots, N\}$,

$$\Big\|\sum_{n\in I} \xi_n x_n\Big\|_{L^p(\Omega;X)} \leqslant \Big\|\sum_{n=1}^N \xi_n x_n\Big\|_{L^p(\Omega;X)}.$$

By a limiting argument, a similar inequality holds for convergent random series. A more general inequality, the Kahane contraction principle, will be discussed in the next section.

Of special interest are Rademacher and Gaussian random variables.

Definition 6.1.6 (Rademacher variables and sequences). *A* Rademacher variable *is a random variable ε uniformly distributed over $\{z \in \mathbb{K} : |z| = 1\}$. A* Rademacher sequence *consists of independent Rademacher variables ε_n.*

In the real case, a Rademacher variable ε takes the values ± 1 with equal probability $\frac{1}{2}$, i.e.,

$$\mathbb{P}(\varepsilon = 1) = \mathbb{P}(\varepsilon = -1) = \frac{1}{2}.$$

In the complex case, a Rademacher variable is a random variable with uniform distribution on the unit circle in the complex plane. Classically, such variables are sometimes called *Steinhaus random variables.* If we wish to use real Rademachers in a complex setting, this will be explicitly mentioned and the notation r will be used instead of ε.

The notion of a Gaussian random variable is discussed at length in Appendix E.2, and we will not repeat it here. We only point out that this notion is adapted to the scalar field in the same way as the notion of symmetry and that of a Rademacher variable. If we wish to use real Gaussian variables when working over the complex scalars, we will always mention this explicitly. In this situation we will use the notation g for real Gaussians and reserve the notation γ for complex Gaussians. Whereas Rademacher variables are intimately connected with unconditionality and provide the tool for randomisation techniques, the main virtue of Gaussians is their invariance under the orthogonal group. This is the basis of the principle of covariance domination, the consequences of which will be explored in later chapters.

Definition 6.1.7 (Gaussian sequences). *A* Gaussian sequence *consists of independent standard Gaussian variables* γ_n.

The reader should keep in mind that both the definition of symmetry and that of Rademacher and Gaussian variables depends on the choice of the underlying scalar field. Although this convention is perhaps not entirely standard in the Probability literature, it is analogous to the standard convention in Functional Analysis that a *linear* map is a *real-linear* map in the real case and a *complex-linear* map in the complex case. We wish to emphasise that this is not an accidental observation but a key point: our definition of symmetry is consistent with the scalar multiplication of the underlying Banach space X. The advantages of this approach will become clear along the way.

The Rademacher variables are often hiding in the background, even when their presence is not immediately obvious, thanks to the following basic lemma, which may be regarded as a toy model of the *randomisation* technique discussed below.

Lemma 6.1.8 (Polar decomposition). *Let ξ be a symmetric X-valued random variable, and ε an independent Rademacher variable. Then ξ and $\varepsilon\xi$ are identically distributed.*

If $X = \mathbb{K}$, they are also identically distributed with $\varepsilon|\xi|$.

Proof. Preserving the joint distributions, we may assume that the independent random variables are defined on different probability spaces Ω_ξ and Ω_ε. We denote the corresponding probabilities and expectations by obvious subscripts. Then

$$\mathbb{P}(\varepsilon\xi \in B) = \mathbb{E}\mathbf{1}_{\{\varepsilon\xi\in B\}} = \mathbb{E}_\varepsilon\mathbb{E}_\xi\mathbf{1}_{\{\varepsilon\xi\in B\}} \overset{(*)}{=} \mathbb{E}_\varepsilon\mathbb{E}_\xi\mathbf{1}_{\{\xi\in B\}} = \mathbb{P}(\xi \in B),$$

where the critical step $(*)$ used the fact that, for each fixed $\omega \in \Omega_\varepsilon$, the random variables $\varepsilon(\omega)\xi$ and ξ on Ω_ξ have equal distribution by the assumed symmetry of ξ.

If $X = \mathbb{K}$ we can make a similar computation

$$\mathbb{P}(\varepsilon\xi \in B) = \mathbb{E}_\xi\mathbb{E}_\varepsilon\mathbf{1}_{\{\varepsilon\xi\in B\}} \overset{(*)}{=} \mathbb{E}_\xi\mathbb{E}_\varepsilon\mathbf{1}_{\{\varepsilon|\xi|\in B\}} = \mathbb{P}(\varepsilon|\xi| \in B).$$

In $(*)$ we used the fact that, for each fixed $\omega \in \Omega_\xi$, the random variables $\xi(\omega)/|\xi(\omega)| \cdot \varepsilon$ and ε, and therefore also $\xi(\omega)\varepsilon$ and $|\xi(\omega)|\varepsilon$, have equal distribution by the assumed symmetry of ε. $\square$

Randomisation

We will now present an extremely useful result that permits one to change the signs of the coefficients in a sum of independent symmetric variables in a deterministic or random way. For its formulation we need a definition.

Definition 6.1.9. *Let I and J be index sets.*

- *Two families of random variables $(\xi_i)_{i\in I}$ and $(\eta_i)_{i\in I}$ are called* identically distributed *if for all indices $i_1,\dots,i_N \in I$ the random variables $(\xi_{i_1},\dots,\xi_{i_N})$ and $(\eta_{i_1},\dots,\eta_{i_N})$ are identically distributed.*
- *Two families of random variables $(\xi_i)_{i\in I}$ and $(\eta_j)_{j\in J}$ are called* independent *if for all indices $i_1,\dots,i_N \in I$ and $j_1,\dots,j_M \in J$ the random variables $(\xi_{i_1},\dots,\xi_{i_N})$ and $(\eta_{j_1},\dots,\eta_{j_M})$ are independent.*

By Dynkin's lemma (Lemma A.1.3), the random variables $(\xi_{i_1},\dots,\xi_{i_N})$ and $(\eta_{i_1},\dots,\eta_{i_N})$ are identically distributed if and only if

$$\mathbb{P}(\xi_{i_1}\in B_1,\dots,\xi_{i_N}\in B_N) = \mathbb{P}(\eta_{i_1}\in B_1,\dots,\eta_{i_N}\in B_N)$$

for all Borel sets $B_1,\dots,B_N$. In particular, two families of *independent* random variables are identically distributed if and only if for every $i\in I$ the random variables ξ_i and η_i are identically distributed.

Likewise, the random variables $(\xi_{i_1},\dots,\xi_{i_N})$ and $(\eta_{j_1},\dots,\eta_{j_M})$ are independent if and only if

$$\begin{aligned}\mathbb{P}(\xi_{i_1}&\in B_1,\ \dots\ ,\xi_{i_N}\in B_N,\ \eta_{j_1}\in C_1,\ \dots\ ,\eta_{j_M}\in C_M)\\ &= \mathbb{P}(\xi_{i_1}\in B_1,\ \dots\ ,\xi_{i_N}\in B_N)\mathbb{P}(\eta_{j_1}\in C_1,\ \dots\ ,\eta_{j_M}\in C_M)\end{aligned}$$

for all Borel sets $B_1,\dots,B_N$ and $C_1,\dots,C_M$. It is not true, however, that two families of *independent* random variables $(\xi_i)_{i\in I}$ and $(\eta_i)_{i\in I}$ are independent if for all $i,i'\in I$ the random variables ξ_i and $\eta_{i'}$ are independent.

Example 6.1.10. Let ε_1 and ε_2 be independent Rademacher variables and let $\eta := \varepsilon_1\varepsilon_2$. Then clearly $\{\varepsilon_1,\varepsilon_2\}$ and $\{\eta\}$ are not independent; however, η is independent of both ε_1 and ε_2. In fact, assuming without loss of generality that ε_1 and ε_2 are defined on different probability spaces Ω_1,Ω_2, we have

$$\begin{aligned}\mathbb{P}(\varepsilon_1\in A,\eta\in B) &= \mathbb{E}_1\mathbb{E}_2(\mathbf{1}_{\{\varepsilon_1\in A\}}\mathbf{1}_{\{\varepsilon_1\varepsilon_2\in B\}}) = \mathbb{E}_1(\mathbf{1}_{\{\varepsilon_1\in A\}}\mathbb{E}_2\mathbf{1}_{\{\varepsilon_1\varepsilon_2\in B\}})\\ &= \mathbb{E}_1(\mathbf{1}_{\{\varepsilon_1\in A\}}\mathbb{E}_2\mathbf{1}_{\{\varepsilon_2\in B\}}) = \mathbb{E}_1(\mathbf{1}_{\{\varepsilon_1\in A\}})\mathbb{E}_2(\mathbf{1}_{\{\varepsilon_2\in B\}}),\end{aligned}$$

where $\mathbb{E}_1(\mathbf{1}_{\{\varepsilon_1\in A\}}) = \mathbb{P}(\varepsilon_1\in A)$ and

$$\mathbb{E}_2(\mathbf{1}_{\{\varepsilon_2\in B\}}) = \mathbb{E}_1\mathbb{E}_2(\mathbf{1}_{\{\varepsilon_2\in B\}}) = \mathbb{E}_1\mathbb{E}_2(\mathbf{1}_{\{\varepsilon_1\varepsilon_2\in B\}}) = \mathbb{P}(\eta\in B).$$

Proposition 6.1.11 (Randomisation). *Let $(\xi_n)_{n\geqslant1}$ be a sequence of independent and symmetric X-valued random variables, let $(\epsilon_n)_{n\geqslant1}$ be a sequence in $\{z\in\mathbb{K}:\ |z|=1\}$, and let $(\varepsilon_n)_{n\geqslant1}$ be a Rademacher sequence which is independent of $(\xi_n)_{n\geqslant1}$.*

(1) *The sequences* $(\xi_n)_{n\geqslant 1}$, $(\epsilon_n\xi_n)_{n\geqslant 1}$, *and* $(\varepsilon_n\xi_n)_{n\geqslant 1}$ *are identically distributed.*
(2) *If the random variables* ξ_n *are scalar-valued, then the sequences* $(\xi_n)_{n\geqslant 1}$ *and* $(\varepsilon_n|\xi|_n)_{n\geqslant 1}$ *are identically distributed.*

The same result holds if one replaces {*symmetric,* $\mathbb{K}$*, Rademacher*} *by* {*real-symmetric,* $\mathbb{C}$*, real Rademacher*}*.*

Proof. (1): By independence, we may assume that each ξ_n and ε_n is defined on a different probability space, say ξ_n on Ω_n and ε_n on Ω_n'. From this it is clear that the sequence $(\varepsilon_n\xi_n)_{n\geqslant 1}$ since the random variables are defined on the different probability spaces $\Omega_n \times \Omega_n'$.

Thus it suffices to show that for each fixed $n \geqslant 1$, the random variables ξ_n, $\epsilon_n\xi_n$, and $\varepsilon_n\xi_n$ are identically distributed. For ξ_n and $\epsilon_n\xi_n$ this is the definition of symmetry. For ξ_n and $\varepsilon_n\xi_n$, this is Lemma 6.1.8.

(2): As above, it suffices to prove that ξ_n and $\varepsilon_n|\xi_n|$ are identically distributed. This follows from Lemma 6.1.8. □

A first application of these ideas is the following simple maximal estimate.

Proposition 6.1.12 (Lévy's inequality). *Let* $\xi_1,\dots,\xi_n$ *be independent real-symmetric* X*-valued random variables, and put* $S_k := \sum_{j=1}^k \xi_j$ *for* $k = 1,\dots,n$*. Then for all* $r \geqslant 0$ *we have*

$$\mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \|S_k\| > r\Big) \leqslant 2\mathbb{P}(\|S_n\| > r).$$

Proof. Put

$$\begin{aligned} A &:= \Big\{\max_{1\leqslant k\leqslant n} \|S_k\| > r\Big\}, \\ A_k &:= \{\|S_1\| \leqslant r, \dots, \|S_{k-1}\| \leqslant r, \|S_k\| > r\}; \quad k = 1,\dots,n. \end{aligned}$$

The sets $A_1,\dots,A_n$ are disjoint and $\bigcup_{k=1}^n A_k = A$.

The identity $S_k = \frac{1}{2}(S_n + (2S_k - S_n))$ implies that

$$\{\|S_k\| > r\} \subseteq \{\|S_n\| > r\} \cup \{\|2S_k - S_n\| > r\}.$$

By Proposition 6.1.11, $(\xi_1,\dots,\xi_n)$ and $(\xi_1,\dots,\xi_k,-\xi_{k+1},\dots,-\xi_n)$ are identically distributed, which, in view of the identities

$$S_n = S_k + \xi_{k+1} + \dots + \xi_n, \qquad 2S_k - S_n = S_k - \xi_{k+1} - \dots - \xi_n,$$

implies that $(\xi_1,\dots,\xi_k,S_n)$ and $(\xi_1,\dots,\xi_k,2S_k - S_n)$ are identically distributed. Hence,

$$\begin{aligned} \mathbb{P}(A_k) &\leqslant \mathbb{P}(A_k \cap \{\|S_n\| > r\}) + \mathbb{P}(A_k \cap \{\|2S_k - S_n\| > r\}) \\ &= 2\mathbb{P}(A_k \cap \{\|S_n\| > r\}). \end{aligned}$$

Summing over k, we obtain

$$\mathbb{P}(A) = \sum_{k=1}^{n} \mathbb{P}(A_k) \leqslant 2 \sum_{k=1}^{n} \mathbb{P}(A_k \cap \{\|S_n\| > r\}) = 2\mathbb{P}(\|S_n\| > r).$$

□

6.1.b Kahane's contraction principle

One of the most useful tools for estimating random sums is the following contraction principle due to Kahane, which asserts that scalar sequences act as multipliers with respect to the L^p-norm. The randomisation technique gives us this immediate extension from a version already covered in Proposition 3.2.10.

Theorem 6.1.13 (Kahane's contraction principle). *Let $1 \leqslant p \leqslant \infty$ and let $(\xi_n)_{n=1}^N$ be a sequence of independent random variables in $L^p(\Omega; X)$.*

(i) *If all ξ_n are $\mathbb{K}$-symmetric, then for all sequences $(a_n)_{n=1}^N$ in $\mathbb{K}$ we have*

$$\Big\| \sum_{n=1}^{N} a_n \xi_n \Big\|_{L^p(\Omega;X)} \leqslant \max_{1 \leqslant n \leqslant N} |a_n| \, \Big\| \sum_{n=1}^{N} \xi_n \Big\|_{L^p(\Omega;X)}.$$

(ii) *If all ξ_n are $\mathbb{R}$-symmetric, then for all sequences $(a_n)_{n=1}^N$ in $\mathbb{C}$ we have*

$$\Big\| \sum_{n=1}^{N} a_n \xi_n \Big\|_{L^p(\Omega;X)} \leqslant \frac{\pi}{2} \max_{1 \leqslant n \leqslant N} |a_n| \, \Big\| \sum_{n=1}^{N} \xi_n \Big\|_{L^p(\Omega;X)}.$$

The constants in these inequalities are sharp.

Proof. The special case that $\xi_n = \varepsilon_n x_n$ in (i) or $\xi_n = r_n x_n$ in (ii) has already been treated in Proposition 3.2.10. We will use randomisation to reduce the general statement to this special case.

Indeed, letting $(\varepsilon_n)_{n=1}^N$ be an independent Rademacher sequence on another probability space Ω', we have

$$\begin{aligned}
\Big\| \sum_{n=1}^{N} a_n \xi_n \Big\|_{L^p(\Omega;X)} &= \Big\| \sum_{n=1}^{N} a_n \varepsilon_n \xi_n \Big\|_{L^p(\Omega;L^p(\Omega';X))} \\
&\leqslant \max_{1 \leqslant n \leqslant N} |a_n| \Big\| \sum_{n=1}^{N} \varepsilon_n \xi_n \Big\|_{L^p(\Omega;L^p(\Omega';X))} \\
&= \max_{1 \leqslant n \leqslant N} |a_n| \Big\| \sum_{n=1}^{N} \xi_n \Big\|_{L^p(\Omega;X)}
\end{aligned}$$

in case (i), and (ii) is proved similarly with obvious modifications. Both equalities above are based on randomisation, and the inequality is the special case of the theorem proved in Proposition 3.2.10, with $\varepsilon_n x_n$ in place of ξ_n. The sharpness has already been observed in this special case.

While the above chain of identities and estimates above is meaningful for all $p \in [1, \infty]$, one can also derive the case $p = \infty$ simply as the limit $p \to \infty$ of finite exponents, using $\|f\|_{L^\infty(\Omega)} = \lim_{p\to\infty} \|f\|_{L^p(\Omega)}$. $\square$

Remark 6.1.14. The symmetry assumption may be dropped from one of the variables ξ_m, say ξ_1, in Theorem 6.1.13. Namely, suppose that $\xi_1, \dots, \xi_N$ are independent and $\xi_2, \dots, \xi_N$ are $\mathbb{K}$-symmetric, and consider a random variable ε uniformly distributed over $\{z \in K : |z| = 1\}$ and independent of all $\xi_1, \dots, \xi_N$. Then $(\xi_1, \varepsilon\xi_2, \dots, \varepsilon\xi_N)$ is equidistributed with $(\xi_1, \xi_2, \dots, \xi_N)$, and

$$\Big\|\sum_{n=1}^N a_n\xi_n\Big\|_{L^p(\Omega;X)} = \Big\|a_1\xi_1+\sum_{n=2}^N a_n\varepsilon\xi_n\Big\|_{L^p(\Omega;X)} = \Big\|a_1\bar{\varepsilon}\xi_1+\sum_{n=2}^N a_n\xi_n\Big\|_{L^p(\Omega;X)},$$

where $\bar{\varepsilon}\xi_1$ is a $\mathbb{K}$-symmetric random variable. In particular, it follows that

$$\Big\|a_1\bar{\varepsilon}\xi_1 + \sum_{n=2}^N a_n\xi_n\Big\|_{L^p(\Omega;X)} \leqslant \|a\|_\infty \Big\|\sum_{n=1}^N \xi_n\Big\|_{L^p(\Omega;X)}.$$

This formulation includes Proposition 6.1.5 as a special case.

6.1.c Norm comparison of different random sums

In this subsection we prove several basic L^p-norm comparisons between the different types of random sums discussed so far.

Rademacher sums versus other symmetric random sums

We begin with the comparison of Rademacher sums against random sums involving general symmetric random coefficients.

Proposition 6.1.15 (Comparison). *Let $1 \leqslant p \leqslant \infty$ and let $(\xi_n)_{n=1}^N$ be a sequence of independent symmetric random variables in L^p and let $(\varepsilon_n)_{n\geqslant 1}$ denote a Rademacher sequence. Then for all $x_1, \dots, x_N \in X$ we have*

$$\Big\|\sum_{n=1}^N \varepsilon_n x_n\Big\|_{L^p(\Omega;X)} \leqslant \max_{1\leqslant n\leqslant N} \frac{1}{\mathbb{E}|\xi_n|} \Big\|\sum_{n=1}^N \xi_n x_n\Big\|_{L^p(\Omega;X)}.$$

Note that in the case ξ_n is real/complex symmetric, the Rademacher sequence in the above result should be real/complex as well.