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Bounds and Asymptotics for Orthogonal Polynomials for Varying Weights

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Contents

Chapter 1
Introduction

Let μ be a finite positive Borel measure with support on the real line containing infinitely many points, and all finite power moments $\int x^j d\mu(x)$, $j = 0, 1, 2, \ldots$. We may then define orthonormal polynomials $p_n(x)$ of degree n, $n = 0, 1, 2, \ldots$, satisfying

$$\int p_m(x) p_n(x) d\mu(x) = \delta_{mn}.$$

The asymptotic behavior of $p_n(x)$ as $n \to \infty$ has been studied for over a century. Beginning around 1918 [47, 48] Szegő analyzed orthonormal polynomials for absolutely continuous measures supported on $[-1, 1]$, or the unit circle, motivated by connections to Hankel and Toeplitz matrices. Plancherel and Rotach in the late 1920s [41] considered the Hermite weight $\mu'(x) = e^{-x^2}$ on $(-\infty, \infty)$, in order to investigate convergence of orthonormal expansions in Hermite polynomials. Plancherel and Rotach applied the method of steepest descent to a contour integral representation of Hermite polynomials. The very precise asymptotics they established are now called Plancherel-Rotach type asymptotics, and continue to be studied for more general measures to this day.

Until the last three decades of the 20th century, there were very few techniques for investigating orthogonal polynomials for non-compactly supported measures. If the orthogonal polynomials admit a contour integral representation, or a simple second order differential equation, or have a generating function as in the case of Pollaczek polynomials, classical asymptotic methods are applicable. However even the rudiments of a general theory were lacking. It was Geza Freud and later Paul Nevai who in the 1970s began to consider general weights $e^{-2Q(x)}$ on $(-\infty, \infty)$, using extremal properties and approximation to develop weaker forms of asymptotics. Nevai and his students, William Bauldry, Stan Bonan, Rong Sheen, and Shing Whu-Jha, obtained precise asymptotics for weights like $\exp\left(-x^{2m}\right)$, where

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m is a positive integer, using a mixture of analyzing differential equations and recurrence relations. Paul Erdős provided valuable insights for the case where $\mu'(x)$ decreases faster than $e^{-|x|^{\alpha}}$ for all $\alpha > 0$. See the still very relevant 1986 survey paper of Nevai [40].

Potential theory with external fields provided a dramatic breakthrough in the 1980s. In landmark papers, E. A. Rakhmanov [42] and Mhaskar and Saff [36–38] showed how to analyze orthogonal and extremal polynomials for quite general weights of the form $e^{-2Q(x)}$ on the real line. A comprehensive and polished development of that theory appears in the celebrated monograph of Saff and Totik [44]. By combining that potential theory with older methods of orthogonal polynomials, such as explicit formulae for Bernstein-Szegő weights, many researchers in orthogonal polynomials were able to analyze asymptotics – including the present authors [25].

An alternative approach to asymptotics for orthogonal polynomials is to place hypotheses on the coefficients in their three term recurrence relation, rather than on the underlying measure or weight. Some model examples of this approach for non-compactly supported measures appear in [14, 15, 55, 57]. Yet another relevant link is to discrete measures associated with indeterminate moment problems, see for example [7].

A second revolution for the case of absolutely continuous weights, came with the Deift-Zhou method in [9, 11, 12]. They developed a steepest descent method for a matrix Riemann-Hilbert problem whose solution includes orthonormal polynomials, and which was first observed by Fokas, Its, and Kitaev. The dramatic ramifications of that method continue to be observed to this day. While it initially dealt primarily with analytic or piecewise analytic weights, it has been extended by McLaughlin and Miller using a $\bar{\partial}$ approximation [34, 35]. A distinguishing feature of results obtained via Riemann-Hilbert methods is that they hold globally, and are far more precise than any general results obtained using any other method. Because they were motivated by problems arising in random matrices, Riemann-Hilbert researchers usually considered varying rather than fixed measures. That brings us to the setting of this monograph, which is the varying weights case.

For $n \geq 1$, let μ_n be a finite positive Borel measure with support $\text{supp}[\mu_n] \subset \mathbb{R}$, containing infinitely many points. Assume also that all power moments $\int x^j d\mu_n(x)$, $j = 0, 1, 2, \ldots$, are finite. We may define orthonormal polynomials

$$p_{n,m}(\mu_n, x) = \gamma_{n,m}(\mu_n)\, x^m + \cdots, \quad \gamma_{n,m}(\mu_n) > 0,$$

$m = 0, 1, 2, \ldots$, satisfying the orthonormality conditions

$$\int p_{n,k}(\mu_n, x)\, p_{n,\ell}(\mu_n, x)\, d\mu_n(x) = \delta_{k\ell}.$$

The nth reproducing kernel for μ_n is

$$K_n(\mu_n, x, y) = \sum_{k=0}^{n-1} p_{n,k}(\mu_n, x)\, p_{n,k}(\mu_n, y). \tag{1.1}$$

We often abbreviate this as $K_n(x, y)$. The nth Christoffel function is

$$\lambda_n(\mu_n, x) = K_n(\mu_n, x, x)^{-1}.$$

The absolutely continuous case, where

$$\mu_n'(x) = e^{-2nQ_n(x)} \tag{1.2}$$

and $\{Q_n\}$ are given functions, plays an important role in random matrices. In this case, we often use the notation $p_{n,k}\left(e^{-2nQ_n}, x\right)$, $\lambda_n\left(e^{-2nQ_n}, x\right)$, and so on. The canonical example is $Q_n(x) = x^2$ for $n \geq 1$. Szegő style asymptotics of the associated orthonormal polynomials have been investigated by many authors, with many of the most spectacular results obtained using the Deift-Zhou steepest descent method. In particular, in celebrated papers [11, 12], Deift, Kriecherbauer, McLaughlin, Venakides, and Zhou considered the case where all $Q_n = Q$, and $Q(x)$ is real analytic on the real line, and grows faster than $\log|x|$ as $|x| \to \infty$. They established uniform asymptotics for the associated orthonormal polynomials in all regions of the complex plane, as well as detailed asymptotics for associated quantities, with applications to universality limits for random matrices. This set the stage for treating a large array of varying weights, such as varying (and sometimes fixed) Jacobi or Laguerre weights - some of the references are [3, 18–23, 56].

In all the earlier Riemann-Hilbert papers, Q was required to be analytic in a neighborhood of the real line, or piecewise analytic. As noted above, using the $\bar{\partial}$-method, McLaughlin and Miller [34, 35] relaxed the requirement of analyticity, and considered the case where Q'' satisfies a Lipschitz condition of order 1, together with some other conditions. In particular, the latter conditions are satisfied when Q is strictly convex in the real line. They established asymptotics for $p_{n,n}$ and $p_{n,n-1}$ in all regions of the complex plane – including asymptotics inside and at the edge of the Mhaskar-Rakhmanov-Saff interval (or equivalently, the support of the equilibrium measure). One of our foci is to further relax their smoothness requirements on Q.

We shall need some concepts from the potential theory for external fields [44], to which we alluded above. Let Σ be a closed set on the real line, and e^{-Q} be an upper semi-continuous function on Σ that is positive on a set of positive linear Lebesgue measure. If Σ is unbounded, we assume that

$$\lim_{|x|\to\infty, x\in\Sigma} (Q(x) - \log|x|) = \infty.$$

Associated with Σ and Q, we may consider the extremal problem

$$\inf_{\nu}\left(\int\int \log\frac{1}{|x-t|}d\nu(x)\,d\nu(t)+2\int Q\,d\nu\right),$$

where the inf is taken over all positive Borel measures ν with support in Σ and $\nu(\Sigma)=1$. The inf is attained by a unique equilibrium measure ω_Q, with support $\mathrm{supp}[\omega_Q]$, characterized by the following conditions: let

$$V^{\omega_Q}(z)=\int\log\frac{1}{|z-t|}d\omega_Q(t) \tag{1.3}$$

denote the logarithmic potential for ω_Q. Then [44, Thm. I.1.3, p. 27]

$$V^{\omega_Q}+Q\geq F_Q \text{ q.e. on } \Sigma; \tag{1.4}$$

$$V^{\omega_Q}+Q=F_Q \text{ q.e. in } \mathrm{supp}\left[\omega_Q\right]. \tag{1.5}$$

Here the number F_Q is a constant, and q.e. stands for quasi everywhere, that is, except on a set of capacity 0. Notice that we are using ω_Q for the equilibrium measure, rather than the more standard μ_W or ν_W, to avoid confusion with μ_n or ν_n. We use

$$\sigma_Q(x)=\omega_Q'(x) \tag{1.6}$$

for the Radon-Nikodym derivative of ω_Q. Sometimes we denote V^{ω_Q} by V^{σ_Q}.

While the Riemann-Hilbert methods yield the strongest results for smooth weights, techniques based on potential theory and Bernstein-Szegő weights allow one to treat more general weights. Indeed, this was the traditional approach for fixed exponential weights adopted in [25, 32, 42, 43, 51, 52]. These methods enabled one to establish asymptotics of the orthonormal polynomials in the complex plane away from the interval of orthogonality, but not usually pointwise asymptotics on the interval. The most general results for varying weights, using these types of tools, were obtained by V. Totik in his 1994 lecture notes [51, Thm. 14.2, p. 99; Thm. 14.4, p. 101]:

Theorem A. *For $n\geq 1$, let e^{-2nQ_n} be a weight function on $[-1,1]$, whose equilibrium measure ω_{Q_n} has support $[-1,1]$. Assume that ω_{Q_n} is absolutely continuous, and its density σ_{Q_n} satisfies*

$$\frac{1}{A}\left(1-t^2\right)^{\beta_0}\leq\sigma_{Q_n}(t)\leq A\left(1-t^2\right)^{\beta_1},\quad t\in(-1,1),$$

where $\beta_1 > -1$, *and* A, β_0, β_1 *are independent of n. Assume also that* $\{\sigma_{Q_n}\}$ *are uniformly equicontinuous in every compact subset of* $(-1, 1)$.

(I) Then for any fixed integer k,

$$p_{n,n+k}\left(e^{-2nQ_n}, x\right) e^{-nQ_n(x)} - \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt[4]{1-x^2}} \cos\left[\left(k+\frac{1}{2}\right) \arccos x + n\pi \int_x^1 \sigma_{Q_n} - \frac{\pi}{4}\right] \tag{1.7}$$

tends to 0 *in* $L_2[-1, 1]$ *as* $n \to \infty$.

(II) Uniformly for z in compact subsets of $\bar{\mathbb{C}} \setminus [-1, 1]$,

$$p_{n,n+k}(z) = \frac{1+o(1)}{\sqrt{2\pi}} \left(z + \sqrt{z^2-1}\right)^{k+\frac{1}{2}} \left(z^2-1\right)^{-1/4} \times \exp\left(nF_{Q_n} - n\int \log \frac{1}{z-t} \sigma_{Q_n}(t)\, dt\right). \tag{1.8}$$

Here F_{Q_n} *is the constant in* (1.4) *for* $Q = Q_n$.

We note that this is not the most general form of Totik's result, and both asymptotics above can be formulated in terms of Szegő functions and their arguments. Indeed, (1.7) is formulated in a different way in [51]. Moreover, for $Q_n(x) = |x|^\alpha$, $\alpha > 1$, all the constants and densities can be given explicit forms. It is also significant that the weights $\left\{e^{-nQ_n}\right\}$ are assumed to be supported on $[-1, 1]$. There are extra difficulties in establishing asymptotics when, for example, the interval of orthogonality is unbounded. Then one has to use restricted range inequalities, and often this requires extra hypotheses.

Another important asymptotic is that for Christoffel functions. One of Totik's celebrated results for asymptotics of Christoffel functions for varying weights is [52]:

Theorem B. *Let* e^{-Q} *be a continuous nonnegative function on the set* Σ, *which is assumed to consist of finitely many intervals. If* Σ *is unbounded, we assume also*

$$\lim_{|x|\to\infty, x\in\Sigma} Q(x) / \log|x| = \infty.$$

Let J be a closed interval lying in the interior of $\mathrm{supp}\left[\omega_Q\right]$, *where* ω_Q *denotes the equilibrium measure for Q. Assume that* ω_Q *is absolutely continuous in a neighborhood of J, and that* σ_Q *is continuous in that neighborhood. Then uniformly for* $x \in J$,

$$\lim_{n\to\infty} \frac{1}{n} \lambda_n\left(e^{-2nQ}, x\right) e^{2nQ(x)} = \sigma_Q(x). \tag{1.9}$$

In particular, when Q' satisfies a Lipschitz condition of some positive order in a neighborhood of J, then [44, p. 209] σ_Q is continuous there, and hence we obtain asymptotics of Christoffel functions there. Note too that when Q is convex in Σ, or $xQ'(x)$ is increasing there, then the support of ω_Q consists of at most finitely many intervals, with at most one interval per component of Σ [44, p. 199, Thm. 1.10(c)]. We used Totik's result to establish universality results for varying weights in [27, p. 747, Thm. 1.1].

Our aim in this paper is especially to establish locally uniform versions of (1.7) in compact subsets of the Mhaskar-Rakhmanov-Saff interval, as well as global bounds on the orthonormal polynomials. We now define the class of weights that we shall use throughout this book:

Definition 1.1. For $n \geq 1$, let $I_n = (c_n, d_n)$, where $-\infty \leq c_n < d_n \leq \infty$. Assume that for some $r^* > 1$, $[-r^*, r^*] \subset I_n$, for all $n \geq 1$. Assume that

$$\mu_n'(x) = e^{-2nQ_n(x)}, \quad x \in I_n, \tag{1.10}$$

where (i) $Q_n(x)/\log(2+|x|)$ has limit ∞ *at* c_n+ *and* d_n-.
(ii) Q_n' is strictly increasing and continuous in I_n.
(iii) There exists $\alpha \in (0,1)$, $C > 0$ such that for $n \geq 1$ and $x, y \in [-r^*, r^*]$,

$$\left|Q_n'(x) - Q_n'(y)\right| \leq C\,|x-y|^{\alpha}. \tag{1.11}$$

(iv) There exists $\alpha_1 \in \left(\frac{1}{2}, 1\right)$, $C_1 > 0$, and an open neighborhood I_0 of 1 and -1, such that for $n \geq 1$ and $x, y \in I_n \cap I_0$,

$$\left|Q_n'(x) - Q_n'(y)\right| \leq C_1\,|x-y|^{\alpha_1}. \tag{1.12}$$

(v) $[-1,1]$ is the support of the equilibrium distribution ω_{Q_n} for Q_n.

Then we write $\{Q_n\} \in \mathcal{Q}$.

Remarks. (a) The convexity and smoothness assumptions can be replaced by implicit assumptions involving bounds and smoothness of the equilibrium distributions such as bounds and smoothness.
(b) The support condition (v) is equivalent to the Mhaskar-Rakhmanov-Saff equations

$$\frac{1}{\pi}\int_{-1}^{1} \frac{xQ_n'(x)}{\sqrt{1-x^2}}\,dx = 1; \tag{1.13}$$

$$\frac{1}{\pi}\int_{-1}^{1} \frac{Q_n'(x)}{\sqrt{1-x^2}}\,dx = 0. \tag{1.14}$$

(c) It may seem strange that we impose a stronger smoothness condition near ± 1 than elsewhere. This is needed to bound the equilibrium density near the endpoints of the Mhaskar-Rakhmanov-Saff interval, as we shall see in Chapter 3. We also show there that something like this is needed to ensure uniform convergence of integrals that arise there, such as

$$\int_0^1 \frac{\left|Q_n'(t) - Q_n'(1)\right|}{(1-t)^{3/2}}\,dt.$$

(d) The Lipschitz condition of order $\alpha_1 > \frac{1}{2}$ in (iv) can be weakened to

$$\left|Q_n'(x) - Q_n'(y)\right| \le C_1 \Omega\left(|x-y|\right), \quad x, y \in I_n \cap I_0,$$

where

$$\int_0^1 \frac{\Omega(t)}{t^{3/2}}\,dt < \infty.$$

Under this weaker condition, we can still prove all the results of the next chapter, but with weaker error terms, no longer with $O\left(n^{-\tau}\right)$, some $\tau > 0$.
(e) For notational convenience, we shall often assume that

$$\alpha \le \alpha_1 - \frac{1}{2}. \tag{1.15}$$

(f) The hypotheses force for some $C > 0$ independent of n,

$$Q_n'(-1) \le -C \text{ and } Q_n'(1) \ge C,$$

see Lemma 3.2 below. Then for some $t_n \in (-1, 1)$,

$$Q_n'(t_n) = 0 \tag{1.16}$$

and then the uniform Lipschitz condition gives

$$\sup_{n\ge 1}\ \sup_{t\in[-r^*,r^*]} \left|Q_n'(t)\right| < \infty. \tag{1.17}$$

We can then divide each $\mu_n' = e^{-2nQ_n}$ by a normalizing constant, and assume that also

$$Q_n(t_n) = 0, \tag{1.18}$$

and hence

$$\sup_{n\ge 1}\ \sup_{t\in[-r^*,r^*]} \left|Q_n(t)\right| < \infty. \tag{1.19}$$

(g) The hypotheses of Definition 1.1 are satisfied if, for example, for $n \geq 1, Q_n(x) = c_n|x|^{\beta_n}$ with all $\{\beta_n\}$ lying in a fixed compact subset of $(1, \infty)$, and $\{c_n\}$ are chosen so that the equilibrium measures have support $[-1, 1]$.

Throughout $C, C_1, C_2, \ldots$ denote constants independent of n, x, t and perhaps other specified parameters. The same symbol does not necessarily indicate the same constant in different occurrences. For sequences $\{x_n\}$ and $\{y_n\}$ of nonzero real numbers, we write $x_n \sim y_n$ if there exists $C > 1$ such that for $n \geq 1$,

$$C^{-1} \leq x_n/y_n \leq C.$$

We shall state some of our main results in the next chapter. The proofs of these will be distributed over Chapters 3 to 15. We shall discuss the organization in more detail in the next chapter.

Chapter 2
Statement of Main Results

We first state our uniform bounds on the orthonormal polynomials and related quantities:

Theorem 2.1. *Assume that* $\{Q_n\} \in \mathcal{Q}$ *and that for* $n, m \geq 1$, $p_{n,m}$ *is the orthonormal polynomial of degree* m *for the weight* e^{-2nQ_n} *on* I_n.
(a) Let $A > 0$. *For* $n \geq 1$, *and*

$$|n - m| \leq An^{1/3}, \tag{2.1}$$

we have

$$\sup_{x \in I_n} |p_{n,m}|(x)\, e^{-nQ_n(x)} \left[|1 - |x|| + n^{-2/3}\right]^{1/4} \sim 1. \tag{2.2}$$

Moreover, uniformly in such m, n,

$$\sup_{x \in I_n} |p_{n,m}|(x)\, e^{-nQ_n(x)} \sim n^{1/6}. \tag{2.3}$$

(b) Let $A > 0$. *Uniformly for* $n \geq 1$, m *satisfying* (2.1), *and* $x \in I_n$ *satisfying* $|x| \leq 1 + An^{-2/3}$, *we have*

$$\lambda_m\left(e^{-2nQ_n}, x\right) e^{2nQ_n(x)} \sim \frac{1}{n} \max\{|1 - |x||, n^{-2/3}\}^{-1/2}. \tag{2.4}$$

Moreover, uniformly for $n \geq 1, m$ *satisfying* (2.1), *and* $x \in I_n$,

$$\lambda_m\left(e^{-2nQ_n}, x\right) e^{2nQ_n(x)} \geq C\frac{1}{n} \max\{|1 - |x||, n^{-2/3}\}^{-1/2}. \tag{2.5}$$

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(c) Let $\{x_{jn}\}$ *denote the zeros of* $p_{n,n}$*, ordered as*

$$x_{nn} < x_{n-1,n} < \cdots < x_{1n}.$$

Uniformly for $n \geq 1$ *and* $1 \leq j \leq n-1$*,*

$$x_{jn} - x_{j+1,n} \sim \frac{1}{n} \max\{\left|1 - \left|x_{jn}\right|\right|, n^{-2/3}\}^{-1/2}. \tag{2.6}$$

Moreover,

$$1 - \frac{C_1}{n^{2/3}} \leq x_{1n} \leq 1 + \frac{C_2}{n}, \tag{2.7}$$

with a similar inequality for x_{nn}*.*

Proof. (a) See Theorems 7.1 and 14.2(d).
(b) See Theorem 5.1.
(c) See Theorems 6.1 and 14.2(c). □

Remarks. We believe that for the uniform bound (2.2) to hold, one really does need Q_n' to satisfy a Lipschitz condition of order at least $\frac{1}{2}$ near ± 1.

Next, we turn to asymptotics on the interval of orthogonality.

Theorem 2.2. *Assume that* $\{Q_n\} \in \mathcal{Q}$*. Let* $\varepsilon \in \left(0, \frac{1}{3}\right)$*. For* $n \geq 1$*, let* σ_{Q_n} *denote the density of the equilibrium measure for* Q_n *on* $[-1, 1]$*. There exists* $\tau > 0$ *such that uniformly for* $n \geq 1$ *and* $|x| \leq 1 - n^{-\tau}$*,* $\theta = \arccos x$*, and for*

$$|m - n| \leq n^{1/3-\varepsilon}, \tag{2.8}$$

we have (a)

$$\sqrt{\frac{\pi}{2}} p_{n,m}(x)\, e^{-nQ_n(x)} \left(1 - x^2\right)^{1/4}$$
$$= \cos\left((m-n)\,\theta + n\pi \int_x^1 \sigma_{Q_n}(t)\, dt + \frac{\theta}{2} - \frac{\pi}{4}\right) + O\left(n^{-\tau}\right). \tag{2.9}$$

(b)

$$\frac{1}{n}\sqrt{\frac{\pi}{2}} p'_{n,m}(x)\, e^{-nQ_n(x)} \left(1 - x^2\right)^{1/4}$$
$$= \pi\sigma_{Q_n}(x) \sin\left((m-n)\,\theta + n\pi \int_x^1 \sigma_{Q_n}(t)\, dt + \frac{\theta}{2} - \frac{\pi}{4}\right)$$
$$+ Q_n'(x) \cos\left((m-n)\,\theta + n\pi \int_x^1 \sigma_{Q_n}(t)\, dt + \frac{\theta}{2} - \frac{\pi}{4}\right) + O\left(n^{-\tau}\right). \tag{2.10}$$

(c)

$$\frac{1}{n}\lambda_n^{-1}\left(e^{-2nQ_n},x\right)e^{-2nQ_n(x)}=\sigma_{Q_n}(x)+O\left(n^{-\tau}\right). \tag{2.11}$$

(d) Uniformly for j with $\left|x_{jn}\right|\leq 1-n^{-\tau}$,

$$n\sigma_{Q_n}\left(x_{jn}\right)\left(x_{jn}-x_{j+1,n}\right)=1+O\left(n^{-\tau}\right). \tag{2.12}$$

Proof. (a), (b) See Theorem 13.2(a), (b).
(c) See Theorem 13.3.
(d) See Theorem 13.5(b). □

Remarks. (a) We expect that one can prove the asymptotic (2.11) for the Christoffel function without assuming the extra Lipschitz condition (1.12) near ± 1.

(b) We also expect that one can prove the asymptotic (2.9) assuming less smoothness on $\{Q_n\}$: instead of (1.11), assume equicontinuity of $\{\sigma_{Q_n}\}$ in $[-1,1]$ (which is true if $\left\{Q_n'\right\}$ satisfy a uniform Dini condition). In addition, replace (1.12) by the conditions in the remarks (d) after Definition 1.1. However, one then loses the $O\left(n^{-\tau_1}\right)$ error term, and the asymptotic would hold in compact subsets of $(-1,1)$.

Finally, we turn to asymptotics for orthonormal polynomials in the plane, and for leading coefficients. We need more notations. Let

$$\phi(z)=z+\sqrt{z^2-1} \tag{2.13}$$

denote the conformal map of the exterior of $[-1,1]$ onto the exterior of the unit ball. For $n\geq 1$, let

$$F_n(\theta)=e^{-2nQ_n(\cos\theta)}\left|\sin\theta\right|. \tag{2.14}$$

Define the associated Szegő function

$$D(F_n;z)=\exp\left(\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log F_n(t)\,dt\right),\ |z|<1. \tag{2.15}$$

Theorem 2.3. *Assume that* $\{Q_n\}\in\mathcal{Q}$. *Let* $\varepsilon\in\left(0,\frac{1}{3}\right)$. *There exists* $\tau>0$ *such that uniformly for* $n\geq 1$ *and m satisfying* (2.8),
(a) For dist $(z,[-1,1])\geq n^{-\tau}$,

$$\left|p_{n,m}(z)\Big/\left\{\frac{1}{\sqrt{2\pi}}\phi(z)^m D^{-1}\left(F_n;\phi(z)^{-1}\right)\right\}-1\right|\leq Cn^{-\tau}. \tag{2.16}$$