Foundations for Undergraduate Research in Mathematics

Aaron Wootton
Valerie Peterson
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Editors

A Primer for Undergraduate Research

From Groups and Tiles to Frames and Vaccines

Birkhäuser

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Contents

Coxeter Groups and the Davis Complex

Timothy A. Schroeder

Suggested Prerequisites. *Group theory, Graph theory, Combinatorial topology.*

1 Introduction

This chapter presents topics in an area of mathematics at the intersection of geometry, topology, and algebra called Geometric Group Theory. It is likely that students have been exposed to geometry and abstract algebra topics as undergraduates. Some reading this may have also been introduced to topology, as well. In this chapter, we will be using terms and concepts from each of these areas, the point being to develop a working knowledge of these terms and concepts that allows us to progress toward our goal: A reflection group acting on a topological space. We will not spend much time on topological details (students are certainly encouraged to pursue that subject formally). Geometric details, referenced throughout, but specifically in Section 3.2, are suggested as a project in Section 6. That leaves algebra. Perhaps the best, and most familiar place to begin. We recall the definition of a group G:

Definition 1. A *group* G is a set G, together with a closed binary operation, denoted $\cdot$, such that the following hold:

- Associativity: For all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- Identity: There is an element e in G such that for all $x \in G$, $e \cdot x = x \cdot e = x$.
- Inverse: For every $a \in G$, there is an element $a' \in G$ such that $a \cdot a' = a' \cdot a = e$. Such an element a' is unique, and is denoted a^{-1}.

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Students in an abstract algebra course begin with this definition, and likely embark on a journey through group theory: element orders, subgroups, cosets, homomorphisms, isomorphisms, cyclic groups, generators, etc. Eventually, students may be introduced to group actions, along with their stabilizers and orbits. All the while, most likely, using the favorite dihedral groups as prototypes.

As useful and correct and edifying as that exposure is, it may strike students as somewhat sterile. The groups are axiomatically presented, they are thought of abstractly, and pictures (aside from regular polygons or the occasional Cayley graph of a finite group) may be few and far between. It is our intention that the groups we present in this context will have a decidedly constructive, and even geometric, flavor to them.

In this chapter, we will study *finitely presented* groups, specifically *Coxeter groups*, and we will present a constructive view of a topological space on which these Coxeter groups act. We begin with an overview of group presentations and graph theory, then define Coxeter groups and the associated spaces. The reader should note that many technical terms are italicized, and can be referenced in the included citations or elsewhere.

2 Group Presentations and Graphs

Let A be a set and define the set $A^{-1} = \{a^{-1} \mid a \in A\}$. Define W_A to be the collection of all finite length *words* in $A \cup A^{-1}$. A "word" being a finite string of elements from the set $A \cup A^{-1}$. (We are thinking of the set A as our "alphabet.") For example, if $A = \{a, b\}$, then a, ab, ba, aba^{-1}, $bb^{-1}ab$ are all elements of the set of words W_A. Perhaps you sense that this list is redundant. That is, you may already be thinking that two of the words are actually equivalent. You're right, of course, for as the notation suggests, elements of the set A^{-1} are to play the role of inverses. To make this clear, let's get a little more formal.

First, to the set W_A, we include the word comprised of no elements, the so-called *empty word*, denoted by the symbol 1.

Now, to have a group, we must have an associative operation, usually described and understood as some sort of multiplication. So, on the set W_A, we define multiplication by *concatenation*. (In other words, "put next to each other.") In the example above, to multiply ba and aba^{-1}, we have $ba \cdot aba^{-1} = baaba^{-1} = ba^2ba^{-1}$. You should verify that, in general, concatenation is in fact associative.

Next, we say $w_2 \in W_A$ is obtained from $w_1 \in W_A$ by an *elementary reduction* (or *expansion)* if w_2 is obtained from w_1 by deleting (or inserting) a sub-word of the form aa^{-1} or $a^{-1}a$, for some element $a \in A$. We say that two words w and w' are *equivalent* if we may pass from w to w' by a finite sequence of elementary reductions or expansions (both are allowed), and we write $w \sim w'$. (As you may have guessed, this means that in our list above, $ab \sim bb^{-1}ab$.) The relation $\sim$ defines an equivalence relation on the set W_A. (See Exercise 1.)

Finally, let F_A be the collection of equivalence classes of $\sim$, where for $w \in W_A$, $[w]$ denotes the equivalence class containing w. Concatenation on W_A induces a well-defined operation on F_A: For $u, v \in W_A$, $[u] \cdot [v] = [uv]$. (See Exercise 2.) In fact, F_A is a group with identity the equivalence class containing 1. It is called the *free group on A*.

2.1 Group Presentations

It could be the case that within a set of words, or within a group, there could be many ways, besides using elementary reductions or expansions, to represent a given element. To handle this more complicated situation, we add what are called *relators* to our group, and study *presentations* of groups.

Let A be a set and consider again F_A, the free group on A. Let R be a set of words in the alphabet $A \cup A^{-1}$, and define $N(R)$ to be the smallest normal subgroup of F_A containing the equivalence classes of the elements of R. (This normal subgroup is formed by taking the collection of all finite products of conjugates of elements of R and their inverses in F_A.) We have the following definition.

Definition 2. Let A, R and $N(R)$ be as above. The *group defined by the presentation* $\langle A \mid R \rangle$ is the quotient group $F_A/N(R)$.

That is, a group defined by a presentation is the quotient of a free group by the normal subgroup generated by the words in some set R. We equate a group with its presentation, writing $F_A/N(R) = \langle A \mid R \rangle$, but note that it is the case that a given group can have multiple presentations.

If A is a finite set (for it could be the case that A is an infinite set), we say the corresponding group is *finitely generated*. If R is also finite, we say that the group is *finitely presented*. For reference on group presentations, see [6] or [10].

In practice we often take a slightly more constructive approach to defining and working with the elements of a finitely presented group. We highlight this perspective next.

2.1.1 A Constructive Approach

Resetting the table, consider the set W_A of words in the alphabet $A \cup A^{-1}$, where A is some finite set, and let R be a finite set of words contained in W_A. The elements of R, again called "relators," will serve to identify certain words in W_A, besides those identified in the free group F_A. Indeed, we say a word w_2 in W_A is obtained from $w_1 \in W_A$ by a *simple R-reduction* (or *R-expansion*) if w_2 is obtained from w_1 by deleting (or inserting) a sub-word r, where $r \in R$. We then say two words w and w' are *R-equivalent*, and write $w \sim_R w'$, if there exist a finite sequence of simple R-reductions, simple R-expansions, elementary reductions, and elementary expansions leading from w to w'. As before, concatenation induces an operation on the set of equivalence classes of R-equivalent words of W_A, and we have the structure of a group G with presentation $\langle A \mid R \rangle$. We again equate the group with its presentation and write $G = \langle A \mid R \rangle$.

Now, for an element $w \in W_A$, we think of w as representing its entire equivalence class of R-equivalent words and drop the equivalence class notation (or the associated coset), understanding that other words may be equivalent to w. That is, we think of the words themselves as elements of the group; but in that set, there is much redundancy. In particular, we write $w = w'$ *as group elements* when $w \sim_R w'$ as words (or when w and w' are in the same coset of $N(R)$). Multiplication is still represented by concatenation, and the identity element still represented by 1. This means that the generating set A is considered a subset of the group itself. Also, R-equivalence means that we are able to insert or delete each $r \in R$ into any part of a word w without changing the group element. This amounts to equating each relator r with the identity element 1, a perspective often indicated in the presentation as each $r \in R$ will often be equated to 1 in the right hand side of the presentation. Finally, note that if $A = \{a_1, a_2, \ldots, a_n\}$ and $R = \{r_1, r_2, \ldots, r_k\}$, then we may write $\langle a_1, a_2, \ldots, a_n \mid r_1, r_2, \ldots, r_k \rangle$ to denote $\langle A \mid R \rangle$.

2.2 Some Basic Graph Theory

In order to fully explore group presentations, and the structures of the resulting groups, it does us well to review (or introduce) some basic graph theory.

Let V be a set, and let E be a collection of two element subsets of V, where an individual set can be repeated in the collection, and an individual element can be repeated to form a two-element subset. Such sets define a graph $\Gamma = (V, E)$ where the elements of V are the *vertices* of the graph and the elements of E the *edges*. A set $\{v, w\} \in E$ indicates an edge between vertices v and w, and $\{v, v\} \in E$ defines an edge in Γ from v to itself, i.e. a *loop* in Γ. If a subset in E has multiplicity n, then we include n edges between the associated vertices in Γ. Here, Γ is said to have *multi-edges*.

We define a *directed graph* by taking $E \subseteq V \times V$ where the ordered pair (v, w) indicates a directed edge between the vertices v and w. We indicate this pictorially by placing an arrow on the associated edge. Of course, this sort of structure can indicate an orientation to the edges in the graph, where traversing the associated edge in different directions has different implications. For our purposes, it is possible to have a graph with both directed edges and undirected edges. In this case, the edge set E will denote undirected edges by two element sets, and directed edges by ordered pairs.

A graph or directed graph $\Gamma = (V, E)$ is said to be *labeled*, or *weighted*, if there is a function from the set of edges E to a set of labels.

A graph Γ is said to be *finite* if both V and E are finite sets. Γ is said to be *simple* if Γ includes no loops nor multi-edges.

Two graphs $\Gamma_1 - (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are *isomorphic* if there exists a bijection $f : V_1 \to V_2$ such that if $\{u, v\} \in E_1$, then $\{f(u), f(v)\} \in E_2$. A similar definition exists for directed edges, that is if $(u, v) \in E_1$, then $(f(u), f(v)) \in E_2$. If the bijection f maps V_1 to itself, and $E_1 = E_2$, then the map f defines an automorphism of the graph $\Gamma_1(V_1, E_1)$.

Example 1. Let $V = \{a, b, c, d, e, f\}$, and consider edge sets

$$E_1 = \{\{a,b\},\{b,c\},\{c,d\},\{d,e\},\{e,f\},\{f,g\}\}\text{ , and}$$

$$E_2 = \{(a,b),(b,c),(c,a),(d,e),(e,f),(f,d),\{a,d\},\{c,e\},\{b,f\}\}\ .$$

The corresponding graphs $\Gamma_1 = (V, E_1)$ and $\Gamma_2 = (V, E_2)$ are shown Figure 1.

Example 2. Let $V = \mathbb{Z}$, and $E = \{\{n, n+1\} \mid n \in \mathbb{Z}\}$. Then $\Gamma = (V, E)$ can be understood as the real line, with vertices at every integer.

2.3 Cayley Graphs for Finitely Presented Groups

We now present the contruction of the *Cayley graph* associated to a finitely presented group. The Cayley graph is a very useful graph, endowed with some of the additional structure discussed above. Let $G = \langle A \mid R \rangle$, we create the corresponding labeled Cayley graph Γ as follows:

- $V = G$; that is, we include one vertex for every element of G.
- E: Given any $v \in G$ and $a \in A$, then there is a directed edge (v, va) from v to the element $va \in G$. We label this edge by the generator a.

This means that in Γ, each vertex has one edge emanating from it for each element of A, and it will also have one edge entering for each element of A. It should be noted that we view the vertices of the graph as elements of the group, that is, we view $G \subseteq \Gamma$. So, from a given vertex v, traversing an edge labeled a with the orientation corresponds to multiplying v on the right by a, and traversing an edge labeled a against the orientation corresponds to multiplying v on the right by a^{-1}.

Example 3. Let $G = \langle a \mid a^6 \rangle$. Then the Cayley graph of G can be viewed as the graph Γ_1 in Figure 1, but replace the edges there with directed edges all oriented to flow counter-clockwise around the hexagon, and with $a = a$, $b = a^2$, $c = a^3$, ... etc.

Fig. 1 Γ_1 and Γ_2.

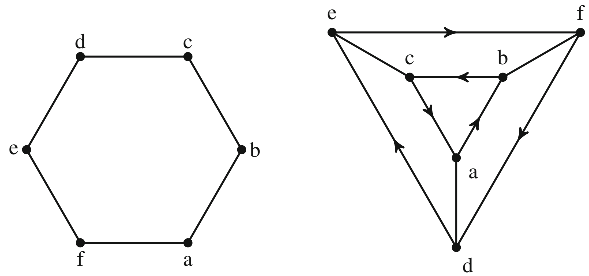
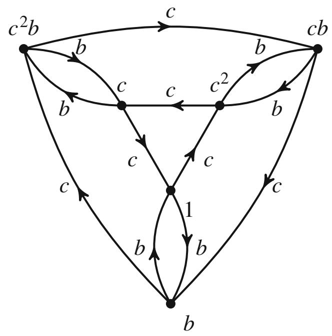


Fig. 2 The directed Cayley graph for G in Example 4.



Example 4. Let $G = \langle b, c \mid b^2, c^3, bcb^{-1}c \rangle$. G has Cayley graph shown in Figure 2.

In Example 4, due to the generator b having order 2 (we have $b^2 = 1$ in G), the description of the Cayley graph given above prescribes multi-edges. For at any $g \in G$, since $gb^2 = g$, there is an edge labeled b emanating from g to gb, and an edge entering g from gb. To avoid these multi-edges, and to reflect the fact that $b^{-1} = b$, it is our convention that when a generator b has order 2 we will identify incoming and outgoing edges corresponding to b; and instead of two directed edges, we will include one undirected edge. Thus, the Cayley graph for the group in Example 4 will actually be viewed as the graph Γ_2 in Figure 1, with directed and undirected edges.

Finally, we remark that the vertices of a Cayley graph correspond to elements of the group, hence *equivalence classes* of words. As a result, it can be very difficult to tell when two words represent the same element. So, constructing the Cayley graph is not as straightforward as one might expect. However, the description above enables us to understand a local picture of the Cayley graph at any vertex.

Exercise 1. Show that the relations $\sim$ and $\sim_R$ on the set W_A described above generate *equivalence relations*.

Exercise 2. Show that concatenation is well-defined on equivalence classes. Then, show that F_A is a group, with identity element 1. (For a given $u \in W_A$, what is $[u]^{-1}$?) In general, is F_A abelian?

Exercise 3. Let $A = \{a, b\}$, $R = \{aba^{-1}b^{-1}\}$. Show that $ab \sim_R ba$.

Exercise 4. Construct the Cayley diagram for the following presented groups.

(a) $\langle a \mid \quad \rangle$
(b) $\langle a, b \mid \quad \rangle$
(c) $\langle a, b \mid aba^{-1}b^{-1} \rangle$
(d) $\langle a, b \mid aba^{-1}b \rangle$
(e) $\langle a, b \mid b^2 \rangle$

(f) $\langle a, b \mid a^2b^{-2} \rangle$
(g) $\langle a, b \mid a^4, b^2 \rangle$
(h) $\langle a, b \mid a^2b^{-3} \rangle$
(i) $\langle a, b, c, d \mid aba^{-1}b^{-1}, cdc^{-1}d^{-1} \rangle$

Challenge Problem 1. A very strong connection between group theory and topology is now at hand. Indeed, given a group $G = \langle A \mid R \rangle$ with Cayley graph Γ (with single edges for generators of order 2), Γ can be given the "path-metric" by making each edge isometric to the unit interval. Define then the distance d between two points to be the minimum path length (or infimum) over all paths connecting the two points. Show that this defines a metric on Γ. Notice that for vertices which correspond to $u, v \in G$, $d(u, v) \in \mathbb{Z}$. Show that in this case, $d(u, v)$ is the minimum length of words w in the alphabet $A \cup A^{-1}$ representing the element $u^{-1}v$. The restriction of the metric to $G \subseteq \Gamma$ is the so called "word-length metric."

Challenge Problem 2. A free group has a more 'universal' definition: Let G be a group and let $A \subseteq G$. We say that G is *free on* A, denoted G_A, if for every group H, and any function $f : A \to H$, then there exists a unique homomorphism $h : G \to H$ s.t. $h|_A = f$.

For a given set A, show that

(a) If G is free on A, then A generates G. (That is, every element of G is a product of elements of A and their inverses.)
(b) If G is free on A, then A contains no elements of finite order.
(c) The group F_A (defined above) is free on A. (That is, for any group H and function $f : A \to H$, we must verify that f extends uniquely to a homomorphism.)
(d) Let G_A be a group that is free on A, then $F_A \cong G_A$. (That is, the definitions are equivalent!)

3 Coxeter Groups

In this section, we'll explore a special type of finitely presented group called a *Coxeter* group. Put succinctly, Coxeter groups are groups that are generated by elements of order 2, often viewed as reflections in some geometric space. As you may recall from an undergraduate geometry course, or can find in a standard geometry text such as [15, Chapter 10], isometries of geometric spaces can be understood as compositions of reflections. Therefore, one can see the importance of groups generated by reflections, and their natural connection to geometry. It should also be noted that the study of such *finite* groups, generated by reflections acting on $\mathbb{R}^2$, are essential in classifying Lie groups and Lie algebras, and the classification of regular polytopes. Coxeter groups are generalizations of this idea, where the order 2 generators are not necessarily viewed as reflections on $\mathbb{R}^n$, but will almost certainly be viewed as some sort of homeomorphism of a topological space.

As we'll see, the review of some basic graph theory in Section 2.2 will be useful, since the presentations of Coxeter groups can be encoded as finite, simple, labeled graphs; and because the topological spaces on which we will have these groups act are intimately related to their Cayley graphs.

3.1 The Presentation of a Coxeter Group

Let $\Gamma = (S, E)$ be a finite simple graph with vertex set S and with edges labeled by integers ≥ 2. Denote by m_{st} the label on the edge $\{s,t\}$. Γ encodes the data for a presentation of a Coxeter group W_Γ

$$W_\Gamma = \left\langle S \mid s^2 = 1 \text{ for each } s \in S \text{ and } (st)^{m_{st}} = 1, \text{ for each edge } \{s,t\} \text{ of } \Gamma \right\rangle . \tag{1}$$

The pair (W_Γ, S) (or simply (W, S) when the graph Γ is clear) is called a *Coxeter system*. We call such a labeled graph Γ a *Coxeter graph*. Throughout this chapter, we will take such a graph as the defining data for our Coxeter groups, noting that the vertices of the graph correspond to the generators of the group. This is standard convention: To simultaneously view each $s \in S$ as a generator of the group, an element of the group, and a vertex of the defining Coxeter graph. See [4, 9] for further reference on Coxeter groups and Coxeter systems. See [14] for further treatment on the defining Coxeter graphs.

Observe that the Coxeter graph Γ is not the Cayley graph associated to the group. Rather, it is an efficient way to encode the presentation of the group. There are other ways of defining Coxeter groups; for example Coxeter matrices or Dynkin diagrams. But, any such effort is just encoding the above type of presentation in another way. Our focus will be on the so-called Coxeter graphs.

In summary, a Coxeter group is generated by a set of elements that have order 2, and the only other relators are of the form $(st)^{m_{st}}$, where $s \neq t$, where m_{st} is the order of the element (st). Also, since the generators each have order two, they are their own inverses and we refer to them as *reflections* or *involutions*. That is to say, a Coxeter group is a group that is generated by reflections.

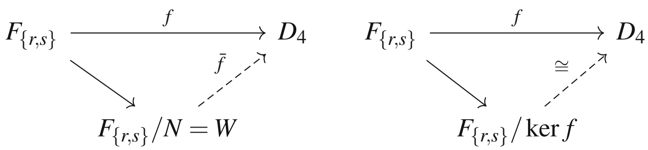
As noted in Section 2.1, the relators of the type r^2 and $(st)^{m_{st}}$ amount to equating these words to the identity element of the corresponding Coxeter group. See Exercises 5, 6, 7, and 8 to explore the implications of these relators, and the ensuing structure of the Coxeter group. Further, the reader should note that if two vertices s and t are not connected by an edge in Γ, then they do not define a relator and themselves generate an infinite subgroup, as Example 5 illustrates.

Example 5. Let Γ be a graph with two vertices, and no edges. Then

$$W = \left\langle r, s \mid r^2, s^2 \right\rangle ,$$

and its elements can be algorithmically listed: $1, r, s, rs, sr, rsr, srs, rsrs, srsr, \ldots$. Is it clear why the generators alternate within each word in this list? This group is called the infinite dihedral group, and denoted D_∞. Is it clear why the group is infinite?

Example 6. Let D_4 denote the dihedral group of order 8, that is, D_4 is the group of isometries of a square, and let $W = \langle r, s \mid r^2, s^2, (rs)^4 \rangle$. We will show $W \cong D_4$. To do this, recall the definition of a group defined by a presentation in Definition 2, and let $F_{\{r,s\}}$ denote the free group on generators r and s, and N the normal subgroup generated by the relators $\{r^2, s^2, (rs)^2\}$. That is, N is the smallest normal subgroup of $F_{\{r,s\}}$ containing the relators. Define a map $f : F_{\{r,s\}} \to D_4$ by mapping r to a reflection across a diagonal of a square, s to a reflection across an adjacent side bisector, and extending f to the rest of $F_{\{r,s\}}$ by requiring that f be a homomorphism. That is, a word in the alphabet $\{r, s, r^{-1}, s^{-1}\}$ is mapped to the corresponding composition of the reflections on the square described above. In particular, the student can verify that f is surjective and that all the relators are all in $\ker f$. From these observations, we can conclude two things: (1) Since $\ker f$ is a normal subgroup of $F_{\{r,s\}}$ containing the relators, we must have that $N \le \ker f$; and (2) By the universal property of quotient groups, [10, 5.6], f descends to a map $\bar{f} : F_{\{r,s\}}/N = W \to D_4$. We have two commuting diagrams:



The diagram on the right being the classical 'first' or 'fundamental' isomorphism theorem. It gives us that $\left[F_{\{r,s\}} : \ker f\right] = 8$. Since $N \le \ker f$, and we have that $\left[F_{\{r,s\}} : N\right] = \left[F_{\{r,s\}} : \ker f\right] \cdot [\ker f : N]$, we know that

$$|W| = \left[F_{\{r,s\}} : N\right] \le 8.$$

But, by simply observing words in the alphabet $\{r, s, r^{-1}, s^{-1}\}$ subject to the relators, it is clear that any element of the group has as a representative a word of length at most 4. Indeed, since $(rs)^4 = 1$, we know $rsrsrsrs = 1$, $rsrsrsr = s$, $rsrsrs = sr$, $rsrsr = srs$, $rsrs = srsr$, and so on. This means that $\{1, r, s, rs, sr, rsr, srs, rsrs = srsr\}$ exhausts the set of words that represent the distinct elements of W, and so $|W| \le 8$. Thus, $|W| = 8$, which means that $[\ker f : N] = 1$, and there is actually only one diagram. In particular, we have that $\bar{f} : W \to D_4$ is an isomorphism.

Example 7. One can show, using an argument similar to that in Example 6, that the Coxeter group $W = \langle r, s \mid r^2, s^2, (rs)^n \rangle$ is isomorphic to D_n, the dihedral group of order $2n$.

Example 8. Let $W = \langle r, s, t \mid r^2, s^2, t^2, (rs)^2, (st)^2, (rt)^2 \rangle$. It is a Coxeter group and in a similar manner to that above, one can show that W is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (See Figure 3).

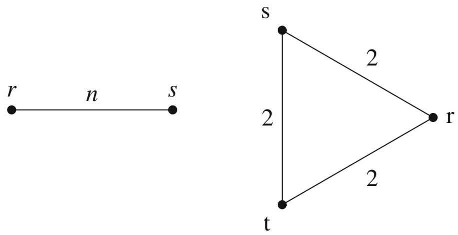


Fig. 3 Coxeter graphs for Examples 7 and 8 respectively.

Example 9. Consider the group

$$W = \left\langle r, s, t, u, v \mid r^2, s^2, t^2, u^2, v^2, (rs)^2, (st)^2, (tu)^2, (uv)^2, (rv)^2 \right\rangle .$$

Verify that W has corresponding Coxeter graph Γ where Γ is a pentagon with each edge labeled 2. Note that W is infinite, as it contains copies of D_∞ as subgroups.

3.2 Coxeter Groups and Geometry

To set up a discussion of the relationship between geometry and Coxeter groups, we recall the following facts, often covered in an undergraduate transformational geometry course.

Fact 1 Let P be a (2-dimensional) geometric space meeting the axioms of so-called "Neutral geometry." Given a line $l \subset P$, the reflection over the line l, denoted r_l is an isometry of the space. That is, the distance between two points p and q is the same as the distance between their reflected images p' and q'.

Fact 2 With P as above, if $\gamma : P \to P$ is an isometry, then γ can be understood as a composition of 1, 2, or 3 reflections over a line.

Fact 3 The composition of reflections over lines whose (acute) angle between them is α is a rotation through an angle of 2α.

(The student is encouraged to recall the construction of a reflection in neutral, plane geometry and to verify that the composition of a reflection with itself is the identity isometry on the geometric plane. In other words, a reflection defined in this way has order 2. For reference, see [15, Chapter 10].)

The first two facts provide great motivation for the study of Coxeter groups and their inherent connection to geometry. Indeed, they give us that the group of isometries of a geometric plane is generated by elements of order 2. The third fact gives context for the other relators present in the presentation of a Coxeter group.

In later chapters, and in the suggested project 2, we give a Coxeter group as our given data, and from it try to determine an appropriate geometric model. Here and in Challenge problem 5, we turn this around. Namely, we present some geometric models and reflections, and ask the reader to determine the associated group.

3.2.1 Euclidean Space and Reflections

View $\mathbb{R}^2$ as the set of 2-dimensional vectors over $\mathbb{R}$ equipped with the usual dot product

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2$$

where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are vectors in $\mathbb{R}^2$. A line l in $\mathbb{R}^2$ through $\mathbf{x}_o$ in the direction of $\mathbf{v}$ has parametric equation $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}t$, where $t \in \mathbb{R}$. Any such line defines a reflection r_l with formula

$$r_l(\mathbf{x}) = \mathbf{x} - 2 \langle \mathbf{u}, \mathbf{x} - \mathbf{x_0} \rangle \mathbf{u}, \tag{2}$$

for any $\mathbf{x} \in \mathbb{R}^2$ and where $\mathbf{u}$ is a unit vector orthogonal to $\mathbf{v}$.

3.2.2 Spherical Geometry and Reflections

Denote by $\mathbb{S}^2$ the subset of $\mathbb{R}^3$ of points (x_1, x_2, x_3) for which $x_1^2 + x_2^2 + x_3^2 = 1$ and call it the "2-sphere." With the usual dot product defined above extended to $\mathbb{R}^3$, we see that $\mathbb{S}^2$ can be viewed as the set of vectors $\mathbf{x}$ for which $\langle \mathbf{x}, \mathbf{x} \rangle = 1$.

A line l in $\mathbb{S}^2$ is defined as the intersection of $\mathbb{S}^2$ with a plane through the origin in $\mathbb{R}^3$. Any such line defines a reflection r_l of $\mathbb{S}^2$ in a similar way to that above. Indeed, for any $\mathbf{x} \in \mathbb{S}^2$, we have

$$r_l(\mathbf{x}) = \mathbf{x} - 2 \langle \mathbf{u}, \mathbf{x} \rangle \mathbf{u}, \tag{3}$$

where $\mathbf{u} \in \mathbb{R}^3$ is unit vector orthogonal to the plane defining the line l.

3.2.3 Hyperbolic Geometry and Reflections

Define a modified inner product on $\mathbb{R}^3$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle_M = x_1 y_1 + x_2 y_2 - x_3 y_3.$$

(The "M" stands for Minkowski, and the student should note that this formula does not technically define an "inner product," as it's not the case that $\langle \mathbf{x}, \mathbf{x} \rangle_M \geq 0$ for all vectors $\mathbf{x}$.) Let $\mathbb{H}^2$ denote the subset of $\mathbb{R}^3$ for which $\langle \mathbf{x}, \mathbf{x} \rangle_M = -1$ and $x_3 > 0$. This is just the upper sheet of the two-sheeted hyperboloid in $\mathbb{R}^3$ defined by equation $x^2 + y^2 - z^2 = -1$, and called the *hyperboloid model* for hyperbolic space.

A line l in $\mathbb{H}^2$ is defined as the intersection of $\mathbb{H}^2$ with a plane through the origin in $\mathbb{R}^3$, and, as above, any such line defines a reflection in $\mathbb{H}^2$. In particular, let $\mathbf{u} \in \mathbb{R}^3$ denote a unit vector orthogonal to plane defining l, with respect to the modified inner product $\langle \quad , \quad \rangle_M$. (This means that $\langle \mathbf{u}, \mathbf{v} \rangle_M = 0$ for any vector $\mathbf{v}$ in the plane defining l.) For any $\mathbf{x} \in \mathbb{H}^2$, the reflection r_l is defined by

$$r_l(\mathbf{x}) = \mathbf{x} - 2 \langle \mathbf{u}, \mathbf{x} \rangle_M \mathbf{u}. \tag{4}$$

3.2.4 The Poincaré Disk Model for Hyperbolic Space

There is another model we'll consider for hyperbolic space called the "Poincaré disk model," denoted by $\mathbb{H}^2_P$. In terms of its points, $\mathbb{H}^2_P$ consists of $(x,y) \in \mathbb{R}^2$ for which $x^2+y^2 < 1$, the interior of the unit disk in $\mathbb{R}^2$. However, distance in $\mathbb{H}^2_P$ is calculated in such a way that the distance from the origin to the boundary of the disk is infinite. As a result, lines come in two forms:

1. The portion of lines through the origin in $\mathbb{R}^2$ contained in the interior of the unit disk, or
2. The portion of circles in $\mathbb{R}^2$ orthogonal to the boundary circle $x^2+y^2 = 1$ contained in the interior of the unit disk.

Since lines come in two forms, the corresponding reflections come in two forms.

1. If l is a line of type (1) above, then r_l is a restriction to the interior of the unit disk of the standard Euclidean reflection defined in Equation 2 with $\mathbf{x}_0 = (0,0)$.
2. If l is a line of type (2), then r_l is the inversion in the circle orthogonal to the boundary circle, applied to the points in the interior of the unit disk.

The student may recall that the inversion $\mathbf{x}'$ of a point $\mathbf{x}$ in a circle of radius k with center $\mathbf{x}_0$ is given by

$$\mathbf{x}' = \mathbf{x}_0 + \frac{(\mathbf{x}-\mathbf{x}_0)}{\langle \mathbf{x}-\mathbf{x}_0, \mathbf{x}-\mathbf{x}_0 \rangle}. \tag{5}$$

From the above formulas, one can easily see the how the hyperboloid model is completely analogous to the Euclidean and spherical model. An advantage of the Poincaré model over the hyperboloid model is that it is *conformal*. That is, angles between lines in $\mathbb{H}^2_P$ are equal to the angles between the corresponding lines or circles in $\mathbb{R}^2$.

Exercise 5. Let $r \in A$ and suppose $rr = r^2 \in R$. Show that $r \sim_R r^{-1}$.

Exercise 6. Let $r, s \in A$ and suppose $r^2, s^2, (rs)^2 \in R$. Show that $rs \sim_R sr$.

Exercise 7. Let $r, s \in A$ and suppose $(rs)^3 = rsrsrs, r^2, s^2 \in R$. Show that $rsr \sim_R srs$.

Exercise 8. Let $r, s \in A$ and suppose $r^2, s^2, (rs)^n \in R$, for some $n \in \mathbb{Z}$, $n \geq 2$. Then $(sr)^n = 1$. (That is, if you have the relator $(rs)^n$. You also have the relation $(sr)^n$.)

Exercise 9. Write the presentation of a Coxeter group for each of the three Coxeter graphs shown in Figure 4.

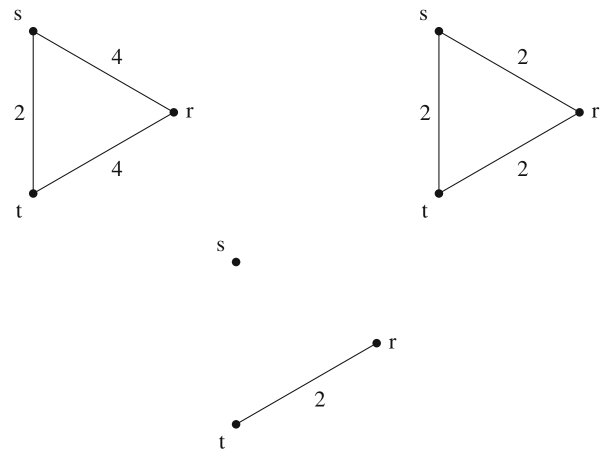


Fig. 4 The Coxeter graphs for Exercise 9.

Exercise 10. Sketch the Coxeter graph that defines the following presentations.

(a) $\langle r, s \mid r^2, s^2 \rangle$
(b) $\langle r, s, t \mid r^2, s^2, t^2, (rs)^2, (st)^3, (rt)^6 \rangle$
(c) $\langle r, s, t \mid r^2, s^2, t^2, (rs)^2, (st)^4, (rt)^4 \rangle$
(d) $\langle r, s, t \mid r^2, s^2, t^2, (rs)^4, (st)^4, (rt)^4 \rangle$
(e) $\langle r, s, t, u, v \mid r^2, s^2, t^2, u^2, v^2, (rs)^2, (st)^2, (tu)^4, (uv)^2, (rv)^4 \rangle$

Exercise 11. Cayley graphs for Coxeter groups: Recall the construction of Cayley graphs described in Section 2.2. Note that the generators of a Coxeter group always have order 2, so at each vertex of the Cayley graph of a Coxeter group, the incoming and outgoing edges are identified to reflect the idea that for a given generator r, $r = r^{-1}$. Construct the Cayley graph for the groups in Examples 7,8, and 9. Do the same for the groups presented in Exercises 9 and 10.

Exercise 12. By using an argument similar to that in Example 6, identify each of the presented groups with a familiar group. Namely the student should attempt to find a map from an appropriate free group to a familiar group, then show that this map descends to an isomorphism of the given presented group to the familiar group.

(a) $\langle r, s \mid r^2, s^2, (rs)^3 \rangle$
(b) $\langle r, s \mid r^2, s^2, (rs)^n \rangle$ (See Example 7)
(c) $\langle r, s, t \mid r^2, s^2, t^2, (rs)^2, (st)^2, (rt)^2 \rangle$ (See Example 8)
(e) $\langle r, s, t \mid r^2, s^2, t^2, (rs)^3, (st)^3, (rt)^2 \rangle$ (Hint: This finite Coxeter group is quite famous. Make an educated guess at its order and try to think of a group with the same order.)

Challenge Problem 3. Given a Coxeter graph $\Gamma = (V, E)$, show that an automorphism of Γ which preserves edge labels (that is, if m is the label on $\{r, s\}$, then m is the label on the edge defined by the images of r and s) induces an automorphism of the corresponding Coxeter group.

Challenge Problem 4. Consider the formulas for reflections in Euclidean, spherical, and hyperbolic space given in equations 2, 3, 4, and 5. Let r_l generically denote one of these reflections. Show that

(a) r_l is appropriately defined for the non-Euclidean models. (Specifically, show that it sends points on the sphere to points on the sphere, points on the hyperboloid to points on the hyperboloid, and points in the interior of the unit disk, to points in the interior of the unit disk.)
(b) For all the models, verify that $r_l \circ r_l =$ the identity map on the space.

Challenge Problem 5. Consider again the 2-dimensional geometric models discussed above.

(a) Find a presentation for the group generated by a reflection over the x-axis, and the line $y = \sqrt{3}x$ in Euclidean space.
(b) Find a presentation for the group generated by reflections over the lines formed by the planes $z = 0$, $y = 0$ and $y = x$ in $\mathbb{S}^2$.
(c) Find a presentation for the group generated by reflections over the lines formed by the planes $y = 0$ and $y = x$ in $\mathbb{H}^2$.
(d) Find a presentation for the group generated by reflections over the x-axis, the line $y = x$, and the hyperbolic line connecting the points

$$\left(\frac{\sqrt{\cos^3\frac{\pi}{4}}}{\cos\frac{\pi}{8}}, 0\right) \quad \text{and} \quad \left(\frac{\sqrt{\cos^5\frac{\pi}{4}}}{\cos\frac{\pi}{8}}, \frac{\sqrt{\cos^5\frac{\pi}{4}}}{\cos\frac{\pi}{8}}\right)$$

in $\mathbb{H}^2_P$.

For each of these, besides considering the angle between the given reflections as a clue to appropriate group element orders, the student is also encouraged to use a computer algebra system to calculate the order of the composition of reflections directly.

4 Group Actions on Complexes

In Geometric Group Theory, the idea is to study the interplay between a finitely presented group G, and a corresponding topological, or geometric, space X. In particular, we view the elements of a group G as *homeomorphisms* of the space X.

That is, each element of the group corresponds to a continuous, bijective function $X \to X$. For example, if $G = D_3$, the dihedral group of order 6, then each element of the group can be viewed as a function that maps a regular 3-gon X to itself. Perhaps, the element reflects the triangle over an altitude, or rotates the triangle by $120°$. In any case, each element of D_3 is viewed as a function of the triangle to itself.

The language we use to encompass this sort of relationship between a group G and a topological space X is to say that the group G *acts* on X. Group actions are one of the richest areas of mathematics, for if the action is appropriate, many of the properties of the group (orders of elements, subgroups, cosets, etc.) manifest themselves in the topological space; and properties of the topological space (e.g. any geometric structure) manifest themselves in the group. In this chapter, the approach we take is to focus on finitely presented groups, mostly Coxeter groups, acting on a special type of topological space called a *CW-complex*. For an overview of Geometric Group Theory, see [2]. For more references on topological spaces, see [12], and for references on CW-complexes, see [8] or [7].

4.1 CW-Complexes

CW-complexes are topological spaces whose construction can be done in a step-by-step manner, using very simple topological spaces as building blocks. The blocks are referred to as n-balls, and the steps in the process correspond to dimension. The idea is to build a space out of points, then edges, then disks/squares/triangles/5-gons/etc., then tetrahedron/cubes/prisms/etc.,... and so on. Let us get a bit more specific.

Let D^n denote a *topological n-ball*, with *boundary* $\partial D^n = S^{n-1}$. In particular,

- D^0 is just a point, S^{-1} is the empty set.
- D^1 is just a line segment, S^0 is the two endpoints of the edge.
- D^2 is a disk, S^1 is the circle bounding the disk.
- D^3 is a (filled) ball, S^2 is the surface of the ball.

$\vdots$

In general, D^n can be viewed as the set $\{(x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 \leq 1\}$, with boundary S^{n-1} viewed as the set of points $\{(x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = 1\}$. It is important to note, however, that these balls and their boundaries are thought to be completely independent of any sort of Cartesian space. They are considered to be their own spaces. This isn't difficult to picture in low dimensions, for it is easy to think of collections of points, segments, disks, and balls as independent of any sort of coordinate axes. Though harder to picture (or even impossible?), the same applies in higher dimensions. But for the sake of this chapter, picturing things in dimensions ≤ 3 will suffice.

We should also note that the n-balls used as building blocks do not have to be round. For example, D^2 could be a traditional disk, or it could be a square, or a pentagon, or a very strange non-convex shape. See Figure 5. These are all

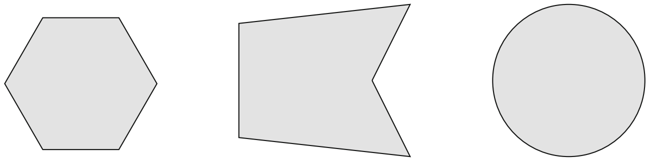


Fig. 5 Homeomorphic examples of a 2-ball.

topological 2-balls because any one of them can be stretched, bent, squashed, etc..., but not torn; in a such a way as to match any of the others. This is the idea of a homeomorphism. The study of homeomorphisms in topology is as central as the study of isomorphisms in group theory, or even as central as the study of differentiable functions in Calculus I. For a nice introductory reference on the centrality of homeomorphisms to the study of topology, see [11].

The idea of a CW-complex, then, is to take n-balls of various dimensions, and use them as the building blocks to construct a topological space. It's almost as if we are playing LEGOs®, and the n-balls are our pieces. Within the context of the larger space, we refer to the individual n-balls as *n-cells*. Now, just as a LEGO® construction can be given in steps, the construction of a CW-complex X, can be described in steps, corresponding to dimension:

0. Start with a (discrete) set of 0-cells. Denote this set X^0, called the 0-skeleton.
1. To X^0, attach a set of 1-cells along their boundaries; forming X^1 called the 1-skeleton. (At this stage, your complex looks like a graph.)
2. To X^1, attach a set of 2-cells along their boundaries; forming X^2 called the 2-skeleton.
3. To X^2, attach a set of 3-cells along their boundaries; forming X^3 called the 3-skeleton.

$\vdots$

The "attaching" described above should be described a bit more formally. Inductively, for $n \geq 1$, we form the n-skeleton X^n from the $(n-1)$-skeleton X^{n-1} by attaching n-balls via functions $f : \partial D^n = S^{n-1} \to X^{n-1}$ (called attaching maps), one function for each of the attached n-balls mapping its boundary to the lower dimensional skeleton. If $\{D^n\}$ is the collection of n-balls attached at step n, then the n-skeleton X^n is understood as the disjoint union $X^{n-1} \cup \{D^n\}$ under some identifications. In particular, for each attached n-ball D^n, each point x in the boundary is identified with its image under f. In other words, new cells are "glued" along their boundaries to the existing space. More formally, we have

$$X^n := \left(X^{n-1} \cup \{D^n\}\right) / \sim,$$

where $x \sim f(x)$ for each attached n-ball D^n, each x in ∂D^n, and corresponding attaching map f. Precisely, X^n is defined as a *quotient space*. Set $X = \cup_n X^n$, the union of all n-skeleta. It is a CW-complex. The "CW" stands for "Closure-Weak" in reference to the topology of such a space. We will not get specific with the topology of these spaces, only noting that each cell carries its own topology homeomorphic to the unit ball in $\mathbb{R}^n$.

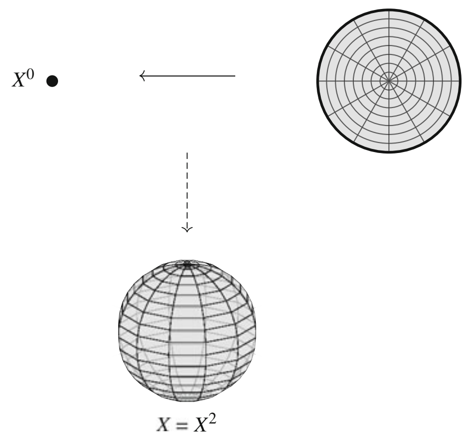
If the process described above stops at some dimension n (that is, if there are no m-balls attached for $m > n$), than we say X is finite dimensional. In particular, if the process stops after X^1, that is $X = X^1$ and there are no m-balls for $m > 1$, then X is a graph with vertices the 0-cells, and edges the 1-cells of X. (It may help to think of CW-complexes as graphs generalized to higher dimensions.)

If X contains finitely many cells, we say X is finite. Note, however, that in any given step, there may be infinitely many cells, or there could be no cells to attach. While we will look at some constructions in Examples 10 and 11, we often are given the total space X, and understand the above process to have taken place, and refer to the resulting *cellulation* of X.

Example 10. We can view a 2-sphere S^2 as a CW-complex in many ways. One way is with one 0-cell, no 1-cells, and one 2-cell attached to the 0-cell by identifying all of its boundary to the 0-cell – like pulling a draw string on a bag to tighten the opening. The attaching map is indicated with a solid arrow in Figure 6. Here we see that since there are no 1-cells, $X^0 = X^1$. In general, it can be the case in a CW-complex that the i-skeleton can equal the $i + 1$-skeleton.

Example 11. Figure 7 depicts a cellulation of a torus by one 0-cell, two 1-cells, and one 2-cell. The attaching maps are indicated with solid arrows in the figure, resulting skeleta indicated with dashed arrows. Two 1-cells are attached to the indicated 0-cell. One 2-cell, viewed as a rectangle, is attached to the 1-skeleton where the corners are all identified with the 0-cell, and the edges are identified to the 1-cells with orientations as shown. (Though it is not shaded, the rectangle shown represents a 2-cell.)

Fig. 6 A cellulation of a 2-sphere.



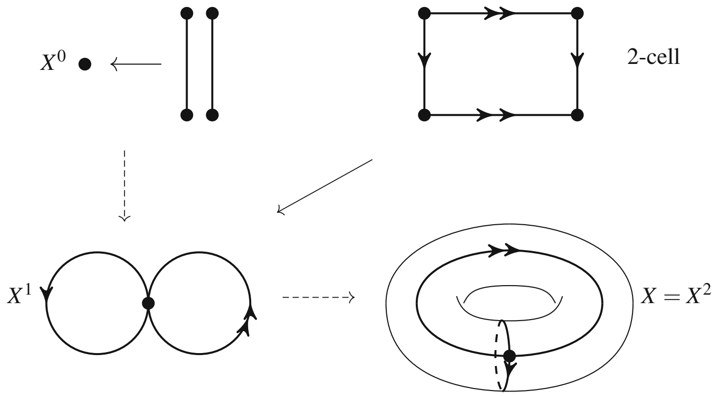


Fig. 7 The cellular decomposition of the torus.

The Euler Characteristic To a finite CW-complex X we can attach a number called the *Euler characteristic* of X, denoted $\chi(X)$, where

$$\chi(X) = \sum_{\text{cells } \sigma} (-1)^{\dim \sigma}. \tag{6}$$

That is, $\chi(X)$ is the alternating sum of the number of cells in each dimension. It is an interesting result of Algebraic Topology that the Euler characteristic of a space X does not depend on the specific cellulation one uses, rather only on the *homotopy class* of the space. In particular, homeomorphic spaces (regardless of cellulation) will have the same Euler characteristic. See [8] for reference. For example, using the cellulation of the sphere in Example 10, we get $\chi(S^2) = 2$. But it is the case that no matter the cellulation of S^2, we still get $\chi(S^2) = 2$. In fact, it is a fundamental result of algebraic topology that the Euler characteristic classifies surfaces, both orientable and non-orientable. The interested student should refer to Project Idea 3 that further investigates the Euler characteristic.

4.2 Group Actions on CW-Complexes

As discussed in the introductory comments of Section 4, an important aspect of group theory (and in fact all of mathematics) is the study of *group actions*. In an undergraduate abstract algebra course, you may have studied group actions on generic sets, or in the context of the Sylow Theorems. The automorphism groups of graphs are commonly studied in this environment, as group elements can be viewed as bijections of the vertex set or edge set of an associated graph. Some students may have even studied how a group can "act" on its own Cayley graph. But as we noted above, graphs are just 1-dimensional CW-complexes, and hence topological spaces.

So a group "acting" on a graph is really an example of the more general sort of group action we study here. That is, a group acting on a topological space.

Definition 3. An *action* of a group G on a topological space X is a homomorphism $\phi : G \to \mathrm{Homeo}(X)$. Where $\mathrm{Homeo}(X)$ is the set of homeomorphisms $X \to X$, under composition.

Once again, this is very similar to the definition of group action you may see in an abstract algebra text like [6], however, instead of each element of the group corresponding to simply a bijection of a set X, we carry a topological requirement that each element $g \in G$ corresponds to a homeomorphism $\phi_g : X \to X$. For $g \in G$, we write $g \cdot x$ for $\phi_g(x)$. That is, we think of g itself as sending $x \in X$ to some other point of the space. Similarly, for a subset $Y \subseteq X$, $g \cdot Y$ stands for "the points to which g sends all of the points of the set Y."

Since the topological spaces we consider in this chapter are CW-complexes, we require our actions be *cellular*. That is, we consider the action on the level of cells and require that, for $g \in G$, and for any n-cell σ in X, $g \cdot \sigma$ is another n-cell of X. So a cellular action is one that sends n-cells to n-cells, and all adjacency relationships are maintained. In other words, if two cells are adjacent before being acted upon, then they are adjacent after being acted upon. In this context, the familiar definitions of *orbit* and *stabilizer* take on this cellular theme.

Definition 4. For an n-cell σ, the *stabilizer* of σ is

$$\mathrm{Stab}_G(\sigma) = \{g \in G \mid g \cdot \sigma = \sigma\}$$

and the *G-orbit* of σ is the set of n-cells γ for which $\tau = g \cdot \sigma$ for some $g \in G$.

In the examples we consider, the actions are also be *proper* and *co-compact.*

Definition 5. An action of a group G on a complex X is *proper* if $|\mathrm{Stab}_G(\sigma)| < \infty$ for each cell σ of X.

The next term we would like to define, "co-compact", requires some set-up. For a cellular action of a group G on a complex X, we define the *quotient space* X/G to be a CW-complex where each cell σ of X is identified with its orbit. There are some topological issues that we are bypassing, but for reference, the space X/G is the endowed with the *quotient topology*. (See [12] or [8].) Put simply, X/G is a CW-complex comprised of an n-cell representing each G-orbit of n-cells (for each n), and containment relationships from the parent space X are maintained: If two n-cells are in the same orbit, then they will identify as a single cell in X/G. If an n-cell σ contains an m-cell τ in X, then they will have corresponding n- and m-cells in the same containment relationship in X/G. We are now ready to define a co-compact action.

Definition 6. An action of a group G on a complex X is *co-compact* if the quotient space X/G is a finite complex.

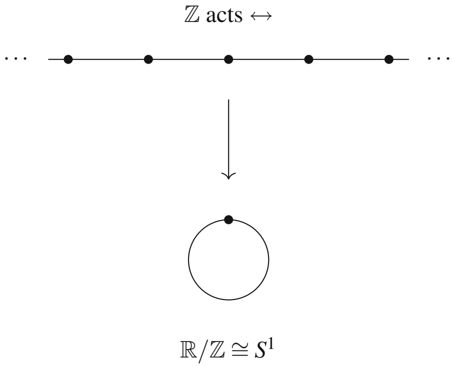


Fig. 8 $\mathbb{Z}$ acts via translations, $\mathbb{R}/\mathbb{Z}$ is shown.

Before exploring some examples, we have one last, closely related, term to define for the action of a group G on a complex X.

Definition 7. Let G act on the CW-complex X. A closed subset C of X is a *fundamental domain* for the action if $G \cdot C$, the union of all orbits of cells in C, contains X. A fundamental domain C of X is a *strict* fundamental domain if the G-orbit of each cell intersects C in exactly one cell.

Example 12. The real line $\mathbb{R}$ can be thought of as a CW-complex, where each integer point on the line corresponds to a 0-cell, and each resulting interval corresponds to a 1-cell attached at two endpoints. The group $(\mathbb{Z}, +)$ acts on $\mathbb{R}$ by translation. For example, if we take $5 \in \mathbb{Z}$, and $x \in \mathbb{R}$, $5 \cdot x = 5 + x$. That is, the action of 5 on $\mathbb{R}$ slides every point on the line 5 units right. $-3 \in \mathbb{Z}$ slides each point 3 units left. This action is cellular, with (non-strict) fundamental domain any interval; it is proper, the only cell-stabilizer is the identity $0 \in \mathbb{Z}$; and it is co-compact. There is one orbit of 0-cells and one orbit of 1-cells. Thus, the space $\mathbb{R}/\mathbb{Z}$ should consist of exactly two cells: one 0-cell, and one 1-cell. And, since each 1-cell is connected to 0-cells on both ends, in the quotient space, the one 1-cell must be connected to the one 0-cell at both ends. Thus, $\mathbb{R}/\mathbb{Z}$ is a circle as shown in Figure 8.

The idea of a strict fundamental domain of an action is that it is a subset of the space whose orbit covers the whole space, but does so efficiently. For example, the interval $[0, 1]$ is a fundamental domain for the action of $\mathbb{Z}$ on $\mathbb{R}$ described in example 12, but it is not strict, since the endpoints of the interval are in the same orbit. A similar situation occurs in the next example.

Example 13. The real plane $\mathbb{R} \times \mathbb{R}$ can be thought of as a CW-complex, where each integer grid point corresponds to a 0-cell, horizontal and vertical segments connecting the grid points correspond to 1-cells, and 2-cells correspond to the resulting squares. $\mathbb{Z} \times \mathbb{Z}$ acts on $\mathbb{R} \times \mathbb{R}$ by horizontal and vertical translation. For example, for $(3, -2) \in \mathbb{Z} \times \mathbb{Z}$ and $(x, y) \in \mathbb{R} \times \mathbb{R}$, $(3, -2) \cdot (x, y) = (3 + x, -2 + y)$. This action is cellular, proper, and co-compact. To see the quotient space, realize that the square of the form $[0, 1] \times [0, 1]$ can be translated to cover the whole plane.

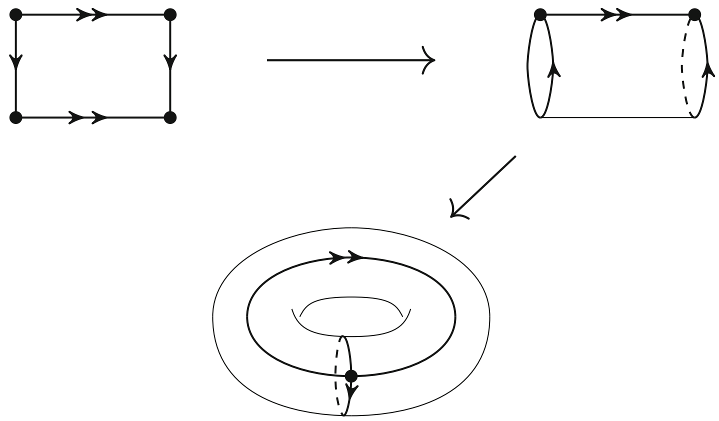


Fig. 9 $\mathbb{R}/(\mathbb{Z} \times \mathbb{Z}) \cong$ a torus.

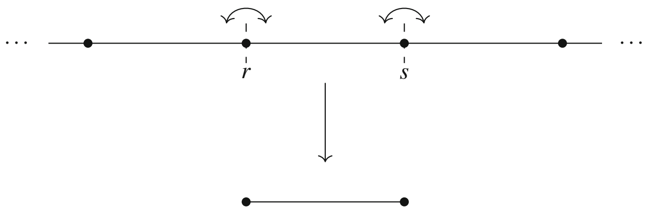


Fig. 10 D_∞ acting on $\mathbb{R}$ by reflections. $\mathbb{R}/D_\infty$ is shown.

That is, this square is a fundamental domain of the action, but it is not strict. There is one orbit of 2-cells, and thus one 2-cell in the quotient space. Within this one square, the top and bottom edges are identified, as they are within the same orbit, and the left and right edges are identified, as they are within the same orbit. However, the horizontal edges are not translated to the vertical edges under this action. So there are two orbits of 1-cells and thus two 1-cells in the quotient space. Finally, the four corners are identified as a single point, as there is one orbit of 0-cells, and so the quotient space has one 0-cell. $(\mathbb{R} \times \mathbb{R})/(\mathbb{Z} \times \mathbb{Z})$ is the torus, the identifications shown in Figure 9. Note that this is a slightly different perspective on the cellulation of the torus described in Example 11.

Example 14. The infinite dihedral group $D_\infty = \langle r, s \mid r^2, s^2 \rangle$ acts on $\mathbb{R}$ where we take r to correspond to a reflection about $0 \in \mathbb{R}$, and s to be a reflection about $1 \in \mathbb{R}$. See Figure 10. This action is cellular, proper (the stabilizers of the 0-cells have order 2), and co-compact. There are two orbits of 0-cells, one orbit of 1-cells. The quotient space then is a closed interval. Note that it can be identified with a strict fundamental domain of the action.

Orbihedral Euler Characteristic If a group G acts properly and co-compactly on a complex X, then the *orbihedral Euler characteristic* of X/G is the rational number

$$\chi^{\text{orb}}(X/G) = \sum_{\sigma} \frac{(-1)^{\dim \sigma}}{|\operatorname{Stab}_G(\sigma)|}, \tag{7}$$

where the sum is over the cells of X/G. (See [4] or [14] for reference on the orbihedral Euler characteristic.) Note that (1) The orbihedral Euler characteristic is the usual Euler characteristic in the case all cell stabilizers are trivial and (2) The orbihedral Euler characteristic is multiplicative. That is, if $H \leq G$ of index m, then

$$\chi^{\text{orb}}(X/H) = m\chi^{\text{orb}}(X/G). \tag{8}$$

(See Exercise 15 below.)

In Example 14 with $\mathbb{R}/D_\infty$, we have a 1-cell stabilized by the trivial subgroup, and two 0-cells stabilized by order two subgroups. So

$$\chi^{\text{orb}}(\mathbb{R}/D_\infty) = \underbrace{1/2 + 1/2}_{0-\text{cells}} - \underbrace{1}_{1-\text{cell}} = 0.$$

Exercise 13. Calculate $\chi(X)$ in the case X is

(a) An empty tetrahedron.
(b) An empty octahedron.
(c) An empty cube.
(d) The torus described in Example 11.

The cube, octahedron, and tetrahedron are "regular" cellulations of $\mathbb{S}^2$. Can you think of any others? Calculate the corresponding Euler characteristic. Sketch an non-regular cellulation of $\mathbb{S}^2$ and calculate χ.

Exercise 14. Calculate $\chi^{orb}(X/G)$ for each of the quotient spaces in Examples 12, 13, and 14.

Exercise 15. Prove Equation 8 above: If $H \leq G$ of index m, then $\chi^{\text{orb}}(X/H) = m\chi^{\text{orb}}(X/G)$.

Exercise 16. Using the idea of Example 13, describe an action of $D_\infty \times D_\infty$ on $\mathbb{R} \times \mathbb{R}$. Sketch the quotient space $\mathbb{R} \times \mathbb{R}/(D_\infty \times D_\infty)$ and calculate

$$\chi^{\text{orb}}\left((\mathbb{R} \times \mathbb{R})/(D\infty \times D\infty)\right).$$

Exercise 17. A group $G = \langle A \mid R \rangle$ acts on its Cayley graph Γ in the following way: For $g \in G$, v a vertex (which is also an element of the group), we have $g \cdot v = gv$. This is a generalization of the (left) action of a group on itself. Verify that this defines a cellular action on the Cayley graph. (That is, edges are sent to edges). Consider the following examples.

(a) Describe the action of $F_{a,b} = \langle a, b \mid \quad \rangle$ on its Cayley graph Γ. Sketch $\Gamma/F_{a,b}$.
(b) Describe the action of $G = \langle a, b \mid aba^{-1}b^{-1} \rangle$ on its Cayley graph Γ. Sketch Γ/G. (Compare with Example 13).
(c) Describe the action of $W = \langle r, s \mid r^2, s^2, (rs)^3 \rangle$ on its Cayley graph Γ. Sketch Γ/W.
(d) Describe the action of $W = \langle r, s, t \mid r^2, s^2, t^2, (rs)^2, (st)^2, (rt)^2 \rangle$ on its Cayley graph Γ. Sketch Γ/W.
(e) Describe the action of

$$W = \langle r, s, t, u, v \mid r^2, s^2, t^2, u^2, v^2, (rs)^2, (st)^2, (tu)^2, (uv)^2, (rv)^2 \rangle$$

on its Cayley graph Γ. Sketch Γ/W.

Exercise 18. Look up a cellular decomposition of a two-holed torus X (sometimes called a genus 2 surface). It can be understood as an "identification space" similar to the torus in Example 13, though beginning with an octagon rather than a rectangle. Calculate $\chi(X)$. Do the same with a "Klein bottle" and any "genus g-surface."

Challenge Problem 6. Consider the element $rs \in D_\infty$ and the subgroup generated by it, $\langle rs \rangle \leq D_\infty$.

(a) What is the order of $\langle rs \rangle$?
(b) What is the index of $\langle rs \rangle$ in D_∞.
(c) Under the action of D_∞ on $\mathbb{R}$ described in example 10, describe the action of the element rs on $\mathbb{R}$. That is, for $x \in \mathbb{R}$, what is $(rs) \cdot x$? What is $(rs)^{-1} \cdot x$? Deduce the action of $(rs)^n$ on $\mathbb{R}$.
(d) Sketch the quotient space $\mathbb{R}/\langle rs \rangle$ and verify equation 8.

The group $\langle rs \rangle$ is called a *finite index torsion free subgroup* of D_∞. This means that it contains no elements of finite order, besides the identity. Project idea 3 asks the student to consider such subgroups and similar questions in the context of different Coxeter groups.

Challenge Problem 7. Repeat Problem 6 for $D_\infty \times D_\infty$ acting on $\mathbb{R} \times \mathbb{R}$. That is, find a finite index subgroup in $D_\infty \times D_\infty$ acting on $\mathbb{R} \times \mathbb{R}$, and answer the included questions.

5 The Cellular Actions of Coxeter Groups: The Davis Complex

In several papers (e.g., [3], [4], and [5]), M. Davis describes a construction which associates to any Coxeter system (W,S), a complex $\Sigma(W,S)$, or simply Σ when the Coxeter system is clear, on which W acts properly and co-compactly. This is the Davis complex. We describe the construction here.

5.1 Spherical Subsets and the Strict Fundamental Domain

Let (W,S) be a Coxeter system with defining graph Γ. For a subset of generators U, denote by W_U the subgroup of W generated by the elements of U. Of interest are subsets of generators (vertices of the graph) that generate finite groups. We call these *spherical subsets*. These spherical subsets will be the key to defining an action of the corresponding Coxeter group on a complex.

5.1.1 Spherical Subsets

Finite Coxeter groups are completely classified, codified by the so-called "Dynkin diagrams;" and in general, one can detect if a given subset of generators of a Coxeter group defines a finite subgroup. But for us to work through our low-dimensional examples (dimension ≤ 3), we need only detect spherical subsets with three or fewer elements, and we can do that in a way that doesn't directly rely on knowing Dynkin diagrams.

Let (W,S) be a Coxeter system with corresponding Coxeter graph Γ. First note that every vertex of Γ corresponds to an order 2 generator, so every vertex defines a spherical subset of order 1. Next, recall that any two vertices connected by an edge generate a finite group, so all edges define a spherical subset of order 2. Furthermore, these are the only spherical subsets of order 2, since any two vertices not connected by an edge generate D_∞. Also, from this we deduce that the vertices of *any* spherical subset must be pairwise connected. Finally, we present the following fact for spherical subsets of order 3: For pairwise connected vertices r, s, and t of Γ with edge labels m_{rs}, m_{st}, and m_{rt}, the subgroup $\{r,s,t\}$ is finite if and only if

$$\frac{1}{m_{rs}} + \frac{1}{m_{st}} + \frac{1}{m_{rt}} > 1.$$

The reader is invited to check this fact against the Coxeter group examples and exercises worked at the end of Section 3, and investigate the geometric implications of such an inequality.

5.1.2 The Strict Fundamental Domain

With (W,S) a Coxeter system with Coxeter graph Γ, we define a finite complex K that will be the strict fundamental domain of the action of W on the Davis complex. The perspective we take is in some ways the inverse of the process laid out in Section 4.2, where given a group acting on a space X, we calculated the quotient space X/G. Here, we will construct the strict fundamental domain first and use it