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Sepideh Stewart
Christine Andrews-Larson
Avi Berman
Michelle Zandieh *Editors*

Challenges and Strategies in Teaching Linear Algebra

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Sepideh Stewart · Christine Andrews-Larson
Avi Berman · Michelle Zandieh
Editors

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Editors
Sepideh Stewart
Department of Mathematics
University of Oklahoma
Norman, OK
USA

Avi Berman
Department of Mathematics
Technion—Israel Institute of Technology
Haifa
Israel

Christine Andrews-Larson
College of Education
Florida State University
Tallahassee, FL
USA

Michelle Zandieh
Arizona State University
Mesa, AZ
USA

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Foreword

Linear algebra is arguably one of the more interesting (and complex) domains that students encounter in their first 2 years at university. The reasons for this include both the nature of a typical first course in linear algebra and the vitality of linear algebra beyond the first course. At the course level, linear algebra is one of the first opportunities for students to wrestle with definitions and proofs. In comparison to much of their prior mathematical experiences that emphasize procedural competency, linear algebra includes a rich array of new ideas, including linear independence, span, linear transformations, eigen theory, vector spaces, invertibility, rank, kernel, etc. These new concepts require students to carefully and consciously use definitions and to prove fundamental statements related to these ideas, all of which is something new for first-year university students. The extensive use of definitions and reliance on theorems often gives the first course an abstract and theoretical flavor, something that experts relish but which many students find distasteful. Linear algebra is also one of the richest domains for making connections between course concepts. For example, what many refer to as the invertible matric theorem relates over a dozen equivalent concepts. Thus students must not only understand the ideas themselves, but they must also develop reasons for how and why ideas are related. It is no wonder then that the literature is replete with studies that examine the challenges and difficulties that students encounter in linear algebra.

The importance of connections extends well beyond course-specific concepts. Indeed, linear algebra is also a vital area of mathematics, both within the discipline and across disciplines. For example, in differential equations, eigen theory plays an essential role in understanding linear homogeneous systems, which then provide useful tools for analyzing nonlinear systems. The concepts in linear algebra also play important roles in more advanced mathematics, including functional analysis and abstract algebra. Linear algebra also plays a vital role in other disciplines such as physics, engineering, and economics.

Despite the growth of research focused on the learning and teaching of linear algebra, there is still tremendous need for work that further examines students' difficulties, the underlying reasons for these difficulties, and instructional sequences and pedagogical approaches that have promise to promote student progress and deep understanding of the ideas in linear algebra and its widespread applicability. This book makes a significant contribution in addressing these needs that span research and practice. In terms of research, several of the chapters in this volume illuminate particular theoretical developments about learning as they relate to linear algebra, while other chapters offer a wide range of interesting and challenging problems that promise to engage students and promote deep understanding of core ideas. Just as these problems will be interesting for students, so will this volume be for readers.

San Diego, USA

Chris Rasmussen
San Diego State University

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Introduction

This book stems from the work of a Discussion Group (*Teaching Linear Algebra*) that was held at the 13th International Conference on Mathematics Education (ICME-13). The organizers of this Discussion Group (who are also the co-editors of this volume) aimed to orchestrate a conversation that would highlight current efforts regarding research and practice on teaching and learning of linear algebra from around the world. Their ultimate goal was to initiate a multinational research project on how to foster conceptual understanding of Linear Algebra concepts. This conversation was organized around a theme of problems and issues, with a particular focus on mathematical problems that are productive for learning. Key questions and issues discussed were as follows:

a. How can applications of Linear Algebra be used as motivation for studying the topic?
b. What are the advantages of proving results in Linear Algebra in different ways?
c. In what ways can a linear algebra course be adapted to meet the needs of students from other disciplines, such as engineering, physics, and computer science?
d. How can challenging problems be used in teaching Linear Algebra?
e. In what way should technology be used in teaching Linear Algebra?
f. What is the role of visualization in learning Linear Algebra?
g. In what order (pictures, symbols, definitions, and theorems) should we teach Linear Algebra concepts?
h. How can we educate students to appreciate the importance of deep understanding of Linear Algebra concepts?

While this rich list of questions was motivating, at the time of the 2-day meeting at ICME, the conversations gave rise to a common theme focusing on problems and issues in Linear Algebra instruction and ultimately the making of this book.

This volume offers insights into recent work related to the teaching and learning of linear algebra across a range of countries and contexts, drawing on expertise of mathematics educational researchers and research mathematicians with experience teaching linear algebra. The 18 chapters of this book represent work from nine countries: Austria, Germany, Israel, Ireland, Mexico, Slovenia, Turkey, USA, and Zimbabwe. Chapters share a thread of commonality in their focus on the use of challenging problems or tasks that are supportive of student learning. The chapters are organized in four sections: Chapters highlighting a theoretical perspective on the teaching and learning of Linear Algebra, chapters based on empirical analyses related to learning of particular content in linear algebra, chapters focusing on the use of technology and dynamic geometry software, and chapters featuring examples of challenging problems that experienced practitioners have found to be pedagogically useful.

Theoretical Perspectives Elaborated Through Tasks

The first three chapters in this volume focus on pedagogical aspects of Linear Algebra theoretically. In his chapter, Guershon Harel builds on his instructional framework, which is organized around notions of Duality, Necessity, and Repeated reasoning (DNR). Specifically, he considers the role of cognitive and pedagogical aspects of Linear Algebra through the lenses of two main DNR concepts, namely, *intellectual need and epistemological justification*, and exemplifies them through a variety of Linear Algebra tasks. Harel invites the mathematics community to reflect on whether instruction that is organized around this theoretical viewpoint will have an effect in advancing Linear Algebra students' performance.

Continuing the theoretical conversation, Maria Trigueros's chapter proposes a teaching approach that builds on theory about Actions, Processes, Objects, and Schema (APOS) through the use of several challenging modeling situations and tasks designed to introduce some main linear algebra concepts. The results reveal crucial moments as students develop new strategies, resulting in further understanding of the concepts.

Based on her work with research mathematicians, Sepideh Stewart believes that creating opportunities to move between Tall's (2013) Worlds of mathematical thinking will encourage students to think in multiple modes of thinking and increases their abilities in dealing with problems from different angles. In her chapter, she proposes a set of Linear Algebra tasks designed to move learners among Tall's Worlds.

Analyses of Learners' Approaches and Resources

The empirical analyses section of this book offers an exciting variety of findings across populations and topic areas in linear algebra. Data is taken from populations ranging from middle and high school students in Mexico to undergraduates in North and Central America, to current teachers updating their certification through undergraduate coursework in Zimbabwe. Topics include systems of linear equations, matrix multiplication, determinants, vector spaces, eigenvectors, and eigenvalues.

Asuman Oktac provides a synthesis of three previously unpublished thesis studies examining student reasoning about systems of linear equations across middle school, high school, and university contexts (Mora Rodríguez 2001; Cutz Kantún 2005; Ochoviet Filguieras 2009). All three studies were conducted in Mexico and written in Spanish. This chapter identifies points of commonality across these studies and leverages a common theoretical framework, making these findings available to an English-speaking audience.

John Paul Cook, Dov Zazkis, and Adam Estrup point to conceptual underpinnings entailed in matrix multiplication as motivation for analyzing how matrix multiplication is introduced and motivated in 24 introductory linear algebra textbooks. This work provides a timely update to Harel's (1987) textbook analysis and expands the corresponding framework to include computational efficiency. Additionally, this piece offers insight into the variety of ways current texts address the issue of matrix multiplication, considers aspects of reasoning emphasized and valued in each approach, and draws connections between textbook approaches and current research on student reasoning.

The chapter by Cathrine Kazunga and Sarah Bansilal, as well as the chapter by Lillias Mutambara and Sarah Bansilal, draws on data from a population of current mathematics teachers who were part-time students at a Zimbabwean university to meet new teacher certification requirements in the country. Their chapters provide analyses of participants' understanding of determinants and vector spaces, respectively.

The chapter by David Plaxco, Michelle Zandieh, and Megan Wawro, and the chapter by Khalid Bouhjar, Christine Andrews-Larson, Muhammad Haider, and Michelle Zandieh both offer insights into student reasoning about eigenvectors and eigenvalues in the context of inquiry-oriented instruction. The Plaxco et al. chapter offers insights into student reasoning in a guided reinvention approach drawn from classroom data, whereas the Bouhjar et al. chapter documents the effectiveness of this approach by comparing written assessment data of students who learned through this approach with students who learned the material in more standard ways.

Dynamic Geometry Approaches

Three of the chapters in this book discuss ways that technology can influence the learning of Linear Algebra. Hamide Dogan's chapter compares learners who were exposed to dynamic visual representations to those who were exposed to the traditional instructional tools. She found notable differences in the nature of the mental schemes displayed by learners in the two groups. In addition, those students exposed to dynamic visual representations were able to use this geometry-based knowledge to make sense of more abstract algebraic ideas. Melih Turgut's chapter uses the theory of semiotic mediation to describe how the tools and functions of a dynamic geometric system affect student learning. In particular, he focuses on how these tools mediated the evolution of student reasoning about linear transformations from personal meanings based on work in R^2 to new mathematical meanings in R^3 and R^n. Ana Donevska-Todorova's chapter takes a broader perspective in considering which technology-enhanced environments may best affect student learning of different competencies. She suggests a nested model that illustrates how three modes of thinking in linear algebra can be related to the design of tasks or teaching environments.

Challenging Tasks with Pedagogy in Mind

The last six chapters of the book involve challenging tasks that illustrate the beauty and usefulness of linear algebra and feature many applications. Barak Pearlmutter and Helena Smigoc show how nonnegative factorization of data matrices can motivate the study of basic Linear Algebra. They give a simple example stopping at discussion points. Their chapter includes an example of factoring a data matrix of module descriptors for 62 mathematics modules that were taught in their school. The chapter by Avi Berman uses formulas on Fibonacci numbers, a proof of the uniqueness of Lagrange polynomials, a periodicity two property of neural networks, and a computer game on lights as examples of challenging problems that can be used as motivation in teaching Linear Algebra. Franz Pauer describes a computational approach to teaching systems of linear equations. He gives an example of electric circuits and concludes his chapter with geometric interpretation. Frank Uhlig demonstrates his successful experience of holistic teaching and holistic learning with a Linear Algebra example of plane rotation. David Strong suggests how to motivate a course, how to motivate a chapter, and how to motivate an idea. He describes many motivational applications including systems of equations, discrete dynamical systems, QR factorization, traffic flow, and investments. Damjan Kobal describes how basic linear algebra concepts can be used for a smooth transformation from intuitive to abstract cognition and to deepen students' understanding. The applications in his chapter include Brower fixed point theorem, projective spaces, and barycentric and trilinear coordinates.

In its breadth of perspectives, this book offers a tremendous number of resources on teaching linear algebra, while also bringing together a community of those interested in pedagogical issues in linear algebra from around the world. It is our intention to continue the work started with the ICME-13 Discussion Group on *Teaching Linear Algebra* as we meet at other international conferences to further these discussions.

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Part I
Theoretical Perspectives Elaborated Through Tasks

The Learning and Teaching of Linear Algebra Through the Lenses of Intellectual Need and Epistemological Justification and Their Constituents

Guershon Harel

Abstract *Intellectual need* and *epistemological justification* are two central constructs in a conceptual framework called *DNR-based instruction in mathematics*. This is a theoretical paper aiming at analyzing the implications of these constructs and their constituent elements to the learning and teaching of linear algebra. At the center of these analyses are classifications of intellectual need and epistemological justification in mathematical practice along with their implications to linear algebra curriculum development and instruction. Two systems of classifications for intellectual need are discussed. The first system consists of two subcategories, *global* need and *local need*; and the second system consists of five categories of needs: *need for certainty, need for causality, need for computation, need for communication,* and *formalization,* and *need for structure.* Epistemological justification is classified into three categories: *sentential epistemological justification* (*SEJ*), *apodictic epistemological justification* (*ASJ*), and *meta epistemological justification* (*MEJ*).

Keywords Intellectual need · Epistemological justification

DNR-based instruction in mathematics (*DNR*, for short; Harel, 1998, 2000, 2008a, b, c, 2013a, b) is a theoretical framework for the learning and teaching of mathematics—a framework that provides a language and tools to formulate and address critical curricular and instructional concerns. *DNR* can be thought of as a system consisting of three categories of constructs: *premises*—explicit assumptions underlying the *DNR* concepts and claims; *concepts*—constructs defined and oriented within these premises; and *claims*—statements formulated in terms of the *DNR* concepts, entailed from the *DNR* premises, and supported by empirical studies.

G. Harel (✉)
University of California, San Diego, USA
e-mail: harel@math.ucsd.edu

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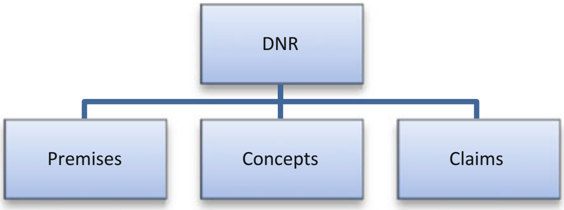


Fig. 1 *DNR's* three categories of constructs

The main goal of this paper is to discuss cognitive and pedagogical aspects of linear algebra through the lenses of two central *DNR* concepts: *intellectual need* and *epistemological justification*. As the above list of references indicates, *DNR* has been discussed extensively elsewhere, and so in this paper we only reiterate briefly the definitions of these concepts along with their essential constituent elements: the concepts of *ways of understanding* and *ways of thinking* and four out of the eight premises of *DNR*.

We begin in Sect. 1 with the concepts of *ways of understanding* and *ways of thinking*. Following this, in Sect. 2, we discuss the four *DNR* premises. With these concepts and premises in hand, we turn, in Sect. 3, to the definition of *intellectual need* and *epistemological justification*. The fourth and fifth sections present, respectively, more refined analyses into various categories of the latter two concepts. The sixth, and last, section concludes with reflections and research questions. In each section, the discussion is accompanied with observations made in teaching experiments in linear algebra we have conducted during the years. In this respect, this is a theoretical, not empirical, paper. That is, the purpose of the paper is to theorize and illustrate the role and function of *intellectual need* and *epistemological justification* and their constituent elements in the learning and teaching of linear algebra.

To help the reader navigate through the various *DNR* terms introduced in this paper, we end each section with a figure depicting the network of terms accrued up to that section. Figure 1, for example, depicts the three categories of constructs comprising *DNR* outlined in this introduction. The rest of the figures in the paper will be expansions of this figure.

1 Ways of Understanding and Ways of Thinking

The notions of *way of understanding* and *way of thinking* have technical definitions (see Harel, 2008c). However, for the purpose of this paper it is sufficient to think of them as two different categories of knowledge, the first refers to one's conceptualization of "subject matter," such as the way one interprets particular definitions, theorems, proofs, problems and their solutions; and the second refers to "conceptual tools," such as deductive reasoning, empirical reasoning, attention to structure and precision, and problem-solving approaches (e.g., heuristics). One of the central claims of *DNR*, called the *duality principle*, asserts that (a) one's ways of thinking impacts her or his ways of understanding; and, (b) it is the acquisition of appropriate ways of understanding that brings about a change and development in one's ways of thinking.

To illustrate, consider the following example. A mathematically mature student who possesses *definitional reasoning*—the way of thinking by which one examines concepts and proves assertions in terms of *well-defined* statements—is likely to understand the concept of *dimension of a subspace* as intended—the number of vectors in a basis of the subspace—but he or she would also realize that such a definition is meaningless without answering the question whether all bases of a subspace have the same number of vectors. Another student, for whom definitional reasoning has not yet reached full maturity, may have the same understanding without realizing the need to settle this question. Yet another student whose conceptualization of mathematics is principally action-based (in the sense of APOS theory),[1] is likely to understand the concept of dimension in terms of a rule applied to n-tuples. For such a student, the dimension of a span of a set of vectors in R^n amounts to carrying out a procedure of, for example, setting up these vectors as the columns of a matrix, row reducing the matrix, and determining, accordingly, the number of pivot columns the matrix has. We observed each of these three conceptualizations among students on various occasions, even in upper division linear algebra courses. And scenarios corresponding to these three conceptualizations have occurred throughout our teaching experiments when attention to a well-defined concept was called for. For example, when the instructor concluded that the *projection matrix onto a subspace* V *of* R^n is the matrix $P = W(W^TW)^{-1}W^T$, where W is a basis matrix[2] of V, there were a few students who fully understood, and some even independently raised, the concern that P

[1]APOS theory (Arnon et al., 2014; Dubinsky, 1991) will be used to provide conceptual bases for some of these observations. Given how widely this theory has been studied during the last three decades, there is no need to allocate more than a brief illustration to the four levels of conceptualizations, *action, process, object*, *and schema* offered by the theory and used in this paper. Briefly, consider the phrase "the coordinates of a vector of x with respect to a basis-matrix A in R^n," denoted by $[x]_A$. At the level of action conception, the learner might be able to deal with $[x]_A$ only in the context of a specific vector and a specific suitable basis-matrix, by following step-by-step instruction to compute the respective coordinate vector. At the level of process conception one is capable of imagining taking *any* vector x in R^n, representing it as a linear combination of the columns of A, and forming a column vector whose entries are the coefficient of, and are sequenced in the order they appear in, the combination. With this conceptualization, the learner is able to carry out this process in thought and with no restriction on the vector x considered. At the level of object conception, one is aware of the process of relating the two coordinate vectors as a totality, for example, in finding the relation between two coordinate vectors of x, one with respect to a basis-matrix A_1, $[x]_{A_1}$, and one with respect to a basis-matrix A_2, $[x]_{A_2}$, whereby being able to express the relation in terms of a transition matrix $S = A_2^{-1}A_1$ between the two vectors. Among the ways of thinking that are essential to cope with linear algebra, in particular, and mathematics, in general, are the abilities to construct concepts at the levels of process conception and object conception, as it is demonstrate throughout the paper. (See also Trigueros, this volume.)

[2]A matrix whose columns form a basis for a subspace.

might be dependent on the choice of W. For most of the students, however, the conclusion engendered no concern.

The implication of the second part of the duality principle is that students acquire a particular way of thinking only by repeatedly dealing with specific ways of understanding associated with that way of thinking. For example, students develop definitional reasoning not by preaching but by repeatedly using definitions in the process of mathematical argumentations and by dealing in a multitude of contexts with the question whether a concept is well defined.

The examples of ways of thinking we have listed above are general—they pertain to mathematics as a discipline. Different areas or sub-areas of mathematics, however, can be branded by ways of thinking specific to them. The conceptualizations of matrix theory and the theory of general vector spaces share ways of thinking (e.g., axiomatic proof schemes (Harel & Sowder, 1998) and structural reasoning (Harel & Soto, 2016), and yet each is branded by a set of ways of thinking unique to it. For example, while thinking in terms of row reduction and block matrices is part of elementary matrix theory, it is often not applicable to coordinate-free, vector spaces.

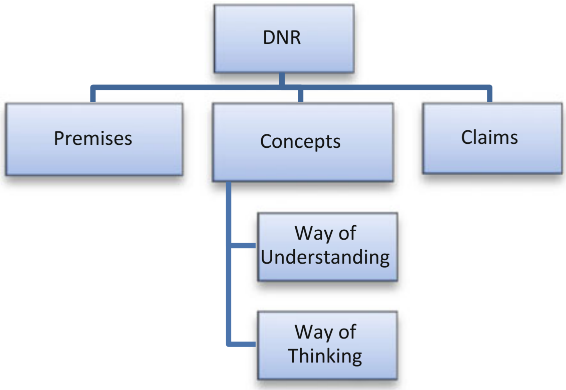
Problem-solving approaches are instances of ways of thinking (Harel, 2008c). Therefore, "reasoning in terms of __ in solving problems" is an instance of a way of thinking. For example, reasoning in terms functions, reasoning in terms of row reduction, reasoning in terms of block matrices, reasoning in terms of linear combinations are all problem-solving approaches, and hence are ways of thinking. In our experience, the acquisition and application of such ways of thinking is difficult for students. Consider, for example, reasoning in terms of block matrices (known also as partitioned matrices) and linear combination. It is one thing applying block-matrices rules to multiply matrices; it is another using block matrices to represent and solve problems. Typically, students are able to perform at the level of action conception (ala APOS theory) algebraic operations using block matrices, but they experience major difficulties when block matrices are constructed to represent relations and prove theorems. Consider the simple case of the product $A_{m\times n}x_{n\times 1}$ as a linear combination of the columns of A. We repeatedly observed students having difficulties representing a common statement such as "$v_1, v_2, \ldots, v_k \in span(u_1, u_2, \ldots, u_m) \subseteq R^n$" in a matrix form: $[\,v_1 \quad v_2 \quad \ldots \quad v_k\,] = [\,u_1 \quad u_2 \quad \ldots \quad u_m\,]Q$ for some $Q_{m\times k}$.

This difficulty manifested itself on numerous occasions, for example in comprehending the following proof of the theorem, "Any set of m linearly independent vectors in an m-dimensional subspace H of R^n spans H". The proof presented in class was an elaboration of the following lines:

> Let $u_1, u_2, \ldots, u_m$ be linearly independent vectors in H, and set $U = [\,u_1 \quad u_2 \quad \ldots \quad u_m\,]$. Let $V = [\,v_1 \quad v_2 \quad \ldots \quad v_m\,]$ be a basis matrix of H. There exists a matrix $Q_{m\times m}$ such that $U = VQ$. Since the columns of U are linearly independent, Q is invertible, and so $UQ^{-1} = V$. Hence $u_1, u_2, \ldots, u_m$ span H.

We point to two obstacles students typically encounter in comprehending this proof. The first revolves around the equation $U = VQ$; students indicate that they do

Fig. 2 Two of the *DNR* concepts: Ways of understanding and ways of thinking—



not understand where the matrix Q came from, even after they are presented with an explanation by their group mates or the instructor. A possible conceptual basis for this difficulty is that the construction of Q requires performance at the level of process conception (ala APOS theory), a form of abstraction known to be cognitively demanding (Dubinsky, 1991). Specifically, one needs to construct, successively and in thought, each column of Q out of the coefficients of the expression representing its corresponding column of U as a linear combination of the columns of V (i.e., $Q_i = [q_{i1} \quad q_{i2} \quad \ldots \quad q_{im}]^T$, where $U_i = \sum_{j=1}^{m} q_{ij} V_i$, $i = 1, 2, \ldots, m$). Even students who overcome this difficulty express discomfort with the claim that the result $UQ^{-1} = V$ completes the proof. At the heart of this claim, and the difficulty, is the fact that linear combination of linear combinations is a linear combination—that since each vector in H is a linear combination of the columns of V and each column of V is a linear combination of the columns of U, by $UQ^{-1} = V$, $ColU = H$. Here too performance at the level of process conception seems essential, in that one has to carry out this chain of relations in thought in order to fully bring oneself to a firm conviction about the validity of the claim.

Figure 2 expands Fig. 1 to include the *DNR* constructs discussed in this section.

2 *DNR* Premises

DNR has eight premises; they are philosophical stands appropriated from existing theories, such as the Piagetian theory of equilibration (Piaget, 1985), Brousseau's (1997) theory of didactical situation, and Aristotle. Relevant to this paper are four of these premises; they are: the *knowledge of mathematics premise,* the *knowing premise,* the *knowledge-knowing linkage premise,* and the *subjectivity premise.*

The ***knowledge of mathematics premise*** states: *knowledge of mathematics consists of two related but different categories of knowledge: the ways of understanding and ways of thinking that have been institutionalized throughout history*.[3] The significance of this premise to mathematics instruction is that while knowledge of and focus on ways of understanding is indispensable for quality teaching, it is not sufficient. Mathematics instruction should also attend to ways of thinking. With this instructional view one would teach, for example, row reduction not only as a tool to solve systems of linear equations but be cognizant of and explicit about the value of this tool in analyzing and answering theoretical questions. In accordance to the duality principle stated earlier, the development of such a way of thinking is facilitated by instruction that persistently models it in proving theorems and solving problems, as the following episode illustrates.

The episode occurred in an elementary linear algebra class. The instructor defined column rank and row rank. It turned out that the class as a whole dealt with these concepts in an add-on Matlab component to the course (entirely not coordinated with the instructional pace of the course), where the students have used the fact that $\dim ColA = \dim RowA$ without proof. Before the instructor turned to prove this statement, one of the students in the class exclaimed publically that she found this fact fascinating—that for any array of numbers, "no matter what" (her words), the maximum number of linearly independent columns equals the maximum number of linearly independent rows. Then she added: "I kept thinking about it for some time until I found why". In response to the instructor's question, "What was the explanation you have found?" she said: "… by reducing the matrix into *rref* … I always bring up *rref* … it helps me solve the homework problems". Then she proceeded by explaining how in *rrefA* the number of columns with a leading 1 is necessarily equal to the number of rows with a leading 1, from which she concluded that $\dim ColA = \dim RowA$, using the previously proved facts that row reduction preserves dependence/independence of the columns of A as well as *RowA*.

We posit that the instructor's explicit and persistent effort to present row reduction as a conceptual tool in proving theorems and solving problems contributed to conceptualizations as the one articulated by this student.

The next two premises are inextricably linked; one is about *knowing* and the other about the linkage between *knowing* and *knowledge*. The ***knowing premise*** states: *The means of knowing is the process of assimilation and accommodation*. According to Piaget (1985), disequilibrium, or perturbation, is a mental state when one fails to assimilate. Equilibrium, on the other hand, is a state in which one perceives success in assimilating. In Piaget's terms, equilibrium occurs when one has successfully modified her or his viewpoint (accommodation) and is able, as a result, to integrate new ideas toward obtaining a solution of a problem (assimilation). The ***knowing-knowledge linkage*** premise states: *Any piece of knowledge humans know is an outcome of their resolution of a problematic situation* (Brousseau, 1997; Piaget, 1985). This premise is an extension of the knowing

[3]For the philosophical foundations of this premise, see Harel (2008c).

premise. While the knowing premise is about the mechanism of learning, the knowing-knowledge linkage premise guarantees that row material for the operation of that mechanism (i.e., a problematic situation from the engagement of which knowledge is constructed) exist. Collectively, the last two premises constitute a theoretical foundation for, respectively, the *essentiality* and *viability* of problem-solving based curricula. Namely, these curricula are *essential* because the only way to construct knowledge is by resolution to problematic situations (by the *knowing-knowledge premise*); and they are *viable* because such situations exist (by the *knowing premise*).

The implication for instruction of the view articulated by the last two premises is the *necessity principle,* which states: *For students to learn what we intend to teach them, they must have a need for it, where* '*need*' *refers to intellectual need*. Relevant to curriculum design, the necessity principle entails that new concepts and skills should emerge from problems understood and appreciated as such by the students, and these problems should demonstrate to the student the intellectual benefit of the concept *at the time of its introduction*.

The problematic situations referred to in this premise may or may not be historic. For example, matrices did not grow out of the need to solve systems of linear equations, as typically is done in elementary linear algebra textbooks, but out of the need to develop determinants (in 1848 by J.J. Sylvester). According to Tucker (1993), "array of coefficients led mathematicians to develop determinants, not matrices. Leibniz … used determinants in 1693 about hundred and fifty years before the study of matrices …" (p. 5). Also, most problems studied in linear algebra are not introduced in the context of the field in which they originated initially. For example, Gauss elimination is typically introduced in textbooks in an application-free context, but it initially emerged in the field of geodesy and for years was considered part of the development of this field (Tucker, 1993).

The ***subjectivity premise*** states: *Any observations humans claim to have made are due to what their mental structure attributes to their environment.* This premise orients our interpretations of the actions and views of the learner. It cautions us—teachers—that what might be problematic for one individual or a community may not be so for others. A situation might trigger a mental perturbation with one person and be accepted by another. To illustrate, we continue the discussion about the concept of *dimension* we started in Sect. 1. In one of our teaching experiments, the instructor deliberately defined "dimension" as the number of vectors in a basis without first stating the theorem that all bases of a subspace have the same number of vectors. Following this, he asked the students to discuss in their working groups whether the definition is sound. A while later, when no productive response came from the students, he made the task more explicit by asking whether there is a need to establish a particular property of bases in a subspace for the definition to be meaningful. None of the students found any fault with the definition as stated. Following this, the instructor asked the class to comment on the following hypothetical scenario:

> Two students, John and Mary, are asked to determine the dimension of a particular subspace of a vector space. John identifies a basis of the subspace, counts the number of its elements, and reports that the dimension is 5. Mary identifies a different basis of the subspace, counts the number of its elements, and reports that the dimension is 7.

While some of the students responded as expected—that there is a need to establish that all bases of the subspace have the same number of vectors—astonishingly, there were students who responded by saying something to the effect that for John the dimension of the subspace is 5, and for Mary the dimension is 7. Presumably, these students did not possess the definitional way of thinking, and so the scenario described by their instructor did not cause them the desirable perturbation—their response was an outcome of their current schemes.

The subjectivity premise also cautions us, teachers, that learners' current ways of thinking may lead them to independently generalize faulty knowledge from a correct one. For example, student may, and typically do, erroneously conclude that row reduction preserves the column space, as it does with row space. Furthermore, often due to the level of robustness of certain ways of thinking students possess counterexamples to such faulty generalizations may not be effective (Harel & Sowder, 2007). For example, in our experience, students continue to hold this generalization true even after they are shown counterexamples to the contrary [e.g., for any matrix whose entries are all *1*s, $ColA \neq Col(rrefA)$]. Many scholars (e.g., Confrey, 1991; Dubinsky, 1991; Steffe, Cobb, & Glasersfeld, 1988; Steffe & Thompson, 2000) have articulated essential implications of the *subjectivity premise* to mathematics curriculum and instruction, even if they have not given it an axiomatic status as we do.

Figure 3 expands Fig. 2 to include the *DNR* constructs discussed in this section.

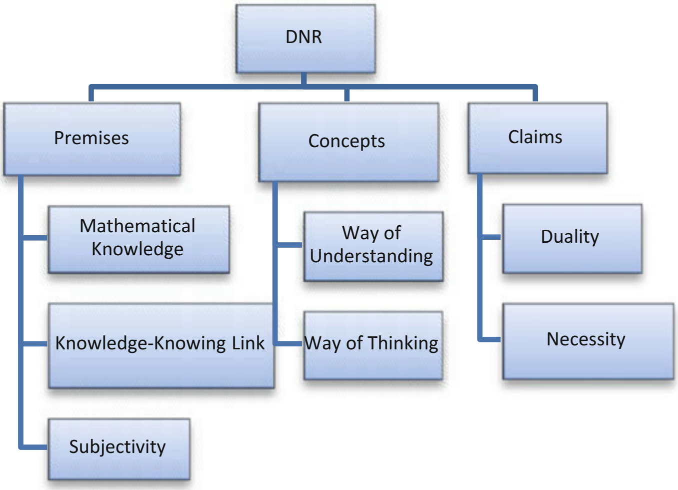


Fig. 3 Three of the eight *DNR* premises and two of *DNR* foundational principles

3 Intellectual Need and Epistemological Justification

With these premises at hand, we now present the definitions of *intellectual need* and its associated concept, *epistemological justification*, as formerly introduced in Harel (2013a), with minor modifications.

[Let] K be a piece of knowledge possessed by an individual or community, then, by the *knowing-knowledge linkage premise*, there exists a problematic situation S out of which K arose. S (as well as K) is subjective, by the *subjectivity premise*, in the sense that it is a perturbational state resulting from an individual's encounter with a situation that is incompatible with, or presents a problem that is unsolvable by, her or his current knowledge. Such a problematic situation S, prior to the construction of K, is referred to as an individual's *intellectual need:* S is the need to reach equilibrium by learning a new piece of knowledge. Thus, intellectual need has to do with disciplinary knowledge being created out of people's current knowledge through engagement in problematic situations conceived as such by them. One may experience S without succeeding to construct K. That is, intellectual need is only a necessary condition for constructing an intended piece of knowledge. Methodologically, intellectual need is observed when we see that (a) one's engagement in the problematic situation S has led her or him to construct the intended piece of knowledge K and (b) one sees how K resolves S. The latter relation between S and K is crucial, in that it constitutes the genesis of mathematical knowledge—the perceived reasons for its birth in the eyes of the learner. We call this relation *epistemological justification*.

Intellectual need and epistemological justification are two sides of the same coin —they are different but inextricably related constructs. Their occurrence is entirely dependent on one's background knowledge. Consider the question: What is a generator for the *ideal of polynomials annihilating a given operator T over an n-dimensional vector space*? Clearly, such a question wouldn't occur unless one possesses a cluster of ways of understanding for the concepts: *ideal, generator of an ideal, operator annihilating ideal* etc. Less trivial is the question, what ways of thinking facilitate the emergence of such a question with an individual? Or put in another way, how can we educate students to develop the habit of mind of asking such questions? A critical claim of this paper is that attention to epistemological justifications in generating definitions and proving theorems may pave the road to such habit of minds, as we will see in the next sections.

Even if such a question is raised, its answer hinges upon one's understanding and appreciation of the most foundational concept of linear algebra: *linear combination*. Thinking in terms of this concept and its derivative concepts of linear independence and linear dependence, one may recognize that the ideal of polynomials annihilating an operator T over an n-dimensional vector space is not empty, since it contains an annihilator polynomial of degree n^2. This may not end here if this individual continues to ask: What is a generator for this ideal? And since the degree of such a polynomial is not greater than n^2, can it be n? Is there a polynomial of degree n that annihilates T? If the search for an answer to this question leads the individual, independently or with the help of an expert, to Cayley-Hamilton Theorem ("Any linear operator on a finite-dimensional vector space is annihilated by its characteristic polynomial"), then by definition, the individual has constructed an epistemological justification for the theorem. An epistemological justification for the proof of the theorem—how the proof might be elicited—is a different matter. Such a proof may require additional or different networks of ways of understanding and ways of thinking.

It is important to highlight two points concerning intellectual need and epistemological justification. First, we iterate a point we made earlier, these constructs are not historical; rather, they are pedagogical (and research) tools. Namely, the need which has originally necessitated a particular concept may not—and is usually not—the one used in a curriculum. For example, in one of our teaching experiment, the concept of *linear independence* was necessitated through the question, When does Gaussian Elimination lead to "loss" of equations (i.e., zero equations in a system obtained through the application of elementary operations)?; and in another experiment through the question, When does a consistent system of linear equation have a unique solution? Historically, this concept emerged from generalizations of spatial relationships by Grassmann (Li, 2008).

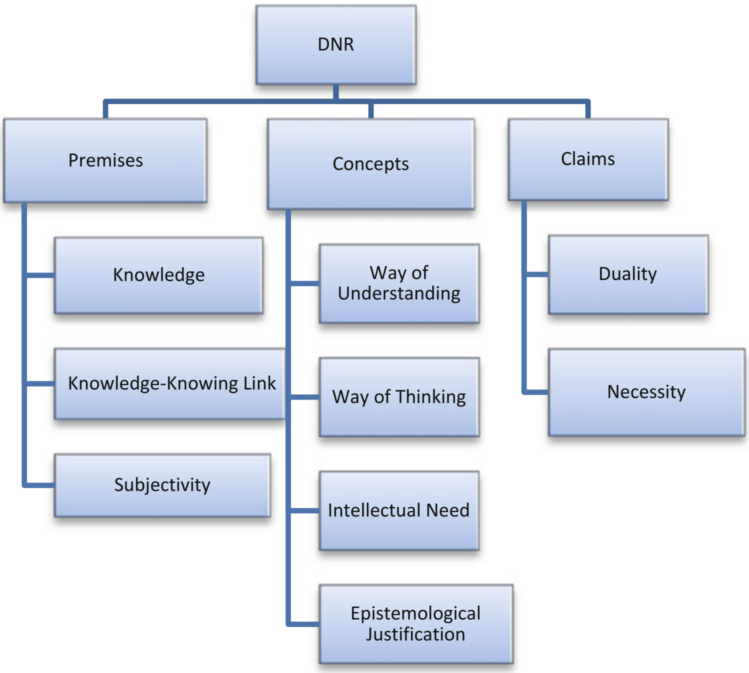


Fig. 4 Two additional foundational *DNR* concepts: Intellectual need and epistemological justification

Second, while problems outside the fields of mathematics can serve as intellectual need for particular mathematical concepts and ideas, as we know from history, intellectual need is not synonymous with application. Cognitively, the term "application" refers to problematic situations aiming at helping students solidify mathematical knowledge they have already constructed or are in the process of constructing. Intellectual need, on the other hand, aims at eliciting knowledge students are yet to learn.

Figure 4 expands Fig. 3 to include the *DNR* constructs discussed in this section.

4 Categories of Intellectual Need

We offer two systems of classifications of intellectual need, each with a particular role in curriculum development and instruction; in this paper, they are instantiated in the context of the learning and teaching of linear algebra. The first system of classification rests on the distinction between *local need* and *global need*; it pertains to the structure of a mathematics curriculum. The second system of classification is more refined, in that it identifies specific types of intellectual needs that emerge in mathematical practice; they are: *need for certainty, need for causality, need for computation, need for communication,* and *need for structure.* These two systems of classifications will be discussed in turn in the next two sections. (For a discussion on the cognitive origins of these needs, see Harel, 2013a.)

4.1 Local Need Versus Global Need

Consider an elementary course in linear algebra structured around a series of investigations, each aimed at answering a particular central question. The course begins with the question: (1) What is linear algebra? And it immediately discusses one of its branches: *systems of linear equations*, both systems in which the unknowns are scalars in a particular field (linear systems of scalar equations) and systems in which the unknowns are functions (linear systems of differential equations). Attending first to linear systems of scalar equations, the course then progressively proceeds by investigating, in this order, the questions: (2) Why is the focus on linear systems? (3) What exactly is the elimination process (which typically students are familiar with its basic form from their high-school mathematics)? (4) Why does the process of elimination work? (5) Why are equations "lost" in the elimination process? (6) Is there an algorithm to solve linear (scalar) systems? (7) What does the reduced echelon form (rref) tell us about the solution set of a system? This is a partial sequence of central questions aimed at helping the students build a coherent global image of the purposes of the study of systems of linear equations. Collectively, not individually, such questions represent a *global intellectual need* for the study of a particular area of mathematics.

An investigation into each of such questions generates specific problems manifesting *local intellectual need*—the need for the construction of particular concepts and ideas. A probe into some of the above questions, generate, for example, the concepts of *linear combination*, *equivalent systems*, *linear independence*, and *basis*, for the purpose of advancing the overarching investigation. To illustrate, consider, for example, Question 4—Why does the process of elimination work? In linear-algebraic terms, this question can be formulated as: Why *elementary operations* preserve the solution set of a system? A probe into the nature of these operations elicits the need for the creation of concepts and ideas. It begins with the following central idea:

Let S be an $m \times n$ system, with equations $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$. For any m scalars $c_1, c_2, \ldots, c_m$, any solution of system S is a solution of the equation $\varepsilon_\Sigma = c_1\varepsilon_1 + c_1\varepsilon_1 + \ldots + c_m\varepsilon_m$.

In turn, this idea elicits the foundational concept of *linear combination* (i.e., the equation ε_Σ is a linear combination of the equations, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$), and with it, the following conclusion, which gives rise to the concept of *equivalent systems*:

Given two systems S_1 and S_2 of the same size, if each equation of S_1 is a linear combination of equations of S_2 and each equation of S_2 is a linear combination of S_1, then the two systems have same solution set.

Thus,

Two systems of equal size are equivalent if each equation in one system is a linear combination of the equations in the second system, and vice versa.

And so:

If two systems are equivalent, then they have the same solution set.

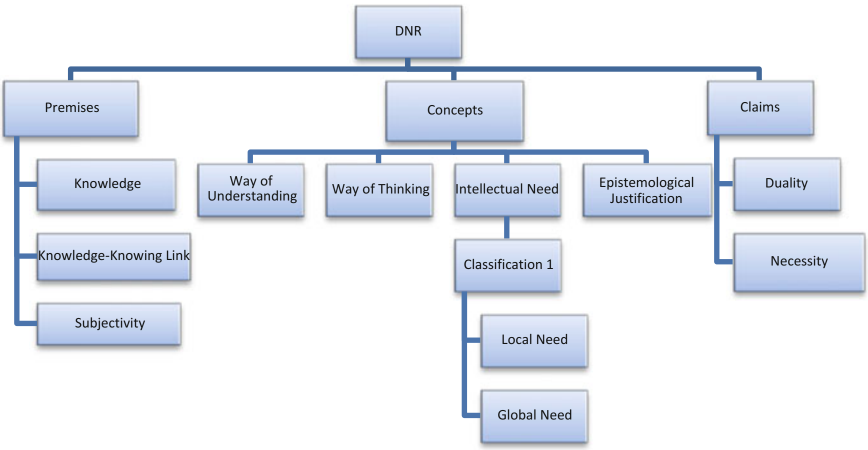


Fig. 5 First classification of intellectual need

These results, then, lay the foundation for the question under consideration (Question 4), which now can be stated as: Do elementary operations preserve equivalency?

The second half of the course turns to linear systems of differential equations (i.e., $Y^{'}(t)=AY(t)$, $Y(0)=C$) where eigen theory is then introduced through the global need to investigate the question, How to solve such systems? This question leads to local needs, as will be discussed in the next section.

Figure 5 expands Fig. 4 to include the *DNR* constructs discussed in this section.

4.2 Intellectual Need in Mathematical Practice

Based on cognitive and historical analyses, we offered in Harel (2013a) five categories of intellectual needs: (1) *need for certainty,* (2) *need for causality,* (3) *need for computation,* (4) *need for communication,* and (5) *need for structure.*

The first two needs are complementary to each other: understanding cause brings about certainty, and certainty might trigger the need to determine cause. The need for certainty is the need to prove—to remove doubts. One's certainty is achieved when one determines, by whatever means he or she deems appropriate, that an assertion is true. The *need for causality,* on the other hand, is the need to explain—to determine a cause of a phenomenon, to understand what makes a phenomenon the way it is. A student might be certain that a particular assertion is true because a teacher or textbook said so or because he or she verified the assertion empirically. The student might even reach certainty on the basis of a proof, and yet lack an insight as to what makes the assertion true—the proof may not be explanatory for her or him. In the next section, we will discuss explanatory proofs in the context of epistemological justification.

The third need is the *need for computation.* It is the need to quantify or calculate values of quantities and relations among them by means of symbolic algebra. For example, the need to quantify the "size" of a solution set of a linear system $Ax=b$ may be addressed by the concept of *rank:* the smaller the rank of a matrix A is the "larger" the solution set of a consistent system $Ax=b$ becomes. Likewise, the need to reduce the data storage of a digitized image without compromising significantly the quality of the image through its electronic transmission may be responded to by decomposing the matrix representing the gray values of the image into a particular sum of rank-1 matrices, what is known as *singular value decomposition* (*svd*; see below for more discussion on this decomposition).

The fourth need is the *need for communication.* This need consists in two reflexive needs: *the need for formulation*—the need to transform strings of spoken language into algebraic expressions—and the *need for formalization*—the need to externalize the exact meaning of ideas and concepts and the logical justification for arguments. It is common that students experience difficulties formalizing a mathematical statement into a symbolic form. For example, students may understand that to find a least square solution to an inconsistent system $Ax=b$, one needs to replace b by $\hat{b}$, such that $\hat{b}$ is the "closest" to $ColA$. The challenge for students is two-fold: first, they have to reformulate this goal into mathematical statements, verbally or symbolically, such as $\hat{b} \in ColA$ and $b - \hat{b} \perp ColA$; and second they have to express these statements in terms of equation-based expressions, $\hat{b} = Ac$ for some vector c and $A^T(b - \hat{b}) = 0$. This latter step is typically challenging for students. Likewise, students may have an intuitive idea of what dimension is—usually in the context of 2- and 3-dimensional Euclidean spaces, but experience difficulty understanding the formalization of their intuition into a well-defined mathematical concept.

The fifth, and final, need is the *need for structure*. The common meaning of the term *structure* is something made up of a number of parts that are held or put together in a particular way. In mathematics the way these "parts" are held together are relations one conceives among different objects. For example, the expression $Ab = 0$ constitutes a structure for a person when he or she is conceives it as a string of symbols put together in a particular way to convey a particular meaning, such as 0 is a linear combination of the columns of A with the entries of b being the weights of the combination; or b is orthogonal to the row space of A.

In mathematics, in general, the need for structure manifests itself as a need to encapsulate (in the sense of APOS theory) occurrences of phenomena. For example, one might encapsulate a series of empirical observations concerning products of square matrices into the patterns, $\det(AB) = \det(A)\det(B)$ or $tr(AB) = tr(BA)$; another may derive such patterns through deduction or may observe them empirically but see a need to establish them deductively. In linear algebra, there is the critical need to encapsulate different structures into a single representation: a vector

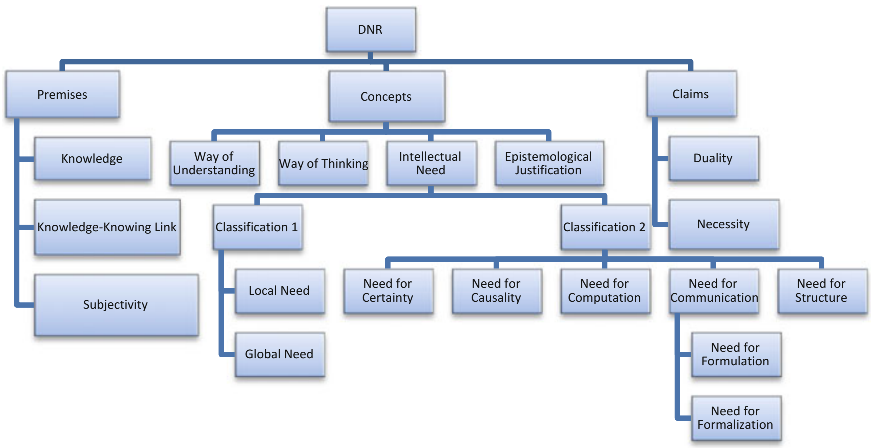


Fig. 6 Second classification of intellectual need

space over the reals as a single representation of all n-tuples of real number, of all polynomials of degree less or equal to n with real coefficients, of all $m \times n$ matrices with real entries, etc. This process of encapsulation assumes, of course, that members of each of these spaces are conceived as conceptual entities (in the sense of APOS theory and Greeno, 1992)) in, respectively, an n-dimensional, $n+1$-dimensional, and mn-dimensional vector space.

Figure 6 expands Fig. 5 to include the *DNR* constructs discussed in this section.

5 Categories of Epistemological Justification

We distinguish among three categories of epistemological justifications: *sentential, apodictic,* and *meta.* While the distinction among these types of epistemological justification is sufficiently clear, as we will now see, it should be noted that they are not mutually exclusive.

5.1 Sentential Epistemological Justification

Sentential epistemological justification (*SEJ*) refers to a situation when one is aware of how a definition, axiom, or proposition was born out of a need to resolve a problematic situation. It is called so because it pertains to *sentences* with objective and logical meaning. To illustrate, consider how linear algebra textbooks typically introduce the pivotal concepts of "eigenvalue," "eigenvector," and "matrix diagonalization". A widely used linear algebra textbook motivates these concepts by saying that the concepts of "eigenvalue" and "eigenvector" are needed to deal with the problem of factoring an $n \times n$ matrix A into a product of the form XDX^{-1}, where D is diagonal, and that this factorization would provide important information about A, such as its rank and determinant. Such an introductory statement aims at pointing out to the student an important problem. While the problem is intellectually intrinsic to its poser (a university instructor), it is most likely to be alien to a student in an elementary linear algebra course, who is unlikely to realize from such a statement the true nature of the problem, its mathematical importance, and the role the concepts to be taught ("eigenvalue," "eigenvector," and "diagonalization") play in solving it.

One of the alternative approaches to this presentation, based particularly on students' intellectual need for computation, is through linear systems of differential equations, which has been experimented successfully several times. In this approach, one begins with an initial-value problem (e.g., a mixture problem)