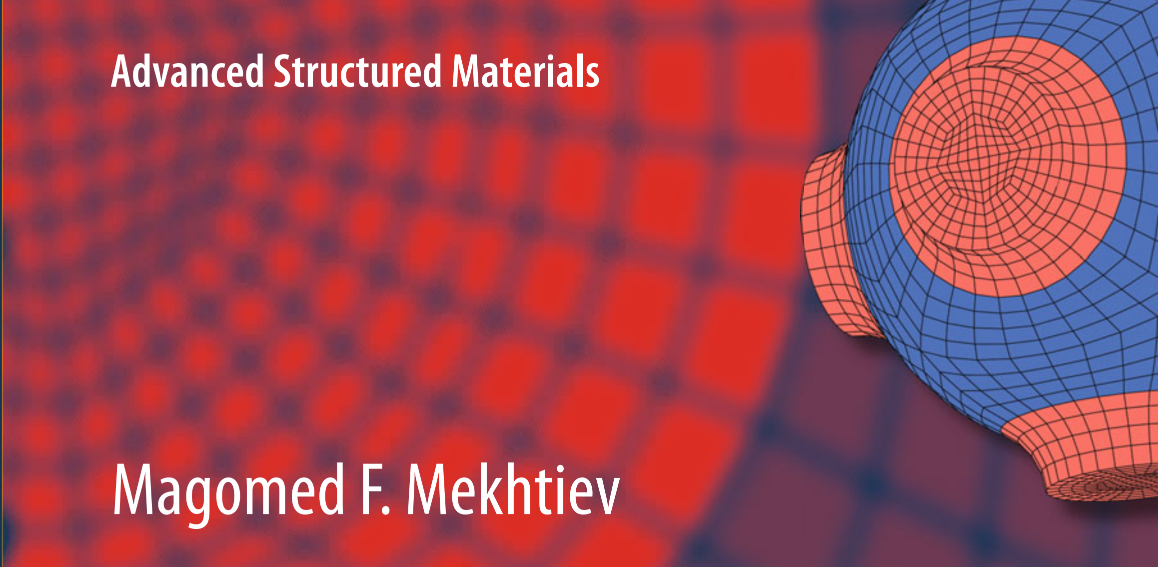

Advanced Structured Materials

Magomed F. Mekhtiev

# Vibrations of Hollow Elastic Bodies

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# Vibrations of Hollow Elastic Bodies

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# Preface

Shell theory is one of the most important fields of modern mechanics. It develops methods for calculating thin-walled structures which are widely used in modern plants and machinery. Requirements of strength, lightness and efficiency of a modern design make thin shells indispensable constructional elements. However, calculation of shells (calculation of stress-strained state of shells) on the basis of three-dimensional equations of elasticity theory involves considerable mathematical difficulties. That is why we have to resort to various approximate methods simplifying the calculation of shells. Here, a fundamental geometrical assumption, i.e. the smallness of the thickness of the shell to its remaining two dimensions, is taken into account. Namely, the problem of reduction of a three-dimensional problem of elasticity theory to a two-dimensional problem is the main content of shell theory. Obviously, there are many possible ways of transforming the problems of elasticity theory to the problems of shell theory for thin-walled structures such as shells. The main relevant results concerning mostly static problems of shell theory are consecrated in Vorovich's (1966, 1975), Goldenveyzer's (1969, 1975), Koiter and Simmonds (1973), Sayir and Mitropoulos (1980), Petraszkievicz (1992) reviews. The problem of reduction of a three-dimensional dynamic problem of elasticity theory to a two-dimensional problem of shell theory is described in particular in the works of Aynola and Nigul (1965) and Grigolyuk and Selezov (1973), Achenbach (1969), Berdichevskii and Khan'Chau (1980), Goldenveizer et al. (1993). Here, we also note the monographs by Kilchevsky's (1963), Berdichevskii (1983), Kaplunov et al. (1998), Le (2012), Aghalovyan (2015) where these issues are rather thoroughly discussed and where a bibliography on the discussed topics can also be found. As it is hardly possible to find a detailed literature survey on the topics mentioned above, it is both practically impossible and there is no need to give an overview of the results on shell theory carried out at different times by means of different methods.

Many of the major achievements in the field of plates and shells are obtained by scientists from the CIS (the Commonwealth of Independent States, or former USSR). The most significant contributions to the development of the foundations of the classical shell theory were made by S. A. Ambartsumyan, V. V. Bolotin, I. N. Vekua,

V. Z. Vlasov, I. I. Vorovich, B. G. Galerkin, K. Z. Galimov, A. L. Goldenveyzer, E. I. Grigolyuk, N. A. Kilchevsky, A. I. Lure, H. M. Mushtari, V. V. Novozhilov, P. M. Ogibalov, Yu. N. Rabotnov, V. V. Sokolovsky, S. P. Timoshenko, K. F. Chernyh, P. M. Nahdi, E. Reissner and others.

It is clear from the ongoing discussion that the modern theory is a well-developed branch of solid mechanics. However, the process of the development of the theory of shells cannot be regarded as a complete one. First of all, there constantly appear new technology designs the calculation of which is impossible in the framework of the existing versions of the theory of shells.

Therefore, the emergence of new variants of the theory of shells is inevitable. One can surmise that the number of such nontraditional structures will increase. It is connected with the modern technology achievement of an extremely high level of exploitational parameters.

On the other hand, the theory of shells as a theory must be internally consistent. As is known, the classical theory of shells is not deprived of contradictions.

Therefore, the major problem of the theory of shells is the formulation of different variants of boundary value problems and evaluation of their applicability scale. The problem, of course, is not only of theoretical interest but of great practical importance. Over many years of development in this particular field of elasticity theory, a considerable amount of material accumulated both on formulation of different variants of applied theories together with the development of solution (calculation) methods. At various stages of its development, the state of the theory of shells was subjected to some critical analysis. A closer look at the current state of the theory reveals a need for additional research on the comparative analysis of various applied theories established in the domain of their applicability.

Below we will consider the ways of bringing the two-dimensional problems to the three-dimensional ones using the smallness of shell thickness in their constructions compared to its other dimensions. Among these methods, the asymptotic method takes a special place. Asymptotic approach, probably for the first time, was applied to the problems of shell theory by Shterman (1924) and later—by Krauss (1929). Asymptotic integration of the equations of two-dimensional theory of shells undergone a great development in the works of A. L. Goldenveyzer (since 1939). The combination of complex transformation of equations of V. V. Novozhilov's shell theory with the asymptotic methods is represented in Chernykh's works (1962, 1964).

Thus, the asymptotic method developed by A. L. Goldenveyzer, I. I. Vorovich and their students made a significant contribution to the development of the theory of plates and shells. This method proved to be very effective in the study of problem of the limiting transition from three-dimensional elasticity problems into two-dimensional ones. Due to works of authors mentioned above, it became possible to solve such important issues as the establishment of the range of applicability of the applied theories of plates and shells, in particular the classical Kirchhoff–Love's theory. Further development of this method allowed to create effective methods for calculating the three-dimensional stress state and to solve practically important problems of stress concentration at the holes in plates and shells of constant thickness in the static case.

However, the question of the relationship between two-dimensional theories and the corresponding three-dimensional problems of elasticity theory for plates and shells of variable thickness has not practically been studied.

The problem of passage to the limit in dynamic problems of elasticity theory is particularly critical. Since there are now a number of dynamic applied theories of shells based on various hypotheses, lack of data on the comparative analysis raises the question of establishing the range of applicability of each of them on the basis of three-dimensional dynamic theory of elasticity. Here, the most important point is the question of determining the natural frequencies and the forms of vibrations of shells from the position of three-dimensional theory of elasticity.

Many of the difficulties associated with the study of these problems are caused by the presence of several parameters in the original problems. For example, the interplay of such parameters as relative thickness, curvature, oscillation frequency, curvature of the holes even for sufficiently smooth external loads can generate a fairly complex stress–strain state which is impossible to calculate correctly not only within the classical theory but also in such approved revised theories as the theory of S. P. Timoshenko, V. Z. Vlasov and others.

In addition, the asymptotic method enables to effectively solve boundary value problems for such elastic bodies that cannot be attributed to shells (e.g. thick cylindrical or conical rings) and for which at the same time the spatial theory is powerless because of the proximity of the parameters describing the boundary surfaces.

This monograph is devoted to these range of questions.

The book consists of four chapters. In the first chapter, an axisymmetric dynamic problem of elasticity theory for a hollow cylinder is investigated by the method of homogeneous solutions. Homogeneous solutions depending on the roots of the dispersion equation are constructed. The classification of the roots of the dispersion equation is presented.

The basis of the classification procedure is at the order of a root with respect to a small parameter $\varepsilon$ characterizing the thinness of the shell depending on the frequency of the driving forces. It is shown that in the high-frequency domain (in terms of spatial problems) in the first term of the asymptotic behaviour, the dispersion equation coincides with the well-known Rayleigh–Lamb equation for an elastic strip.

The classification of homogeneous solutions has been conducted and it is shown that each group of roots of the dispersion equation corresponds to its type of homogeneous solutions. Asymptotic expansions of the homogeneous solutions, allowing the calculation of the stress–strain state at different values of the frequency of the driving forces, are obtained.

A generalized condition of orthogonality of homogeneous solutions for a hollow cylinder which allows the accurate solution of the problem of forced vibrations of a hollow cylinder for certain end boundary conditions is proved. In the general case of the end loading, the original boundary value problem is reduced to the solution of infinite system of linear algebraic equations by means of the Lagrange variational principle.

Also presented is a method for constructing applied theories intended to relieve stresses from cylindrical boundaries of a shell. Together with homogeneous solutions, they permit the solution of the inhomogeneous problem.

The importance of the evaluation of boundary conditions in the formation of the spectrum and vibration shapes of a shell is of great interest in dynamic problems of elasticity theory. Therefore, practically all boundary conditions which can be stated in three-dimensional theory of elasticity are studied. In particular, some solutions of the problem of forced vibrations of a hollow cylinder with a clamped side surface are presented. It is shown that the solution of this problem in the first term of its asymptotic expansion coincides with the known solution for the elasticity theory for an elastic strip. We also discuss the problem of torsional vibrations and the vibrations under mixed boundary conditions on the side surface of a cylinder. These problems appeared to be very simple both from a physical and a mathematical point of view. In fact, these problems, in mathematical terms, are reduced to solving boundary value problems for Helmholtz equation.

The second chapter deals with a three-dimensional dynamic problem of elasticity theory for a spherical layer. Homogeneous solutions are derived for the case of axisymmetric vibrations. In the case of axisymmetric vibrations, homogeneous solutions are constructed. One way of constructing of the heterogeneous solutions is pointed out. An asymptotic analysis of homogeneous solutions for a spherical shell corresponding to different groups of roots of the dispersion equation is performed.

It is shown that in contrast to a cylindrical shell, ultra-low frequency vibrations for a spherical shell are not available. A generalized condition of orthogonality of systems of homogeneous solutions is proved. One class of boundary conditions on the side surface, admitting an exact solution of the problem of forced vibrations of a spherical shell is identified. In the case of a general loading by Hamilton's variational principle, the boundary value problem is reduced to solving an infinite system of linear algebraic equations. Matrices of such systems are known for a spherical shell in the static case, and for an elastic strip in the dynamic elasticity problems. The problem of torsional vibrations of a spherical shell is solved analytically. The need for a detailed discussion of this problem is revealed when considering non-axisymmetric elasticity problems.

Non-axisymmetric dynamic problem of elasticity theory for a spherical layer is considered. Due to spherical symmetry, the general boundary value problem is divided into two problems one of which coincides with the boundary value problem for axisymmetric vibrations of a hollow sphere, and the second one describes the vortex motion of a hollow sphere and coincides with the boundary value problem for purely torsional vibrations of a hollow sphere.

The third chapter provides an asymptotic process for finding the frequencies of free axisymmetric vibrations of an isotropic hollow cylinder and a closed hollow sphere based on the dynamic equations of elasticity theory. An asymptotic process is thoroughly built for a cylinder with free side surfaces and with simply supported at the ends and for a closed hollow sphere with free facial surfaces. These problems are considered to be a model, since the study of the asymptotic processes for other boundary conditions has no fundamental difficulties.

A comparison of the results obtained in Kirchhoff–Love theory with the results obtained by the three-dimensional elasticity theory is given. For a cylinder and a sphere, there are obtained two frequencies in the first term of the asymptotic expansions, coinciding with the frequencies determined by the application of shell theory, and a countable set of frequencies which are not available in the applied theory of shells. The frequencies of the thickness vibrations of cylindrical and spherical shells are determined.

The fourth chapter is dedicated to the development of the asymptotic method of integrating three-dimensional equations of elasticity theory for a conical shell and a plate of variable thickness and the analysis of three-dimensional stress–strain state on the basis of this method.

In the first part of the fourth chapter, the solution of the problem of elasticity theory for a truncated hollow cone of variable thickness is obtained by the method of homogeneous solutions.

An asymptotic analysis of the characteristic equation establishes the existence of three groups of zeros with the following asymptotic properties: $\lambda = O(1)$, $\lambda = O\left(\varepsilon^{-1/2}\right)$ $\lambda = O(\varepsilon^{-1})$ ($\varepsilon$ is a parameter of thin-walledness) each of which corresponds to its type of stress–strain state.

The first group of zeros corresponds to a penetrating solution coinciding with the known Mitchell-Neuber's solution. The stress state, defined by this solution, is equivalent to the resultant vector of forces applied to one end of the shell.

The second group of zeros corresponds to the solution of end-effect type similar to the end effect of the applied theory of shells. The first terms of the asymptotic expansions of the stress state obtained through this solution are equivalent to the bending moment and shearing forces.

The third group of zeros corresponds to the solution of boundary layer type which in the first term of the asymptotic expansion coincides with Saint-Venant's end effect in the theory of thick plates. Using the principle of Lagrange virtual displacements, the boundary value problem is reduced to solving an infinite system of linear algebraic equations. Matrices of these systems are known in the theory of thick plates of constant thickness. Their inversion can be achieved by using the reduction method. A method for constructing applied theories intended to relieve stresses from conical shell boundaries is shown.

The second part of the fourth chapter investigates the asymptotic behaviour of the axisymmetric stress–strain state of the plate, the thickness of which is $h = \varepsilon r$, where $r$ is the distance from the centre of the plate, and $\varepsilon$ is still a small parameter. Here, we are not talking about an arbitrary plate but the particular form of the conical shell discussed in the first part of the fourth chapter which it takes during the degeneration of its midsurface into a flat one. Since this is a special case of degeneration, all the arguments of the previous chapters have to be repeated.

When constructing refined applied theories for the plates of variable thickness, instead of the traditional linear-independent solutions $P_\nu(\cos\theta)$, $Q_\nu(\cos\theta)$ of Legendre's equation, we introduce, for convenience, another set of linearly independent solutions of Legendre equation $T_\nu(\theta) = P_\nu(\cos\theta) + P_\nu(-\cos\theta)$ and

$F_\nu(\theta) = P_\nu(\cos\theta) - P_\nu(-\cos\theta)$ which are respectively odd and even functions with respect to the midplane of the plate. The chosen form of solutions makes it possible to divide the general problem into two independent ones: the problem of the tension–compression of a plate and the plate bending problem. Such a division greatly simplifies the process of building refined applied theories for plates of variable thickness.

Also considered are the problems of equilibrium of an elastic hollow cone with a fixed side surface and with mixed boundary conditions on the side surface.

It is shown that in the case of fixed side surface, the solution of this problem in the first term of the asymptotic expansion coincides with the known solution for the elasticity theory for an elastic strip. A generalized orthogonality condition for a hollow cone is proved.

Proceeding from Papkovich–Neuber's general solution, a stress–strain state of a plate with variable thickness subject to the action of non-axisymmetric loads is studied.

The behaviour of the solution as the parameter of thin-walledness tends to zero is investigated. It is shown that the stress–strain state of a plate consists of the stress–strain state penetrating deep into the plate and the end effect similar to Saint-Venant's end effect. Kirsch's problem for plates of variable thickness is solved.

In the last section of the fourth chapter, torsional vibrations of a conical shell and a plate of variable thickness are discussed. At first, the problem is solved exactly. Then an asymptotic analysis of the problem of harmonic torsional waves spreading in a conical shell and plate of variable thickness is given. Depending on the frequency of the driving forces, a form of wave formation is studied. Asymptotic formulas determining the frequency of torsional vibrations of a conical shell and a plate of variable thickness are obtained.

The derived homogeneous and inhomogeneous solutions not only reveal the qualitative features of three-dimensional solutions in shell theory but they can serve as an effective device of solving specific boundary value problems, as well as a basis for the assessment of simplified theories. As in the general case of loading, the solutions of dynamic and static elasticity problems are reduced to solving infinite systems of linear algebraic equations. So, in the appendix, a solution of the axisymmetric problem on stress concentration around a circular hole in the plate whose boundary is loaded with normal forces of the form $\sigma_r = \chi\left(\eta^p - k_p\eta\right)$ is presented. Here $\chi$ is a constant, $k_p$ is a parameter that has been selected so that the load is self-balanced, $\eta$ is a transverse coordinate. Numerical results for the solution of the problem is presented. This problem can be regarded as a model for the corresponding problems in the theory of shells; it is relatively simple and at the same time contains all the characteristic features of problems in three-dimensional elasticity theory. The exact solution of the problem of axisymmetric vibrations of a cylindrical shell under given mixed boundary conditions at the ends is derived by means of homogeneous solutions. Numerical analysis is carried out for the parabolic distribution of normal stress on the end surface and zero end radial displacement. Some numerical analysis of the problem of dynamic torsion of a spherical layer by forces distributed on the surface of a tapered cut is conducted.

Summarizing the results, we note that the following results of the author are obtained in the monograph:

1. The forced vibrations of a cylinder and spherical layers are investigated by the method of homogeneous solutions. A possible form of wave formation is studied depending on the frequency of the driving forces. A complete asymptotic analysis of solutions of three-dimensional dynamic problems of elasticity theory is conducted as the parameter of thin-walledness tends to zero. A comparison of the asymptotic solution with the solutions obtained by the applied theories of shells is given. A generalized orthogonality condition of homogeneous solutions which allows accurate solutions to the problem of forced vibrations of a hollow cylinder and a spherical layer under specific conditions of shell end bearings are derived. In the general case of loading, the boundary value problem is reduced to solving an infinite system of linear algebraic equations employing Lagrange and Hamilton variational principles.
2. A qualitative study of some applied theories is considered; the limits of their applicability are determined. In particular, it is shown that all existing applied theories of shells inadequately describe the stress–strain state in the vicinity of concentrates and are not suited for the study of high-frequency vibrations of thin and thick shells.

   The detailed study of the properties of homogeneous solutions, in fact, of independent wave motion types, which may be realized in the considered elastic bodies, provides the basis for stating the correct methodological problem of constructing refined theories for thin-walled elements. In this connection, refined applied theories, more accurately describing the processes occurring in thin shells rather than the classical two-dimensional theory of shells, are constructed and these allow obtaining the solutions of inhomogeneous problems to a given degree of accuracy.
3. An asymptotic process for finding the frequencies of free axisymmetric vibrations of a hollow cylinder and a closed hollow sphere is derived. As is well known, even in relatively simple cases the analysis of the frequency equations is quite challenging. It is, therefore, essential to determine all the frequencies in a certain frequency range.

   The author's approaches allow to create algorithms that are able to capture all natural frequencies in a given interval, and this undoubtedly represents a scientific and practical value.

   Comparisons of the results obtained by Kirchhoff–Love and Timoshenko's theories with the results obtained by the three-dimensional elasticity theory are presented. It is shown that in the problems on free vibrations, the applied theory of shells approximately approximates only the lowest part of the frequency spectrum, but is unable to describe the phenomenon of end resonance.
4. An asymptotic method of integrating three-dimensional equations of elasticity theory for a conical shell and plates of variable thickness is developed. Homogeneous and inhomogeneous solutions are established and a generalized orthogonality condition for a cone is proved. Asymptotic analysis of the

problems of harmonic torsional waves spreading in a conical shell and in a plate of variable thickness is conducted through which asymptotic formulas are obtained that allow the determination of the frequencies of the mentioned bodies.

5. It is proved that the derived homogeneous and inhomogeneous solutions not only reveal the qualitative features of the three-dimensional solutions in the theory of shells, but can serve as an effective technique of solving specific boundary value problems, as well as a basis for assessing simplified theories.

Baku, Azerbaijan Magomed F. Mekhtiev

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# About the Book

The book is inspired by justification and refinement of highly diverse approximate dynamic models for engineering structures arising in modern technology, including high-tech domains dealing with nano and metamaterials. The dynamic equations of 3D elasticity are applied to the analysis of harmonic vibrations of hollow bodies of canonical shapes. New exact homogeneous and inhomogeneous solutions are derived for cylinders, spheres, cones, including spherical and conical layers, as well as for plates of variable thicknesses. Associated dispersion relations are subject to detailed asymptotic treatment in case of a small thickness. A classification for vibration spectra is suggested over a broad frequency domain. The range of validity of various existing 2D theories for thin-walled shells is evaluated by comparison with 3D benchmark solutions. A number of numerical examples are presented. Refined versions of 2D dynamic formulations are developed. Boundary value problems for hollow bodies are also considered.

# Contents

# About the Author

**Magomed F. Mekhtiev** graduated from the Department of Mechanics-Mathematics of Baku State University. He defended his Ph.D. thesis at the chair of elasticity theory of Rostov State University and there. In 1989, he defended his Doctoral thesis in Leningrad (St. Petersburg) State University. During 1966–1991, he occupied various positions in the Institute of Mechanics and Mathematics of National Academy of Sciences of Azerbaijan. He has worked in Baku State University since 1991. In 1994, he became professor. Scientific research direction of Prof. M. F. Mekhtiyev is mathematical methods of solid mechanics and qualitative questions of optimal control. He has published over 120 scientific papers and two monographs in this field. M. F. Mekhtiyev is awarded with gold medal by Scientific-Industrial Chamber of European Union. At present, he is Dean of the Department of Applied Mathematics and Cybernetics and heads the chair of Mathematical Methods of Applied Analysis.

# Chapter 1
# Asymptotic Analysis of Dynamic Elasticity Problems for a Hollow Cylinder of Finite Length

**Abstract** In this chapter, we investigate forced vibrations of an isotropic hollow cylinder under the action of axisymmetric loads by the method of homogeneous solutions. Depending on the frequency of the driving forces a possible form of wave formation in a hollow cylinder is explored. The asymptotic behaviour of the solutions of three-dimensional dynamic problems of elasticity theory is studied as the wall-thickness parameter tends to zero hence corresponding to a thin walled structure. The comparison of the asymptotic solutions with the solutions obtained by the applied theories is given. A generalized orthogonality condition of homogeneous solutions is proved which allows an accurate solution of the problem of forced vibrations of a hollow cylinder with mixed end conditions. In the general case of loading of a cylinder by means of the Lagrange variational principle the boundary value problem is reduced to the solution of a system of linear algebraic equations. A method of constructing applied theories designed for stress relief from cylindrical boundaries of the shell is suggested. Together with the construction of the homogeneous solutions the solution of the inhomogeneous problem follows. The problems of torsional vibrations of a hollow cylinder with mixed boundary conditions on the side surface are solved exactly. The vibrations of a cylinder with a fixed side surface are considered as well. It is shown that in the first term of the asymptotic expansion the solution of this problem coincides with the solution of a similar problem in the theory of elasticity for an elastic strip.

## 1.1 Construction of Homogeneous Solutions

Consider the axisymmetric problem of elasticity theory for a hollow cylinder. The position of the points on the cylinder in space is defined by the cylindrical coordinates $r$, $\varphi$, $z$ varying within (Fig. 1.1).

$$R_1 \le r \le R_2,\ 0 \le \varphi \le 2\pi,\ -l \le z \le l \tag{1.1.1}$$

It is assumed that the lateral surface of the cylinder is free from stresses, i.e.



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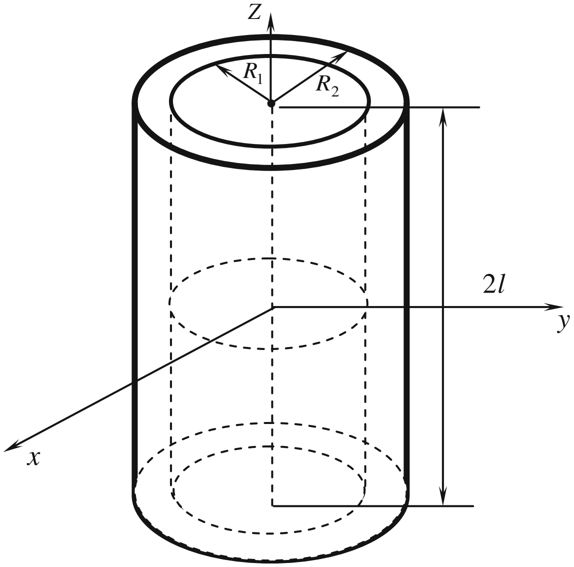
**Fig. 1.1** Hollow cylinder




$$\sigma_r = 0,\ \tau_{rz} = 0$$

at

$$r = R_n, \quad -l \le z \le l, \quad (n = 1, 2) \tag{1.1.2}$$

while the rest of the boundary conditions are given as follows

$$\sigma_z = Q^{\pm}(r)e^{i\omega t}, \tau_{rz} = T^{\pm}(r)e^{i\omega t}$$

at

$$z = \pm l\,(k = 1, 2). \tag{1.1.3}$$

The equations of motion in terms of displacements in a cylindrical coordinate system have the form:

$$\begin{gathered}\frac{1}{1-2v}\frac{\partial x}{\partial \rho} + \Delta U_\rho - \frac{1}{\rho^2}U_\rho = \frac{2g(1+v)R_0^2}{E}\frac{\partial^2 U_\rho}{\partial t^2} \\ \frac{1}{1-2v}\frac{\partial x}{\partial \xi} + \Delta U_\xi = \frac{2g(1+v)R_0^2}{E}\frac{\partial^2 U_\xi}{\partial t^2} \\ x = \frac{\partial U_\rho}{\partial \rho} + \frac{U_\rho}{\rho} + \frac{\partial U_\xi}{\partial \xi}.\end{gathered} \tag{1.1.4}$$

Here $\rho = R_0^{-1}r$, $\xi = R_0^{-1}z$ are dimensionless coordinates, $R_0 = 1/2(R_1 + R_2)$ is the radius of the mid-surface of the shell, $E$ is Young's modulus, $v$ is Poisson's ratio, $g$ is the density of the shell material, $\Delta$ is Laplace operator, and $U_\rho = R_0^{-1}U_z$, $U_\xi = R_0^{-1}U_z$ are the nondimensional displacements.

Stress tensor components are expressed by means of the displacements as follows:

$$\sigma_r = 2G\left(\frac{\partial U_\rho}{\partial \rho} + \frac{v}{1+2v}x\right), \quad \sigma_\varphi = 2G\left(\frac{U_\rho}{\rho} + \frac{v}{1-2v}x\right)$$
$$\sigma_z = 2G\left(\frac{\partial U_\xi}{\partial \xi} + \frac{v}{1-2v}x\right), \quad \tau_{rz} = 2G\left(\frac{\partial U_\xi}{\partial \rho} + \frac{\partial U_\rho}{\partial \xi}\right) \tag{1.1.5}$$

where G is the shear modulus.

The solutions of (1.1.4) will be sought in the form:

$$U_\rho = U(\rho)\frac{dm}{d\xi}\,e^{i\omega t},\ U_\xi = W(\rho)m(\xi)\,e^{i\omega t} \tag{1.1.6}$$

where the function $m(\xi)$ is subject to the condition

$$\frac{d^2m}{d\xi^2} - \mu^2 m(\xi) = 0 \tag{1.1.7}$$

for which the parameter $\mu$ is determined through the fulfilment of the boundary conditions on the surface.

Substituting (1.1.6) into (1.1.4) and separating variables relative to a pair of functions $(U, W)$ we obtain the following system of ordinary differential equations:

$$\begin{aligned} &L_1(\mu,\lambda)(U,W) = U'' + \tfrac{1}{\rho}U' \\ &+ \left(\alpha^2 - \tfrac{1}{\rho^2}\right)U + \tfrac{1}{2(1-v)}\left(W' - \mu^2 U\right) = 0 \\ &L_2(\mu,\lambda)(U,W) = \tfrac{1}{1-2v}\mu^2\left(U' + \tfrac{U}{\rho} + W\right) \\ &+ W'' + \tfrac{1}{\rho}W' + \gamma^2 W = 0 \end{aligned} \tag{1.1.8}$$

$$\lambda^2 = \frac{2g(1+v)R_0^2\omega^2}{E},\ \alpha^2 = \mu^2 + \frac{1-2v}{2(1-v)}\lambda^2,\ \gamma^2 = \mu^2 + \lambda^2.$$

here primes denote derivatives with respect to $\rho$ and $\lambda$ is a frequency parameter.

Taking into account (1.1.6), the formulas (1.1.5) take the form:

$$\begin{aligned} \sigma_r &= 2G\left[U' + \frac{v}{1-2v}\left(U' + \frac{U}{\rho} + W\right)\right]\frac{dm}{d\xi}e^{i\omega t} \\ \sigma_\varphi &= 2G\left[\frac{U}{\rho} + \frac{v}{1-2v}\left(U' + \frac{U}{\rho} + W\right)\right]\frac{dm}{d\xi}e^{i\omega t} \\ \sigma_z &= 2G\left[W + \frac{v}{1-2v}\left(U' + \frac{U}{\rho} + W\right)\right]\frac{dm}{d\xi}e^{i\omega t} \\ \tau_{rz} &= G\left(\mu^2 U + W'\right)m(\xi)e^{i\omega t}. \end{aligned} \tag{1.1.9}$$

Substituting (1.1.9) into (1.1.2) we obtain the following homogeneous boundary conditions for the functions $U(\rho,\mu,\lambda)$, and $W(\rho,\mu,\lambda)$:

$$
\begin{aligned}
&M_1(\mu,\lambda)(U,W)|_{\rho=\rho_n}\\
&=\left[U'+\tfrac{\nu}{1-2\nu}\left(U'+\tfrac{U}{\rho}+W\right)\right]_{\rho=\rho_n}=0,
\end{aligned}
$$

$$
M_2(\mu,\lambda)(U,W)|_{\rho=\rho_n}=\left[W'+\left(W'+\mu^2U\right)\right]_{\rho=\rho_n}=0. \tag{1.1.10}
$$

Thus, the system of Eq. (1.1.8) together with boundary conditions (1.1.10) generates a spectral problem for a pair of functions $(U,\ W)$. Let us undertake the analysis of the suggested spectral problem. Without going into details, we give the final solution of Eq. (1.1.8) in the following form:

$$
\begin{aligned}
U(\rho,\mu,\lambda)&=-\alpha z_1(\alpha\rho)-z_1(\gamma\rho)\\
W(\rho,\mu,\lambda)&=\mu^2z_0(\alpha\rho)+\gamma z_0(\gamma\rho).
\end{aligned} \tag{1.1.11}
$$

Here $z_k(x)=C_{1k}J_k(x)+C_{2k}Y_k(x),\ J_0(x),\ J_1(x),\ Y_0(x)$ are Bessel functions of the first and second kind respectively; $C_i(i=1,2,3,4)$ are arbitrary constants.

On satisfying homogeneous boundary conditions (1.1.10) we obtain a linear system of algebraic equations in the unknowns $C_i$:

$$
\begin{aligned}
&\left[\tfrac{\alpha}{\rho}Z_1(\alpha\rho)-\delta^2Z_0(\alpha\rho)+\tfrac{1}{\rho}Z_1(\gamma\rho)-\gamma Z_0(\gamma\rho)\right]_{\rho=\rho_n}=0\\
&\left[2\mu^2\alpha Z_1(\alpha\rho)+\left(2\mu^2+\lambda^2\right)Z_1(\gamma\rho)\right]_{\rho=\rho_n}=0
\end{aligned} \tag{1.1.12}
$$

$$
\delta^2=\mu^2+1/2\lambda^2
$$

The condition for the existence of nontrivial solutions of system (1.1.12) leads to the following dispersion equation:

$$
\begin{aligned}
\Delta(\mu,\lambda)=\ &8\pi^{-2}\rho_1^{-1}\rho_2^{-1}\mu^2\left(2\mu^2+\lambda^2\right)^2-\lambda^4\alpha^2\rho_1^{-1}\rho_2^{-1}\\
&\times L_{11}(\alpha)L_{11}(\gamma)+2^{-1}\alpha\lambda^2\left(2\mu^2+\lambda^2\right)^2\rho_2^{-1}L_{01}(\alpha)L_{11}(\gamma)\\
&+2^{-1}\alpha\lambda^2\left(2\mu^2+\lambda^2\right)^2\rho_1^{-1}L_{10}(\alpha)L_{11}(\gamma)-2\lambda^2\gamma\mu^2\alpha^2\\
&\times\rho_1^{-1}L_{10}(\gamma)L_{11}(\alpha)-2\lambda^2\gamma\mu^2\alpha^2\rho_2^{-1}L_{01}(\gamma)L_{11}(\alpha)\\
&-4^{-1}\left(2\mu^2+\lambda^2\right)^4L_{00}(\alpha)L_{11}(\gamma)-4\mu^4\alpha^2\gamma^2\\
&\times L_{00}(\gamma)L_{11}(\alpha)+\alpha\gamma\mu^2\left(2\mu^2+\lambda^2\right)^2\\
&\times\left[L_{01}(\gamma)L_{10}(\alpha)+L_{01}(\alpha)L_{10}(\gamma)\right]=0\\
L_{ii}(x)=\ &J_i(x\rho_1)Y_i(x\rho_2)-J_i(x\rho_2)Y_i(x\rho_1)\\
L_{ij}(x)=\ &J_i(x\rho_1)Y_i(x\rho_2)-J_j(x\rho_2)Y_i(x\rho_1)\\
&i,j=0,\ 1
\end{aligned} \tag{1.1.13}
$$

The transcendental Eq. (1.1.13) defines a countable set of roots $\mu_k$ and the corresponding constants $C_1\mu_k$, $C_2\mu_k$, $C_3\mu_k$, $C_4\mu_k$ are proportional to the cofactors of the elements of any row of the determinant of the system. If the matrix of the linear system (1.1.13) is expanded in terms of cofactors in the first row we obtain:

$$\begin{aligned}
C_1\mu_k &= C_k\big[4\pi^{-1}\rho_2^{-1}\alpha_k\mu_k^2\big(2\mu_k^2+\lambda^2\big)Y_1(\alpha_k\rho_1)\\
&\quad -\alpha_k\lambda^2\big(2\mu_k^2+\lambda^2\big)\rho_2^{-1}Y_1(\alpha_k\rho_2)L_{11}(\gamma_k)+2^{-1}\big(2\mu_k^2+\lambda^2\big)^3\\
&\quad \times Y_0(\alpha_k\rho_2)L_{11}(\gamma_k)-2\alpha_k\gamma_k\mu_k^2\big(2\mu_k^2+\lambda^2\big)Y_1(\alpha_k\rho_2)L_{10}(\gamma_k)\big]\\
C_2\mu_k &= C_k\big[4\pi^{-1}\rho_2^{-1}\alpha_k\mu_k^2\big(2\mu_k^2+\lambda^2\big)J_1(\alpha_k\rho_1)\\
&\quad -\alpha_k\lambda^2\big(2\mu_k^2+\lambda^2\big)\rho_2^{-1}J_1(\alpha_k\rho_2)L_{11}(\gamma_k)+2^{-1}\big(2\mu_k^2+\lambda^2\big)^3\\
&\quad \times J_0(\alpha_k\rho_2)L_{11}(\gamma_k)-2\alpha_k\gamma_k\mu_k^2\big(2\mu_k^2+\lambda^2\big)J_1(\alpha_k\rho_2)L_{10}(\gamma_k)\big]\\
C_3\mu_k &= C_k\big[2\pi^{-1}\rho_2^{-1}\mu_k^2\big(2\mu_k^2+\lambda^2\big)Y_1(\gamma_k\rho_1)\\
&\quad +2\rho_2^{-1}\lambda^2\mu_k^2\alpha_kY_1(\gamma_k\rho_2)L_{11}(\alpha_k)+4\gamma_k\mu_k^4\alpha_k^2Y_0(\gamma_k\rho_2)L_{11}(\alpha_k)\\
&\quad -\alpha_k\mu_k^2\big(2\mu_k^2+\lambda^2\big)^2Y_1(\gamma_k\rho_2)L_{10}(\alpha_k)\big]\\
C_4\mu_k &= C_k\big[2\pi^{-1}\rho_2^{-1}\mu_k^2\big(2\mu_k^2+\lambda^2\big)J_1(\gamma_k\rho_1)\\
&\quad +2\rho_2^{-1}\lambda^2\mu_k^2\alpha_kJ_1(\gamma_k\rho_2)L_{11}(\alpha_k)+4\gamma_k\mu_k^4\alpha_k^2J_0(\gamma_k\rho_2)L_{11}(\alpha_k)\\
&\quad -\alpha_k\mu_k^2\big(2\mu_k^2+\lambda^2\big)^2J_1(\gamma_k\rho_2)L_{10}(\alpha_k)\big]
\end{aligned} \tag{1.1.14}$$

Substituting (1.1.14) into (1.1.11), summing through all roots and taking into account formulas (1.1.6) and (1.1.9) we obtain the homogeneous solutions in the following form:

$$\begin{aligned}
U_\rho &= \sum_{k=1}^{\infty} C_kU_k(\rho)\frac{dm_k}{d\xi}e^{i\omega t}\\
U_\xi &= \sum_{k=1}^{\infty} C_kW_k(\rho)m_k(\xi)e^{i\omega t}\\
\sigma_r &= 2G\sum_{k=1}^{\infty} C_kQ_{rk}(\rho)\frac{dm_k}{d\xi}e^{i\omega t}\\
\sigma_\varphi &= 2G\sum_{k=1}^{\infty} C_kQ_{\phi k}(\rho)\frac{dm_k}{d\xi}e^{i\omega t}\\
\sigma_z &= 2G\sum_{k=1}^{\infty} C_kQ_{zk}(\rho)\frac{dm_k}{d\xi}e^{i\omega t}\\
\tau_{rz} &= G\sum_{k=1}^{\infty} C_k\tau_k(\rho)m_k(\xi)e^{i\omega t}
\end{aligned} \tag{1.1.15}$$

Here $C_k$ are arbitrary constants.

$$\begin{aligned} U_k(\rho) &= \alpha_k Z_1(\alpha_k\rho) - Z_1(\gamma_k\rho), \\ W_k(\rho) &= \mu_k^2 Z_0(\alpha_k\rho) - \gamma_k Z_0(\gamma_k\rho), \\ Q_{rk}(\rho) &= \frac{\alpha_k}{\rho} Z_1(\alpha_k\rho) - \delta_k^2 Z_0(\alpha_k\rho) + \frac{1}{\rho} Z_1(\gamma_k\rho) - \gamma_k Z_0(\gamma_k\rho), \\ Q_{\varphi k}(\rho) &= -\frac{\alpha_k}{\rho} Z_1(\alpha_k\rho) - \frac{v}{2(1-v)}\lambda^2 Z_0(\alpha_k\rho) - \frac{1}{\rho} Z_1(\gamma_k\rho), \\ Q_{zk}(\rho) &= \left[\mu_k^2 - \frac{v}{2(1-v)}\lambda^2\right] Z_0(\alpha_k\rho) + \gamma_k Z_0(\gamma_k\rho), \\ \tau_k(\rho) &= -2\mu_k^2\alpha_k Z_1(\alpha_k\rho) - \left(2\mu_k^2 + \lambda^2\right) Z_1(\gamma_k\rho). \end{aligned} \tag{1.1.16}$$

## 1.2 Analysis of the Roots of the Dispersion Equation

Let us undertake the analysis of the roots of the dispersion Eq. (1.1.13). As it is clearly seen from formula (1.1.12), the dispersion equation has a very complicated structure. For an effective study of the location of the roots of (1.1.12) we make some assumptions on the geometric parameters of the cylinder. Suppose:

$$\rho_1 = 1 - \varepsilon,\ \rho_2 = 1 + \varepsilon,\ 2\varepsilon = \frac{R_2 - R_1}{R_0} = \frac{2h}{R_0}. \tag{1.2.1}$$

Let us suggest that $\varepsilon$ is a small parameter. Substituting (1.2.1) into (1.1.13), we obtain

$$D(\mu, \lambda, \varepsilon) = \Delta(\mu, \lambda, \rho_1, \rho_2) = 0 \tag{1.2.2}$$

One can show that $D(\mu, \lambda, \varepsilon)$ is an even function of its arguments.

The case $\lambda_0^2 = gR_0^2\omega^2/E = 1$, $\lambda_0^2 = \frac{1}{1-v^2}$ and $\mu = 0$ is a particular case and is treated separately.

The following statement can be formulated with respect to the zeros of the function $D(\mu, \lambda, \varepsilon)$: the function $D(\mu, \lambda, \varepsilon)$ has three groups of zeros for finite $\lambda[\lambda = 0(1)\, as\, \varepsilon \to 0]$:

(a) The first group consists of two zeros $\mu_k = O(1)(k = 1, 2)$;
(b) The second group consists of four zeros at the order $O\left(\varepsilon^{-1/2}\right)$;
(c) The third group contains a countable set of zeros which are of the order $O(\varepsilon^{-1})$.

Let us prove this assertion. To do this we expand $D(\mu, \lambda, \varepsilon)$ into a series in powers of $\varepsilon$. On doing so we obtain: