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Preface

Set theory provides an excellent foundation for the field of mathematics; however, it suffers from Gödel's incompleteness phenomenon: There are important statements, such as the continuum hypothesis, that remain undecidable using the standard axioms. It is therefore of great value to find well-justified approaches to the discovery of new axioms of set theory.

The Hyperuniverse Project, funded by the John Templeton Foundation (JTF) from January 2013 until September 2015, was the first concerning Friedman's Hyperuniverse Programme, a valuable such approach based on the intrinsic maximality features of the set-theoretic universe. In the course of this project, the participants Carolin Antos, Radek Honzik, Claudio Ternullo and Friedman discovered an optimal form of "height maximality" and generated numerous "width maximality" principles which are currently under intensive mathematical investigation. The project also featured prominently in the important Symposia on the Foundations of Mathematics held in Vienna (7–8 July 2014, 21–23 September 2015) and London (12–13 January 2015); see https://sotfom.wordpress.com/.

The project resulted in 12 chapters, collected in this volume, which together provide the necessary background to gain an understanding of maximality in set theory and related topics.

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Introduction: On the Development of the Hyperuniverse Project

In brief, the Hyperuniverse Programme (HP) aims to generate mathematical principles expressing the maximality of the set-theoretic universe in height and in width, to analyse and synthesise these principles and ultimately to arrive at an optimal maximal principle whose first-order consequences can be regarded as intrinsically justified axioms of set theory.

The primary goal of the Templeton-funded project was to provide a robust and convincing philosophical justification for the Hyperuniverse Programme, which mainly consisted in providing intrinsic evidence for the acceptance of the maximality principles taken into consideration by the programme. And a secondary goal was to systematically formulate mathematical criteria of maximality for the set-theoretic universe and to develop the necessary mathematical tools for analysing them.

We achieved our primary goal, that of providing the HP with a firm foundation, and made significant progress with our secondary goal, the mathematical unfolding of the programme. However, it is now clear that the mathematical challenges for the advancement of the programme are even greater than we had imagined, although we are pleased with the very significant progress that we have made.

The Philosophical Grounding of the HP

At the start of the programme, we in fact considered a number of different features of the set-theoretic universe that might be regarded as "intrinsic".

However, we concluded that in fact the only feature for which there is a definitive case for intrinsicness is the maximality feature of V (= the universe of sets).

Maximality naturally breaks into two forms, height maximality and width maximality. Our initial approach was to treat them analogously, from both a height-potentialist and width-potentialist perspective. However, thanks to the input of several leading scholars in the philosophy of set theory, we later came to realise that the programme is most appropriately (although not exclusively) formulated as a height-potentialist and width-actualist programme.

Height potentialism was further analysed and developed in the Friedman-Honzik theory of sharp generation ("On Strong Reflection Principles in Set Theory", in this volume), what we feel to be the ultimate, strongest formulation of height maximality. However, width maximality presented a serious challenge, as, formally speaking, width actualism does not allow for the existence of thickenings (widenings) of V, blocking the easy formulation of width-maximality principles in which V is compared to wider universes. The resolution of this dilemma constituted a major new discovery of the project: the use of V-logic to internally express, consistently with width actualism, width-maximality principles which refer to possible thickenings of V. A further important point was to realise that the principles expressed in V-logic, although not first-order, are nevertheless first-order over a mild lengthening (heightening) of V called V+ (the least admissible universe past V) and of course such lengthenings are entirely permissible from a height-potentialist perspective. The reason that this point is important is that it then allows the use of the downward Löwenheim-Skolem theorem to reduce the study of width-maximality principles for V to their study over countable transitive models of set theory, quantifying solely over the collection of all countable transitive such models. The latter collection is what is termed the "Hyperuniverse", hence the name of the programme.

In this way, we feel that the HP is well-justified philosophically and its conceptual framework is sound. But of course there remains further work to be done from a philosophical perspective: How is one to justify the "synthesis" of initially conflicting maximality principles? How does one support the claim that the generation and analysis of further maximality principles will ultimately converge upon a single "optimal" maximality criterion? How can the programme be developed from a height-actualist perspective?

The Mathematical Development of the HP

As already mentioned, height maximality is nicely captured using the notion of sharp generation, which has a clean and convincing mathematical formulation. However, the most natural form of width maximality, the inner model hypothesis (IMH), is in conflict with sharp generation. Honzik and I succeeded in "synthesising" the two, arriving at a consistent combined maximality principle IMH-sharp.

However, we did not reach our goal of establishing the consistency of SIMH, the strong IMH. This will be a major achievement, as it will yield a well-motivated form of width maximality that resolves Cantor's continuum problem. Ideally, we aim to then further synthesise the SIMH with sharp generation, arriving at a consistent principle SIMH-sharp, which not only resolves the continuum problem but is also compatible with height maximality (and with large cardinal axioms).

A useful way of organising maximality principles is via the maximality protocol. According to this, maximality is developed by first maximising the ordinals (via sharp generation), then maximising the cardinals through the so-called CardMax principles and finally maximising width via the cardinal-preserving IMH with absolute parameters. This is a satisfying, systematic approach to maximality. However, we have not yet succeeded in finding the mathematical tools needed to establish the consistency of the principles generated in this way. That remains for the further development of the HP.

Two other appealing forms of maximality regard width indiscernibility and omniscience. The former is an analogue for width of sharp generation for height. It postulates that V occurs at stage Ord in a sequence of length Ord+Ord of increasing universes which form a chain under elementary embeddings and which are indiscernible in an appropriate sense. The consistency of this has not yet been established, yet this form of maximality is especially appealing as it helps to restore a symmetry between the notions of maximality in height and in width. Omniscience asserts that the satisfiability of sentences with parameters from V in outer models of V is V-definable. Here we have made definite progress: Honzik and I showed ("Definability of Satisfaction in Outer Models", in this volume) that one can obtain the consistency of the omniscience principle (together with a definable well-order of the universe) from just an inaccessible cardinal. What remains is to verify that it can be successfully synthesised with other forms of maximality, such as the IMH-sharp.

To summarise: The main success of the JTF-funded Hyperuniverse Project was to establish a conceptually sound approach to the discovery of new set-theoretic axioms based on the intrinsic maximality features of V. In addition, significant progress was made on the mathematical formulation of maximality principles, on their synthesis and on establishing their consistency. Thanks to this project, the HP is now well-positioned to make important discoveries regarding set-theoretic truth based on intrinsic evidence and through the use of as yet undiscovered mathematical techniques.

The Chapters in Brief

The 12 chapters of this volume document some of the major advances of the JTF Hyperuniverse Project.

A key technique in the mathematical development of the project is the method of class-forcing. Chapter "Class Forcing in Class Theory" provides the proper setting for class-forcing, which had formerly been done by reducing to versions of ZFC. A further technique is hyperclass-forcing, the foundations for which is provided in Chap. "Hyperclass Forcing in Morse-Kelley Class Theory". Chapter "Multiverse Conceptions in Set Theory" provides a broad analysis of multiverse conceptions in set theory, taking into account different views regarding actualism and potentialism in height and in width. Chapter "Evidence for Set-Theoretic Truth and the Hyperuniverse Programme" is currently the most up-to-date full presentation of the Hyperuniverse Programme. Chapter "On the Set-Generic Multiverse" provides a modern treatment of Bukovsky's characterisation of set-generic extensions, an important feature of the set-generic multiverse. Chapters "On Strong Forms of

Reflection in Set Theory" and "Definability of Satisfaction in Outer Models" are the already-mentioned chapters on height maximality and omniscience. Chapters "The Search for New Axioms in the Hyperuniverse Programme" and "Explaining Maximality Through the Hyperuniverse Programme" take a deeper look at how the HP analyses maximality in set theory. Finally, Chaps. "Large Cardinals and the Continuum Hypothesis", "Gödel's Cantorianism", and "Remarks on Buzaglo's Concept Expansion and Cantor's Transfinite" provide insights into related topics, such as the role of large cardinals, the Cantorian features of Gödel's philosophy of sets and Buzaglo's treatment of concept expansion.

Contents

Class Forcing in Class Theory

Carolin Antos

Abstract In this article we show that Morse-Kelley class theory (MK) provides us with an adequate framework for class forcing. We give a rigorous definition of class forcing in a model $(M, \mathcal{C})$ of MK, the main result being that the Definability Lemma (and the Truth Lemma) can be proven without restricting the notion of forcing. Furthermore we show under which conditions the axioms are preserved. We conclude by proving that Laver's Theorem does not hold for class forcings.

1 Introduction

The idea of considering a forcing notion with a (proper) class of conditions instead of with a set of conditions was introduced by W. Easton in 1970. He needed the forcing notion to be a class to prove the theorem that the continuum function 2^κ, for κ regular, can behave in any reasonable way and as changes in the size of 2^κ are bounded by the size of a set forcing notion, the forcing has to be a class. Two problems arise when considering a class sized forcing: the forcing relation might not be definable in the ground model and the extension might not preserve the axioms. This was addressed in a general way in S. Friedman's book (see [3]) where he presented class forcings which are definable (with parameters) over a model $\langle M, A\rangle$. This is called a model of ZF if M is a model of ZF and Replacement holds in M for formulas which mention A as a predicate. We will call such forcings *A-definable class forcings*, their generics G *A-definable class-generics* and the resulting new model *A-definable class-generic outer model*. Friedman showed that for such A-definable class forcing which satisfy an additional condition called tameness the Definability Lemma, the Truth Lemma and the preservation of the Axioms of ZFC hold.

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In this article we consider class forcing in the framework of Morse-Kelley class theory. In difference to the case of A-definable class forcings we are able to prove in MK that the Definability Lemma holds for all forcing notions (without having to restrict to tame forcings). For the preservation of the axioms however we still need the property of tameness.[1]

In the following we will introduce Morse-Kelley class theory and define the relevant notions like names, interpretations and the extension for class forcing in Morse-Kelley. Then we will show that the forcing relation is definable in the ground model, that the Truth Lemma holds and we characterize P-generic extensions which satisfy the axioms of MK. We will show that Laver's Theorem fails for class forcings.

2 Morse-Kelley Class Theory

In ZFC we can only talk about classes as abbreviations for formulas as our only objects are sets. In class theories like Morse-Kelley (MK) or Gödel-Bernays (GB) the language is two-sorted, i.e. the object are sets and classes and we have corresponding quantifiers for each type of object.[2] We denote the classes by upper case letters and sets by lower case letters, the same will hold for class-names and set-names and so on. Hence atomic formulas for the $\in$-relation are of the form "$x \in X$" where x is a set-variable and X is a set- or class-variable. The models $\mathcal{M}$ of MK are of the form $\langle M, \in, \mathcal{C}\rangle$, where M is a transitive model of ZFC, $\mathcal{C}$ the family of classes of $\mathcal{M}$ (i.e. every element of $\mathcal{C}$ is a subset of M) and $\in$ is the standard $\in$ relation (from now on we will omit mentioning this relation).

The axiomatizations of class theories which are often used and closely related to ZFC are MK and GBC. Their axioms which are purely about sets coincide with the corresponding ZFC axioms such as pairing and union and they share class axioms like the Global Choice Axiom. Their difference lies in the Comprehension Axiom in the sense that GB only allows quantification over sets whereas MK allows

[1] In [2] R. Chuaqui follows a similar approach and defines forcing for Morse-Kelley class theory. However there is a significant difference between our two approaches. To show that the extension preserves the axioms Chuaqui restricts the generic G for an arbitrary forcing notion P in the following way: A subclass G of a notion of forcing P is *strongly P-generic over* a model $(M, \mathcal{C})$ of MK iff G is P-generic over $(M, \mathcal{C})$ and for all ordinals $\beta \in M$ there is a set $P' \in M$ such that $P' \subseteq P$ and for all sequences of dense sections $\langle D_\alpha : \alpha \in \beta\rangle$, there is a $q \in G$ satisfying

$$\forall\alpha(\alpha \in \beta \rightarrow \exists p\,(p \in P' \cap G \wedge \text{the greatest lower bound of } p \text{ and } q \text{ exists}$$
$$\text{and is an element of } D_\alpha)).$$

where a subclass D of a partial order P is a *P-section* if every extension of a condition in D is in D.

[2] There is also an equivalent one-sorted formulation in which the only objects are classes and sets are defined as being classes which are elements of other classes. For reasons of clarity we will use the two-sorted version throughout the paper.

quantification over sets as well as classes. This results in major differences between the two theories which can be seen for example in their relation to ZFC: GB is a conservative extension of ZFC, meaning that every sentence about sets that can be proved in GB can already be proved in ZFC and so GB cannot prove "new" theorems about ZFC. MK on the other hand can do just that, in particular MK implies CON(ZFC)[3] and so MK is not conservative over ZFC. The consistency strength of MK is strictly stronger than that of ZFC but lies below that of ZFC + there is an inaccessible cardinal as $\langle V_\kappa, V_{\kappa+1}\rangle$ for κ inaccessible, is a model for MK in ZFC.

As said above we choose MK (and not GB) as underlying theory to define class forcing. The reason lies mainly in the fact that within MK we can show the Definability Lemma for class forcing without having to restrict the forcing notion whereas in GB this would not be possible. We use the following axiomatization of MK:

A) Set Axioms:

1. Extensionality for sets: $\forall x \forall y (\forall z\, (z \in x \leftrightarrow z \in y) \rightarrow x = y)$.
2. Pairing: For any sets x and y there is a set $\{x, y\}$.
3. Infinity: There is an infinite set.
4. Union: For every set x the set $\bigcup x$ exists.
5. Power set: For every set x the power set $P(x)$ of x exists.

B) Class Axioms:

1. Foundation: Every nonempty class has an $\in$-minimal element.
2. Extensionality for classes: $\forall z\, (z \in X \leftrightarrow z \in Y) \rightarrow X = Y$.
3. Replacement: If a class F is a function and x is a set, then $\{F(z) : z \in x\}$ is a set.
4. Class-Comprehension:

$$\forall X_1 \ldots \forall X_n \exists Y \; Y = \{x : \varphi(x, X_1, \ldots, X_n)\}$$

where φ is a formula containing class parameters in which quantification over both sets and classes are allowed.
5. Global Choice: There exists a global class well-ordering of the universe of sets.

There are different ways of axiomatizing MK, one of them is obtained by using the Limitation of Size Axiom instead of Global Choice and Replacement. Limitation of Size is an axiom that was introduced by von Neumann and says that for every $C \in \mathcal{M}$, C is a proper class if and only if there is a one-to-one function from the universe of sets to C, i.e. all the proper classes have the same size. The two axiomatizations are equivalent: Global Choice and Replacement follow from Limitation of size and

[3]This is because in MK we can form a Satisfaction Predicate for V and then by reflection we get an elementary submodel V_α of V. But any such V_α models ZFC.

vice versa.[4] A nontrivial argument shows that Limitation of Size does not follow from Replacement plus Local Choice.

In the definition of forcing we will use the following induction and recursion principles:

Proposition 1 (Induction) *Let (Ord, R) be well-founded and $\varphi(\alpha)$ a property of an ordinal α. Then it holds that*

$$\forall\alpha \in Ord\,((\forall\beta \in Ord\,(\beta\, R\, \alpha \to \varphi(\beta))) \to \varphi(\alpha)) \to \forall\alpha \in Ord\,\varphi(\alpha)$$

Proof Otherwise, as R is well-founded, there exists an R-minimal element α of Ord such that $\neg\varphi(\alpha)$. That is a contradiction. □

Proposition 2 (Recursion) *For every well-founded binary relation R on Ord and every formula $\varphi(X, Y)$ satisfying $\forall X\, \exists!\, Y\, \varphi(X, Y)$, there is a unique binary relation S on $Ord \times V$ such that for every $\alpha \in Ord$ it holds that $\varphi(S_{<\alpha}, S_\alpha)$, where $S_\alpha = \{x \mid (\alpha, x) \in S\}$ and $S_{<\alpha} = \{(\beta, x) \in S \mid \beta R\alpha\}$.*

Proof By induction on α it holds that for each γ there exists a unique binary relation S^γ on $Ord_{<\gamma} \times V$, where $Ord_{<\gamma} = \{\beta \in Ord \mid \beta R\gamma\}$, such that $\varphi(S^\gamma_{<\alpha}, S^\gamma_\alpha)$ holds for all $\alpha R\gamma$. Then it follows from Class-Comprehension that we can take $S = \bigcup_{\gamma\in Ord} S_\gamma$. □

3 Generics, Names and the Extension

To lay out forcing in MK we have to redefine the basic notions like names, interpretation of names etc. to arrive at the definition of the forcing extension. As we work in a two-sorted theory we will define these notions for sets and classes respectively. Let us start with the definition of the forcing notions and its generics. We use the notation $(X_1, \ldots, X_n) \in \mathcal{C}$ to mean $X_i \in \mathcal{C}$ for all i.

Definition 3 Let $P \in \mathcal{C}$ and $\leq_P \in \mathcal{C}$ be a partial ordering with greatest element 1^P. We call $(P, \leq_P) \in \mathcal{C}$ an $(M, \mathcal{C})$-forcing and often abbreviate it by writing P. With the above convention $(P, \leq_P) \in \mathcal{C}$ means that P and $\leq_P$ are in $\mathcal{C}$.

$G \subseteq P$ is P-generic over $(M, \mathcal{C})$ if

1. G is compatible: If $p, q \in G$ then for some r, $r \leq p$ and $r \leq q$.
2. G is upwards closed: $p \geq q \in G \;\to\; p \in G$.
3. $G \cap D \neq \emptyset$ whenever $D \subseteq P$ is dense, $D \in \mathcal{C}$.

[4]This is because Global Choice is equivalent with the statement that every proper class is bijective with the ordinals.

Note that from now on we will assume M to be countable (and transitive) and $\mathcal{C}$ to be countable to ensure that for each $p \in P$ there exists G such that $p \in G$ and G is P-generic.

We build the hierarchy of names for sets and classes in the following way (we will use capital greek letters for class-names and lower case greek letters for set-names):

Definition 4

$\mathcal{N}_0^s = \emptyset$.
$\mathcal{N}_{\alpha+1}^s = \{\sigma : \sigma \text{ is a subset of } \mathcal{N}_\alpha^s \times P \text{ in } M\}$.
$\mathcal{N}_\lambda^s = \bigcup\{\mathcal{N}_\alpha^s : \alpha < \lambda\}$, if λ is a limit ordinal.
$\mathcal{N}^s = \bigcup\{\mathcal{N}_\alpha^s : \alpha \in ORD(M)\}$ is the class of all set-names of P.
$\mathcal{N} = \{\Sigma : \Sigma \text{ is a subclass of } \mathcal{N}^s \times P \text{ in } \mathcal{C}\}$.

Note that the $\mathcal{N}_\alpha^s$ (for $\alpha > 0$) are in fact proper classes (and indeed $\mathcal{N}$ is a hyperclass) and therefore Definition 4 is an inductive definition of a sequence of proper classes of length the ordinals. The fact that with this definition we stay inside $\mathcal{C}$ follows from Proposition 2.

Lemma 5

a) If $\alpha \leq \beta$ *then* $\mathcal{N}_\alpha^s \subseteq \mathcal{N}_\beta^s$.
b) $\mathcal{N}^s \subseteq \mathcal{N}$.

Proof

a) By induction on β. For $\beta = 0$ there is nothing to prove.
Successor step $\beta \to \beta + 1$. Assume $\mathcal{N}_\alpha^s \subseteq \mathcal{N}_\beta^s$ for all $\alpha \leq \beta$. Let $\tau \in \mathcal{N}_\alpha^s$ for some $\alpha < \beta + 1$. Then we know by assumption that $\tau \in \mathcal{N}_\beta^s$. So by Definition 4 there is some $\gamma < \beta$ such that $\tau = \{\langle \pi_i, p_i \rangle \mid i \in I\}$ where for each $i \in I$, $\pi_i \in \mathcal{N}_\gamma^s$ and $p_i \in P$. By assumption $\pi_i \in \mathcal{N}_\beta^s$ for all $i \in I$ and so $\tau \in \mathcal{N}_{\beta+1}^s$.
Limit step λ. Assume $\mathcal{N}_\alpha^s \subseteq \mathcal{N}_\beta^s$ for all $\alpha \leq \beta < \lambda$. But by Definition 4, $\sigma \in \mathcal{N}_\lambda^s$ iff $\sigma \in \mathcal{N}_\beta^s$ for some $\beta < \lambda$ and so it follows that $\mathcal{N}_\alpha^s \subseteq \mathcal{N}_\lambda^s$ for all $\alpha \leq \lambda$.
b) By Definition 4, $\Sigma \in \mathcal{N}$ iff Σ is a subclass of $\mathcal{N}^s \times P$ iff for every $\langle \tau, p \rangle \in \Sigma$, $\tau \in \mathcal{N}^s$ and $p \in P$ iff for every $\langle \tau, p \rangle \in \Sigma$ there is an ordinal α such that $\tau \in \mathcal{N}_\alpha^s$ and $p \in P$. Let $\sigma \in \mathcal{N}^s$, i.e. there is an ordinal β such that $\sigma \in \mathcal{N}_\beta^s$. Then it holds that for every $\langle \tau, p \rangle \in \sigma$ there is an ordinal $\alpha < \beta$ such that $\tau \in \mathcal{N}_\alpha^s$ and $p \in P$. So $\sigma \in \mathcal{N}$.

□

We define the interpretations of set- and class-names recursively.

Definition 6

$\sigma^G = \{\tau^G : \exists p \in G(\langle \tau, p \rangle \in \sigma)\}$ for $\sigma \in \mathcal{N}^s$.
$\Sigma^G = \{\sigma^G : \exists p \in G(\langle \sigma, p \rangle \in \Sigma)\}$ for $\Sigma \in \mathcal{N}$.

According to the definitions above we define the extension of an MK model $(M, \mathcal{C})$ to be the extension of the set part and the extension of the class part:

Definition 7 $(M, \mathcal{C})[G] = (M[G], \mathcal{C}[G]) = (\{\sigma^G : \sigma \in \mathcal{N}^s\}, \{\Sigma^G : \Sigma \in \mathcal{N}\})$.

Definition 8 If P is a partial order with greatest element 1^P, we define the canonical P-names of $x \in M$ and $C \in \mathcal{C}$:

$\check{x} = \{\langle \check{y}, 1^P \rangle \mid y \in x\}$.
$\check{C} = \{\langle \check{x}, 1^P \rangle \mid x \in C\}$.

From these definitions the basic facts of forcing follow easily:

Lemma 9 *Let $\mathcal{M} = \langle M, \mathcal{C} \rangle$ be a model of MK, where M is a transitive model of ZFC and $\mathcal{C}$ the family of classes of $\mathcal{M}$. Then it holds that:*

a) $\forall x \in M(\check{x} \in \mathcal{N}^s \wedge \check{x}^G = x)$ *and* $\forall C \in \mathcal{C}(\check{C} \in \mathcal{N} \wedge \check{C}^G = C)$.
b) $(M, \mathcal{C}) \subseteq (M, \mathcal{C})[G]$ *in the sense that* $M \subseteq M[G]$ *and* $\mathcal{C} \subseteq \mathcal{C}[G]$.
c) $G \in (M, \mathcal{C})[G]$, *i.e.* $G \in \mathcal{C}[G]$
d) $M[G]$ *is transitive and* $Ord(M[G]) = Ord(M)$.
e) If $(N, \mathcal{C}')$ *is a model of MK,* $M \subseteq N$, $\mathcal{C} \subseteq \mathcal{C}'$, $G \in \mathcal{C}'$ *then* $(M, \mathcal{C})[G] \subseteq (N, \mathcal{C}')$.

Proof

a) Using Definitions 6 and 8 we can easily show this by induction.
b) follows immediately from 1.
c) Let $\Gamma = \{\langle \check{p}, p \rangle : p \in P\}$. Then this is a name for G as $\Gamma^G = \{\check{p}^G \mid p \in G\} = \{p \mid p \in G\} = G$.
d) It follows from Definition 6 and Definition 7 that $M[G]$ is transitive. For every $\sigma \in N^s$ the rank of σ^G is at most rank σ, so $Ord(M[G]) \subseteq Ord(M)$.
e) For each name $\Sigma \in \mathcal{N}$, $\Sigma \in (M, \mathcal{C})$ and therefore $\Sigma \in (N, \mathcal{C}')$. As $G \in \mathcal{C}'$ the interpretation of Σ in $(M, \mathcal{C})[G]$ is the same as in $(N, \mathcal{C}')$.

□

4 Definability and Truth Lemmas

We will define the forcing relation and show that it is definable in the ground model and how it relates to truth in the extension. The main focus will be the Definability Lemma, since it now is possible to prove that it holds for all forcing notions in contrast to A-definable class forcings in a ZFC setting (see [3]). Note that when we talk about a formula $\varphi(x_1, \ldots, x_m, X_1, \ldots, X_n)$ we mean φ to be a second-order formula that allows second-order quantification and we always assume the model $(M, \mathcal{C})$ to be countable.

Definition 10 Suppose p belongs to P, $\varphi(x_1, \ldots, x_m, X_1, \ldots, X_n)$ is a formula, $\sigma_1, \ldots, \sigma_m$ are set-names and $\Sigma_1, \ldots, \Sigma_n$ are class-names. We write $p \Vdash$

$\varphi(\sigma_1, \ldots, \sigma_m, \Sigma_1, \ldots, \Sigma_n)$ iff whenever $G \subseteq P$ is P-generic over $(M, \mathcal{C})$ and $p \in P$, we have $(M, \mathcal{C})[G] \models \varphi(\sigma_1^G, \ldots, \sigma_m^G, \Sigma_1^G, \ldots, \Sigma_n^G)$.

Lemma 11 (Definability Lemma) *For any φ, the relation "$p \Vdash \varphi(\sigma_1, \ldots, \sigma_m, \Sigma_1, \ldots, \Sigma_n)$" of p, $\vec{\sigma}$, $\vec{\Sigma}$ is definable in $(M, \mathcal{C})$.*

Lemma 12 (Truth Lemma) *If G is P-generic over $(M, \mathcal{C})$ then*

$$(M, \mathcal{C})[G] \models \varphi(\sigma_1^G, \ldots, \sigma_m^G, \Sigma_1^G, \ldots, \Sigma_n^G) \Leftrightarrow \exists p \in G\, (p \Vdash \varphi(\sigma_1, \ldots, \sigma_m, \Sigma_1, \ldots, \Sigma_n)).$$

Following the approach of set forcing we introduce a new relation $\Vdash^*$ and prove the Definability and Truth Lemma for this $\Vdash^*$. Then we will show that $\Vdash^*$ equals the intended forcing relation $\Vdash$.

The definition of $\Vdash^*$ consists of ten cases: six cases for atomic formulas, where the first two are for set-names, the second two for the "hybrid" of set- and class-names and the last two for class-names, one for $\wedge$ and $\neg$ respectively and two quantifier cases, one for first-order and one for second-order quantification. By splitting the cases in this way we can see very easily that it is enough to prove the Definability Lemma for set-names only (case one and two in the Definition) and then infer the general Definability Lemma by induction.

Definition 13 $D \subseteq P$ is *dense below* p if $\forall q \leq p\, \exists r\, (r \leq q, r \in D)$.

Definition 14 Let σ, γ, π be elements of $\mathcal{N}^s$ and Σ, Γ elements of $\mathcal{N}$.

1. $p \Vdash^* \sigma \in \gamma$ iff $\{q : \exists \langle \pi, r\rangle \in \gamma$ such that $q \leq r, q \Vdash^* \pi = \sigma\}$ is dense below p.
2. $p \Vdash^* \sigma = \gamma$ iff for all $\langle \pi, r\rangle \in \sigma \cup \gamma$, $p \Vdash^* (\pi \in \sigma \leftrightarrow \pi \in \gamma)$.
3. $p \Vdash^* \sigma \in \Sigma$ iff $\{q : \exists \langle \pi, r\rangle \in \Sigma$ such that $q \leq r, q \Vdash^* \pi = \sigma\}$ is dense below p.
4. $p \Vdash^* \sigma = \Sigma$ iff for all $\langle \pi, r\rangle \in \sigma \cup \Sigma$, $p \Vdash^* (\pi \in \sigma \leftrightarrow \pi \in \Sigma)$.
5. $p \Vdash^* \Sigma \in \Gamma$ iff $\{q : \exists \langle \pi, r\rangle \in \Gamma$ such that $q \leq r, q \Vdash^* \pi = \Sigma\}$ is dense below p.
6. $p \Vdash^* \Sigma = \Gamma$ iff for all $\langle \pi, r\rangle \in \Sigma \cup \Gamma$, $p \Vdash^* (\pi \in \Sigma \leftrightarrow \pi \in \Gamma)$.
7. $p \Vdash^* \varphi \wedge \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$.
8. $p \Vdash^* \neg\varphi$ iff $\forall q \neg \leq p\, (\neg\, q \Vdash^* \varphi)$.
9. $p \Vdash^* \forall x \varphi$ iff for all $\sigma, p \Vdash^* \varphi(\sigma)$.
10. $p \Vdash^* \forall X \varphi$ iff for all $\Sigma, p \Vdash^* \varphi(\Sigma)$.

We have to show that $\Vdash^*$ is definable within the ground model. For this it is enough to concentrate on the first two of the above cases, because we can reduce the definability of the $\Vdash^*$-relation for arbitrary second-order formulas to its definability for atomic formulas $\sigma \in \tau$, $\sigma = \tau$, where σ and τ are set-names. The rest of the cases then follow by induction. So let us restate Lemma 11 for the case of $\Vdash^*$ and set-names:

Lemma 15 (Definability Lemma for the Atomic Cases of Set-Names) *The relation "$p \Vdash^* \varphi(\sigma, \tau)$" is definable in $(M, \mathcal{C})$ for $\varphi =$ "$\sigma \in \tau$" and $\varphi =$ "$\sigma = \tau$".*

Proof We will show by induction[5] on $\beta \in ORD$ that there are unique classes $X_\beta, Y_\beta \subseteq \beta \times M$ which define the $\Vdash^*$-relation for the first two cases of Definition 14 in the following way: for all $\alpha < \beta$, $R_\alpha = (X_\beta)_\alpha, S_\alpha = (Y_\beta)_\alpha$ where $(X_\beta)_\alpha = \{x \mid \langle \alpha, x\rangle \in X_\beta\}$ and

$$R_\alpha = \{(p, \sigma, \in, \tau) \mid p \in P, \sigma \text{ and } \tau \text{ are set } P\text{-names}, \quad (\star)$$
$$\text{rank}(\sigma) \text{ and rank}(\tau) < \alpha, \text{ for all } q \leq p$$
$$\text{there is } q' \leq q \text{ and } \langle \pi, r\rangle \in \tau \text{ such that}$$
$$q' \leq r \text{ and } (q', \pi, =, \sigma) \in S_\alpha\}$$

and

$$S_\alpha = \{(p, \sigma, =, \tau) \mid p \in P, \sigma \text{ and } \tau \text{ are set } P\text{-names}, \quad (\star\star)$$
$$\text{rank}(\sigma) \text{ and rank}(\tau) < \alpha,$$
$$\text{for all } \langle \pi, r\rangle \in \sigma \cup \tau \text{ such that}$$
$$(p, \pi, \in, \sigma) \in R_\alpha \text{ iff } (p, \pi, \in, \tau) \in R_\alpha\}$$

To show that X_β and Y_β are definable we will define the classes R_α and S_α at each step by recursion on the tupel (p, σ, e, τ) according to the following well-founded partial order on $P \times \mathcal{N}^s \times \{\text{“}\in\text{”}, \text{“}=\text{”}\} \times \mathcal{N}^s$.

Definition 16 Suppose $(p, \sigma, e, \tau), (q, \sigma', e', \tau') \in P \times \mathcal{N}^s \times \{\text{“}\in\text{”}, \text{“}=\text{”}\} \times \mathcal{N}^s$. Say that $(q, \sigma', e', \tau') < (p, \sigma, e, \tau)$if

- $max(rank(\sigma'), rank(\tau')) < max(rank(\sigma), rank(\tau))$, or
- $max(rank(\sigma'), rank(\tau')) = max(rank(\sigma), rank(\tau))$, and $rank(\sigma) \geq rank(\tau)$ but $rank(\sigma') < rank(\tau')$, or
- $max(rank(\sigma'), rank(\tau')) = max(rank(\sigma), rank(\tau))$, and $rank(\sigma) \geq rank(\tau) \leftrightarrow rank(\sigma') \geq rank(\tau')$, and e is “=” and e' is “∈”.

Note that clause 1 and 2 of Definition 14 always reduce the <-rank of the members of $P \times \mathcal{N}^s \times \{\text{“}\in\text{”}, \text{“}=\text{”}\} \times \mathcal{N}^s$.

“Successor step $\beta \to \beta + 1$.” We know that there are unique classes X_β, Y_β such that for all $\alpha < \beta$, $R_\alpha = (X_\beta)_\alpha, S_\alpha = (Y_\beta)_\alpha$ and ($\star$) and ($\star\star$) hold. We want to show that there are unique classes $X_{\beta+1}, Y_{\beta+1}$ such that for all $\alpha < \beta + 1$, $R_\alpha = (X_{\beta+1})_\alpha, S_\alpha = (Y_{\beta+1})_\alpha$ and ($\star$) and ($\star\star$) hold. So let for all $\alpha < \beta$ $(X_{\beta+1})_\alpha = (X_\beta)_\alpha = R_\alpha$ and $(Y_{\beta+1})_\alpha = (Y_\beta)_\alpha = S_\alpha$ and define $(X_{\beta+1})_\beta = R_\beta$ and $(Y_{\beta+1})_\beta = S_\beta$ uniquely as follows:

[5]To show how this induction works in the context of a class-theory we will not simply use Propositions 1 and 2, but rather give the complete construction.

A) $(p, \sigma, \text{“}\in\text{”}, \tau) \in R_\beta$ if and only if for all $q \leq p$ there is $q' \leq q$ and $\langle \pi, r\rangle \in \tau$ such that $q' \leq r$ and $(q', \pi, \text{“}=\text{”}, \sigma) \in S_\beta$.
B) $(p, \sigma, \text{“}=\text{”}, \tau) \in S_\beta$ if and only if for all $\langle \pi, r\rangle \in \sigma \cup \tau$: $(p, \pi, \text{“}\in\text{”}, \sigma) \in R_\beta$ iff $(p, \pi, \text{“}\in\text{”}, \tau) \in R_\beta$.

These definitions clearly satisfy ($\star$) and ($\star\star$) and to see that they are indeed inductive definitions over the well-order defined in Definition 16, we consider the following three cases for each of the definitions A) and B):

1. $\text{rank}(\sigma) < \text{rank}(\tau)$
2. $\text{rank}(\tau) < \text{rank}(\sigma)$
3. $\text{rank}(\sigma) = \text{rank}(\tau)$

Ad A.1: $(q', \pi, \text{“}=\text{”}, \sigma) < (p, \sigma, \text{“}\in\text{”}, \tau)$ because $\text{rank}(\sigma), \text{rank}(\pi) < \text{rank}(\tau)$ (first clause of Denfition 16).

Ad A.2: $(q', \pi, \text{“}=\text{”}, \sigma) < (p, \sigma, \text{“}\in\text{”}, \tau)$ because $\max(\text{rank}(\pi), \text{rank}(\sigma)) = \max(\text{rank}(\sigma), \text{rank}(\tau))$ and $\text{rank}(\sigma) \geq \text{rank}(\tau)$ and $\text{rank}(\pi) < \text{rank}(\sigma)$ (second clause of Definition 16).

Ad A.3: $(q', \pi, \text{“}=\text{”}, \sigma) < (p, \sigma, \text{“}\in\text{”}, \tau)$ because $\max(\text{rank}(\pi), \text{rank}(\sigma)) = \max(\text{rank}(\sigma), \text{rank}(\tau))$ and $\text{rank}(\sigma) \geq \text{rank}(\tau)$ and $\text{rank}(\pi) < \text{rank}(\sigma) = \text{rank}(\tau)$ (second clause of Definition 16).

Ad B.1: $(p, \pi, \text{“}\in\text{”}, \sigma) < (p, \sigma, \text{“}=\text{”}, \tau)$ because $\text{rank}(\sigma), \text{rank}(\pi) < \text{rank}(\tau)$ and $(p, \pi, \text{“}\in\text{”}, \tau) < (p, \sigma, \text{“}=\text{”}, \tau)$ because $\max(\text{rank}(\pi), \text{rank}(\tau)) = \max(\text{rank}(\sigma), \text{rank}(\tau))$ and $\text{rank}(\sigma) < \text{rank}(\tau)$ and $\text{rank}(\pi) < \text{rank}(\tau)$ (third clause of Definition 16).

Ad B.2: $(p, \pi, \text{“}\in\text{”}, \sigma) < (p, \sigma, \text{“}=\text{”}, \tau)$ because of the second clause of Definition 16 and $(p, \pi, \text{“}\in\text{”}, \tau) < (p, \sigma, \text{“}=\text{”}, \tau)$ because $\text{rank}(\pi), \text{rank}(\tau) < \text{rank}(\sigma)$.

Ad B.3: $(p, \pi, \text{“}\in\text{”}, \sigma) < (p, \sigma, \text{“}=\text{”}, \tau)$ and $(p, \pi, \text{“}\in\text{”}, \tau) < (p, \sigma, \text{“}=\text{”}, \tau)$ because $\max(\text{rank}(\pi), \text{rank}(\tau)) = \max(\text{rank}(\sigma), \text{rank}(\tau))$ and $\text{rank}(\sigma) \geq \text{rank}(\tau)$ and $\text{rank}(\pi) < \text{rank}(\sigma), \text{rank}(\tau)$ (both second clause of Definition 16).

“Limit step λ.” We know that for every $\beta < \lambda$ there are unique classes X_β, Y_β such that for all $\alpha < \beta$, $R_\alpha = (X_\beta)_\alpha, S_\alpha = (Y_\beta)_\alpha$ and ($\star$) and ($\star\star$) hold. We have to show that there are unique classes $X_\lambda, Y_\lambda \subseteq \lambda \times M$, λ limit, such that for all $\beta < \lambda$, $R_\beta = (X_\lambda)_\beta, S_\beta = (Y_\lambda)_\beta$ and ($\star$) and ($\star\star$) hold respectively. We define the required classes as follows:

$$\begin{aligned}\langle \alpha, x\rangle \in X_\lambda \leftrightarrow \exists \langle \langle R_\gamma, S_\gamma\rangle \mid \gamma \leq \alpha\rangle\, \exists X, Y((\forall \gamma \leq \alpha((X)_\gamma = R_\gamma \text{ and} \\ (Y)_\gamma = S_\gamma \text{ and they satisfy } (\star) \text{ and } (\star\star) \text{ resp.}) \wedge \\ (x \in (X)_\gamma \text{ for some } \gamma \leq \alpha))\end{aligned}$$

$$\begin{aligned}\langle \alpha, x\rangle \in Y_\lambda \leftrightarrow \exists \langle \langle R_\gamma, S_\gamma\rangle \mid \gamma \leq \alpha\rangle\, \exists X, Y((\forall \gamma \leq \alpha((X)_\gamma = R_\gamma \text{ and} \\ (Y)_\gamma = S_\gamma \text{ and they satisfy } (\star) \text{ and } (\star\star) \text{ resp.}) \wedge \\ (x \in (Y)_\gamma \text{ for some } \gamma \leq \alpha))\end{aligned}$$

From the proof of the successor step we see that the sequence $\langle\langle R_\gamma, S_\gamma\rangle \mid \gamma \leq \alpha\rangle$ is unique for every $\alpha < \lambda$ and therefore X_λ, Y_λ are also unique. This definition is possible only in Morse-Kelly with its version of Class-Comprehension and not in Gödel-Bernays, because we are quantifying over class variables (in fact we only need Δ^1_1 Class-Comprehension). □

The general Definability Lemma now follows immediately from this Lemma and Definition 14. We now turn to the Truth Lemma.

In the following a capital greek letter denotes a name from $\mathcal{N}$ (and therefore can be a set- or a class-name), whereas a lower case greek letter is a name from $\mathcal{N}^s$ (and therefore can only be a set-name).

Lemma 17

a) If $p \Vdash^* \varphi$ and $q \leq p$ then $q \Vdash^* \varphi$
b) If $\{p \mid q \Vdash^* \varphi\}$ is dense below p then $p \Vdash^* \varphi$.
c) If $\neg p \Vdash^* \varphi$ then $\exists q \leq p(q \Vdash^* \neg\varphi)$.

Proof

a) By induction on φ: Let φ be $\Sigma \in \Gamma$, then by Definition 4 $D = \{q' : \exists\langle\pi, r\rangle \in \Gamma$ such that $q' \leq r, q' \Vdash^* \pi = \Sigma\}$ is dense below p. Then for all $q \leq p$, D is also dense below q and therefore $q \Vdash^* \varphi$. The other cases follow easily.
b) By induction on φ. Let φ be $\Sigma \in \Gamma$ and $\{q \mid q \Vdash^* \Sigma \in \Gamma\}$ is dense below p. From Definition 14 it follows that $\{q \mid \{s : \exists\langle\pi, r\rangle \in \Gamma$ such that $s \leq r, s \Vdash^* \pi = \Sigma\}$ is dense below $q\}$ is dense below p and from a well-known fact it follows that $D = \{s : \exists\langle\pi, r\rangle \in \Gamma$ such that $s \leq r, s \Vdash^* \pi = \Sigma\}$ is dense below p. Again by Definition 14 we get as desired $p \Vdash^* \Sigma \in \Gamma$.
   The other cases follow easily; for the case of negation we will use the fact that if $\{p \mid q \Vdash^* \neg\varphi\}$ is dense below p then $\forall q \leq p(\neg q \Vdash^* \varphi)$, using a).
c) follows directly from b).

□

Now, the proofs for the Truth Lemma and $\Vdash^* = \Vdash$ follow similarly to the proofs in set forcing (note that a name $\Sigma \in \mathcal{N}$ can also be a set-name and therefore we don't need to mention the cases for set-names explicitly):

Lemma 18 (Truth Lemma) *If G is P-generic then*

$$(M, \mathcal{C})[G] \models \varphi(\Sigma_1^G, \ldots, \Sigma_m^G) \Leftrightarrow \exists p \in G\,(p \Vdash^* \varphi(\Sigma_1, \ldots, \Sigma_m)).$$

Proof By induction on φ.

$\Sigma \in \Gamma$. "→" Assume $\Sigma^G \in \Gamma^G$ then choose a $\langle\pi, r\rangle \in \Gamma$ such that $\Sigma^G = \pi^G$ and $r \in G$. By induction there is a $p \in G$ with $p \leq r$ and $p \Vdash^* \pi = \Sigma$. Then for all $q \leq p$, $q \Vdash^* \pi = \Sigma$ and by Definition 4 $p \Vdash^* \Sigma \in \Gamma$.
"←": Assume $\exists p \in G(p \Vdash^* \Sigma \in \Gamma)$. Then $\{q : \exists\langle\pi, r\rangle \in \tau$ such that $q \leq r, q \Vdash^* \sigma = \pi\} = D$ is dense below p and so by genericity $G \cap D \neq \emptyset$. So there is a $q \in G$, $q \leq p$ such that $\exists\langle\pi, r\rangle \in \Gamma$

with $q \leq r$, $q \Vdash^* \pi = \Sigma$. By induction $\pi^G = \Sigma^G$ and as $r \geq q$, $r \in G$ and therefore $\pi^G \in \Gamma^G$. So $\Sigma^G \in \Gamma^G$.

$\Sigma = \Gamma$. "$\rightarrow$" Assume $\sigma^G = \Gamma^G$. Then for all $\langle \pi, r \rangle \in \Sigma \cup \Gamma$ with $r \in G$ it holds that $\pi^G \in \Sigma^G \leftrightarrow \pi^G \in \Gamma^G$. Let $D = \{p \mid \text{either } p \Vdash^* \Sigma = \Gamma \text{ or for some } \langle \pi, r \rangle \in \Sigma \cup \Gamma, p \Vdash^* \neg(\pi \in \Sigma \leftrightarrow \pi \in \Gamma)\}$. Then D is dense: By contradiction, let $q \in P$ and assume that there is no $p \leq q$ such that $p \in D$. But if there is no $p \leq q$ such that for some $\langle \pi, r \rangle \in \Sigma \cup \Gamma, p \Vdash^* \neg(\pi \in \Sigma \leftrightarrow \pi \in \Gamma)\}$ then by Lemma 17 $q \Vdash^* (\pi \in \Sigma \leftrightarrow \pi \in \Gamma)$ for all $\langle \pi, r \rangle \in \Sigma \cup \Gamma$ and therefore $q \Vdash^* \Sigma = \Gamma$. So there is a $p \leq q$ such that $p \in D$. Since the filter G is generic, there is a $p \in G \cap D$. If $p \Vdash^* \neg(\pi \in \Sigma \leftrightarrow \pi \in \Gamma)\}$ for some $\langle \pi, r \rangle \in \Sigma \cup \Gamma$ then by induction $\neg(\pi^G \in \Sigma^G \leftrightarrow \pi^G \in \Gamma^G)$ for some $\langle \pi, r \rangle \in \Sigma \cup \Gamma$. But this is a contradiction to $\Sigma^G = \Gamma^G$ and so $P \Vdash^* \Sigma = \Gamma$.

"$\leftarrow$" Assume that there is $p \in G$ $(p \Vdash^* \Sigma = \Gamma)$. By Definition 4 it follows that for all $\langle \pi, r \rangle \in \Sigma \cup \Gamma$ $P \Vdash^* (\pi \in \Sigma \leftrightarrow \pi \in \Gamma)$. Then by induction $\pi^G \in \Sigma^G \leftrightarrow \pi^G \in \Gamma^G$ for all $\langle \pi, r \rangle \in \Sigma \cup \Gamma$. So $\Sigma^G = \Gamma^G$.

$\varphi \wedge \psi$ "$\rightarrow$" Assume that $(M, \mathcal{C})[G] \models \varphi \wedge \psi$ iff $(M, \mathcal{C})[G] \models \varphi$ and $(M, \mathcal{C})[G] \models \psi$. Then by induction $\exists p \in G\ P \Vdash^* \varphi$ and $\exists q \in G$, $q \Vdash^* \psi$ and we know that $\exists r \in G(r \leq p \text{ and } r \leq q)$ such that $r \Vdash^* \varphi$ and $r \Vdash^* \psi$ and so by Definition 4 $r \Vdash^* \varphi \wedge \psi$.

"$\leftarrow$" Assume $\exists p \in G, p \Vdash^* \varphi \wedge \psi$, then $p \Vdash^* \varphi$ and $p \Vdash^* \psi$. So $(M, \mathcal{C})[G] \models \varphi$ and $(M, \mathcal{C})[G] \models \psi$ and therefore $(M, \mathcal{C})[G] \models \varphi \wedge \psi$.

$\neg\varphi$ "$\rightarrow$" Assume that $(M, \mathcal{C})[G] \models \neg\varphi$. $D = \{p \mid p \Vdash^* \varphi \text{ or } p \Vdash^* \neg\varphi\}$ is dense (using Lemma 17 and Definition 4). Therefore there is a $p \in G \cap D$ and by induction $p \Vdash^* \neg\varphi$.

"$\leftarrow$" Assume that there is $p \in G$ such that $p \Vdash^* \neg\varphi$. If $(M, \mathcal{C}) \models \varphi$ then by induction hypothesis there is a $q \in G$ such that $q \Vdash^* \varphi$. But then also $r \Vdash^* \varphi$ for some $r \leq p, q$ and this is a contradiction because of Definition 4. So $(M, \mathcal{C}) \models \neg\varphi$.

$\forall X\varphi$ "$\rightarrow$" Assume that $(M, \mathcal{C})[G] \models \forall X\varphi$. Following the lines of the "$\rightarrow$"-part of the proof for $\Sigma = \Gamma$, there is a dense $D = \{p \mid \text{either } p \Vdash^* \forall X\varphi \text{ or for some } \sigma, p \Vdash^* \neg\varphi(\sigma)\}$. By induction we show that the second case is not possible and so it follows that $p \Vdash^* \forall X\varphi$.

"$\leftarrow$" By induction.

□

Lemma 19 $\Vdash^* = \Vdash$

Proof $p \Vdash^* \varphi(\sigma_1, \ldots, \sigma_n) \rightarrow p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$ follows directly from the Truth Lemma. For the converse we use Lemma 17 c) and note that we assumed the existence of generics. Then from $\neg p \Vdash^* \varphi(\sigma_1, \ldots, \sigma_n)$ it follows that for some $q \leq p$, $q \Vdash^* \neg\varphi(\sigma_1, \ldots, \sigma_n)$ and so $\neg p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$. □

5 The Extension Fulfills the Axioms

We have shown that in MK we can prove the Definability Lemma without restricting the forcing notion as we have to do when working with A-definable class forcing in ZFC (see [3]). Unfortunately we do not have the same advantage when proving the preservation of the axioms. For example, when proving the Replacement Axiom we have to show that the range of a set under a class function is still a set and this does not hold in general for class forcings. In [3] two properties of forcing notions are introduced, namely pretameness and tameness. Pretameness is needed to prove the Definability Lemma and show that all axioms except Power Set are preserved. For the Power Set Axiom this restriction needs to be strengthened to tameness. Let us give the definitions in the MK context:

Definition 20 (Pretameness) $D \subseteq P$ is predense $\leq p \in P$ if every $q \leq p$ is compatible with an element of D.

P is pretame if and only if whenever $\langle D_i \mid i \in a\rangle$ is a sequence of dense classes in $\mathcal{M}$, $a \in M$ and $p \in P$ then there exists a $q \leq p$ and $\langle d_i \mid i \in a\rangle \in M$ such that $d_i \subseteq D_i$ and d_i is predense $\leq q$ for each i.

Definition 21 $q \in P$ meets $D \subseteq P$ if q extends an element in D.

A predense $\leq p$ partition is a pair (D_0, D_1) such that $D_0 \cup D_1$ is predense $\leq p$ and $p_0 \in D_0, p_1 \in D_1 \rightarrow p_0, p_1$ are incompatible. Suppose $\langle (D_0^i, D_1^i) \mid i \in a\rangle$, $\langle (E_0^i, E_1^i) \mid i \in a\rangle$ are sequences of predense $\leq p$ partitions. We say that they are equivalent $\leq p$ if for each $i \in a$, $\{q \mid q \text{ meets } D_0^i \leftrightarrow q \text{ meets } E_0^i\}$ is dense $\leq p$. When $p = 1^P$ we omit $\leq p$.

To each sequence of predense $\leq p$ partitions $\vec{D} = \langle (D_0^i, D_1^i) | i \in a\rangle \in M$ and G is P-generic over $\langle M, \mathcal{C}\rangle$, $p \in G$ we can associate the function

$$f_{\vec{D}}^G : a \rightarrow 2$$

defined by $f(i) = 0 \leftrightarrow G \cap D_0^i \neq \emptyset$. Then two such sequences are equivalent $\leq p$ exactly if their associated functions are equal, for each choice of G.

Definition 22 (Tameness) P is tame iff P is pretame and for each $a \in M$ and $p \in P$ there is $q \leq p$ and $\alpha \in ORD(M)$ such that whenever $\vec{D} = \langle (D_0^i, D_1^i) | i \in a\rangle \in M$ is a sequence of predense $\leq q$ partitions, $\{r \mid \vec{D}$ is equivalent $\leq r$ to some $\vec{E} = \langle (E_0^i, E_1^i) \mid i \in a\rangle$ in $V_\alpha^M\}$ is dense below q.

Theorem 23 *Let $(M, \mathcal{C})$ be a model of MK. Then, if G is P-generic over $(M, \mathcal{C})$ and P is tame then $(M, \mathcal{C})[G]$ is a model of MK.*

Proof Extensionality and Foundation follow because $M[G]$ is transitive (see Lemma 9 d); axioms 2 and 3 from Definitions 4 and 6. For Pairing, let σ_1^G, σ_2^G be such that $\sigma_1, \sigma_2 \in \mathcal{N}^s$. Then the interpretation of the name $\sigma = \{\langle \sigma_1, 1^P\rangle, \langle \sigma_2, 1^P\rangle\}$ in the extension gives the desired $\sigma^G = \{\sigma_1^G, \sigma_2^G\}$. Infinity follows because ω exists in $(M, \mathcal{C})$ and the notion of ω is absolute to any model, $\omega \in (M, \mathcal{C})[G]$. Union follows as in the set forcing case.

Replacement This follows as in [3] from the property of pretameness and we give the proof to make clear where the property of pretameness is needed: Suppose that $F : \sigma^G \to M[G]$. Then for each σ_0 of rank $<$ rank σ the class $D(\sigma_0) = \{p \mid \text{for some } \tau, q \Vdash \sigma_0 \in \sigma \to F(\sigma_0) = \tau\}$ is dense below p, for some $p \in G$ which forces that F is a total function on σ. We now use pretameness to "shrink" this class to a set: so for each $q \leq p$ there is an $r \leq q$ and $\alpha \in Ord(M)$ such that $D_\alpha(\sigma_0) = \{s \mid s \in V^M_\alpha$ and for some τ of rank $< \alpha, s \Vdash \sigma_0 \in \sigma \to F(\sigma_0) = \tau\}$ is predense $\leq r$ for each σ_0 of rank $<$ rank σ. Then it follows by genericity that there is a $q \in G$ and $\alpha \in Ord(M)$ such that $q \leq p$ and $D_\alpha(\sigma_0)$ is predense $\leq q$ for each σ_0 of rank $<$ rank σ. So let $\pi = \{\langle \tau, r\rangle \mid \text{rank}\,\tau < \alpha, r \in V^M_\alpha, r \Vdash \tau \in \text{ran}(F)\}$ and then it follows that $\text{ran}(F) = \pi^G \in M[G]$.

Power Set This follows from tameness as shown in [3].

Class-Comprehension Let $\Gamma = \{\langle \sigma, p\rangle \in \mathcal{N}^s \times P \mid p \Vdash \varphi(\sigma, \Sigma_1, \ldots, \Sigma_n)\}$. Because of the Definability Lemma, we know that $\Gamma \in \mathcal{N}$. By Definitions 4 and 6, $\Gamma^G = \{\sigma^G \mid \exists p \in G(\langle \sigma, p\rangle \in \Gamma)\}$ and we need to check that this equals the desired $Y = \{x \mid (\varphi(x, \Sigma^G_1, \ldots, \Sigma^G_n))^{(M,\mathcal{C})[G]}\}$. So let $\sigma^G \in \Gamma^G$. Then by the definition of Γ^G we know that $p \Vdash \varphi(\sigma, \Sigma_1, \ldots, \Sigma_n)$ and because of the Truth Lemma it follows that $(M, \mathcal{C})[G] \models \varphi(\sigma^G, \Sigma^G_1, \ldots, \Sigma^G_n)$. For the converse, let $x \in Y$. By the Truth Lemma, $\exists p \in G(p \Vdash \varphi(\pi, \Sigma_1, \ldots, \Sigma_n)$, where π is a name for x. By definition of Γ, $\langle \pi, p\rangle \in \Gamma$.

Global Choice Let $<_M$ denote the well-order of M and let σ_x, σ_y be the least names for some $x, y \in M[G]$. As the names are elements of M, we may assume that $\sigma_x <_M \sigma_y$. So we define the relation $<_G$ in $M[G]$ using M and $<_M$ as parameters, so that $x <_G y$ iff $\sigma_x <_M \sigma_y$ for the corresponding least names of x and y. Let $R = \{(x, y) \mid x, y \in M[G] \text{ and } x <_G y\}$. Then by Class-Comprehension the class R exists. □

Friedman [3] gives us a simple sufficient condition for tameness that translates directly into the context of MK:

Definition 24 For regular, uncountable $\kappa > \omega$, P is κ-distributive if whenever $p \in P$ and $\langle D_i \mid i < \beta\rangle$ are dense classes, $\beta < \kappa$ then there is a $q \leq p$ meeting each D_i (p meets D if $p \leq q \in D$ for some q).

P is tame below κ if the tameness conditions hold for P with the added restriction that $Card(a) < \kappa$.

Lemma 25 *If P is κ-distributive then P is tame below κ.*

Proof Analogous to set forcing.[6] □

[6] See [3, p. 37].

6 Laver's Theorem

In the following we will give an example which shows that a fundamental theorem that holds for set forcing can be violated by tame class forcings.

Laver's Theorem (see [5]) shows that for a set-generic extension $V \subseteq V[G]$, $V \models ZFC$ with the forcing notion $P \in V$ and G P-generic over V, V is definable in $V[G]$ from parameter $V_{\delta+1}$ (of V) and $\delta = |P|^+$ in $V[G]$. This result makes use of the fact that every such forcing extension has the approximation and cover properties as defined in [4] and relies on certain results for such extensions.

In general, the same does not hold for class forcing. In fact there are class forcings such that the ground model is not even second-order definable from set-parameters:

Theorem 26 *There is an MK-model $(M, \mathcal{C})$ and a first-order definable, tame class forcing $\mathbb{P}$ with G $\mathbb{P}$-generic over $(M, \mathcal{C})$ such that the ground model M is not definable with set-parameters in the generic extension $(M, \mathcal{C})[G]$.*

Proof We are starting from L. For every successor cardinal α, let P_α be the forcing that adds one Cohen set to α: P_α is the set of all functions p such that

$$dom(p) \subset \alpha, \quad |dom(p)| < \alpha, \quad ran(p) \subset \{0, 1\}.$$

Let P be the Easton product of the P_α for every successor α: A condition $p \in P$ is a function $p \in L$ of the form $p = \langle p_\alpha : \alpha \text{ successor cardinal}\rangle \in \Pi_{\alpha \text{ succ.}} P_\alpha$ (p is stronger then q if and only if $p \supset q$) and p has Easton support: for every inaccessible cardinal κ, $|\{\alpha < \kappa \mid p(\alpha) \neq \emptyset\}| < \kappa$. Then P is the forcing which adds one Cohen set to every successor cardinal.

Let $\mathbb{P} = P \times P = \Pi_{\alpha \text{ succ}} P_\alpha \times \Pi_{\alpha \text{ succ}} P_\alpha$ be the forcing that adds simultaneously two Cohen sets to every successor cardinal.[7] Note that $\Pi_{\alpha \text{ succ.}} P_\alpha \times \Pi_{\alpha \text{ succ.}} P_\alpha$ is isomorphic to $\Pi_{\alpha \text{ succ.}} P_\alpha \times P_\alpha$. Let G be $\mathbb{P}$-generic. Then $G = \Pi_{\alpha \text{ succ.}} G_0(\alpha) \times G_1(\alpha)$ and we let $G_0 = \Pi_{\alpha \text{ succ.}} G_0(\alpha)$ and $G_1 = \Pi_{\alpha \text{ succ.}} G_1(\alpha)$ with G_0, G_1 P-generic over L. We consider the extension $L[G_0] \subseteq L[G_0][G_1]$ and we will show, that $L[G_0]$ is not definable in $L[G_0][G_1]$ from parameters in $L[G_0]$.

The reason that we cannot apply Laver's and Hamkins' results of [5] to this extension is that it does not fulfill the δ approximation property[8]: As the forcing adds a new set to every successor, the δ approximation property cannot hold at successor cardinals δ: the added Cohen set is an element of the extension and a subset of the ground model and all of its $< \delta$ approximations are elements of the ground model but the whole set is not.

[7] It follows by a standard argument that $\mathbb{P}$ is pretame (and indeed tame) over $(M, \mathcal{C})$, see [3].

[8] A pair of transitive classes $M \subseteq N$ satisfies the δ *approximation property* (with $\delta \in Card^N$) if whenever $A \subseteq M$ is a set in N and $A \cap a \in M$ for any $a \in M$ of size less than δ in M, then $A \in M$. For models of set theory equipped with classes, the pair $M \subseteq N$ satisfies the δ *approximation property for classes* if whenever $A \subseteq M$ is a class of N and $A \cap a \in M$ for any a of size less than δ in M, then A is a class of M.

Note that the forcing is weakly homogeneous, i.e. for every $p, q \in \mathbb{P}$ there is an automorphism π on $\mathbb{P}$ such that $\pi(p)$ is compatible with q. This is because every P_α is weakly homogeneous (let $\pi(p) \in P_\alpha$ such that $dom(\pi(p)) = dom(p)$ and $\pi(p)(\lambda) = q(\lambda)$ if $\lambda \in dom(p) \cap dom(q)$ and $\pi(p)(\lambda) = p(\lambda)$ otherwise, then π is order preserving and a bijection) and therefore also P is weakly homogeneous (define π componentwise using the projection of p to p_α). Similar for $P \times P$.

To show that $L[G_0]$ is not definable in $L[G_0][G_1]$ with parameters, assume to the contrary that there is a set-parameter a_0 such that $L[G_0]$ is definable by the second-order formula $\varphi(x, a_0)$ in $L[G_0][G_1]$ from a_0. Let α be such that $a_0 \in L[G_0 \upharpoonright \alpha, G_1 \upharpoonright \alpha]$. Now consider $a = G_0(\alpha^+)$, the Cohen set which is added to α^+ in the first component of $\mathbb{P}$. a is P_{α^+}-generic over $L[G_0 \upharpoonright \alpha, G_1 \upharpoonright \alpha]$ and as a is an element of $L[G_0]$ the formula φ holds for a. So we also know that there is a condition $q \in G$ such that $q \Vdash \varphi(\dot{a}, a_0)$.

Now we construct another generic $G^* = G_0^* \times G_1^*$ which produces the same extension but also an element for which φ holds and which is not an element of $L[G_0]$. This new generic adds the same sets as G, but we switch G_0 and G_1 at α^+ so that the set added by $G_1(\alpha^+)$ is now added in the new first component G_0^*. However we have to make sure that the new generic respects q so that φ is again forced in the extension. We achieve this by fixing the generic G on the length of $q(\alpha^+)$ (we can assume that the length is the same on G_0 and G_1).

It follows that $q \in G_0^* \times G_1^*$ and because of weakly homogeneity $G_0^* \times G_1^*$ is generic and $L[G_0][G_1] = L[G_0^*][G_1^*]$. Because of the construction of G^*, the formula $\varphi(x, a_0)$ holds for the set $b = G_0^*(\alpha^+)$ but b is not an element of $L[G_0]$. That is a contradiction! □

We have seen that there are different ways of approaching class forcing, namely on the one hand as definable from a class parameter A in a ZFC model (M, A) and on the other hand in the context of an MK model $(M, \mathcal{C})$. That presents us with three notions of genericity: set-genericity, A-definable class genericity and class-genericity. One of the questions that arises now is in which way we can define the next step in this “hierarchy” of genericity. To answer this question, Sy Friedman and the author of this paper are currently working on so-called hyperclass forcings in a variant of MK, i.e. forcings in which the conditions are classes (see [1]). We will show in which context such forcings are definable and which application they have to class-theory.

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Hyperclass Forcing in Morse-Kelley Class Theory

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Abstract In this article we introduce and study hyperclass-forcing (where the conditions of the forcing notion are themselves classes) in the context of an extension of Morse-Kelley class theory, called MK^{**}. We define this forcing by using a symmetry between MK^{**} models and models of ZFC^{-} plus there exists a strongly inaccessible cardinal (called SetMK^{**}). We develop a coding between β-models $\mathcal{M}$ of MK^{**} and transitive models M^{+} of SetMK^{**} which will allow us to go from $\mathcal{M}$ to M^{+} and vice versa. So instead of forcing with a hyperclass in MK^{**} we can force over the corresponding SetMK^{**} model with a class of conditions. For class-forcing to work in the context of ZFC^{-} we show that the SetMK^{**} model M^{+} can be forced to look like $L_{\kappa^*}[X]$, where κ^* is the height of M^{+}, κ strongly inaccessible in M^{+} and $X \subseteq \kappa$. Over such a model we can apply definable class forcing and we arrive at an extension of M^{+} from which we can go back to the corresponding β-model of MK^{**}, which will in turn be an extension of the original $\mathcal{M}$. Our main result combines hyperclass forcing with coding methods of Beller et al. (Coding the universe. Lecture note series. Cambridge University Press, Cambridge, 1982) and Friedman (Fine structure and class forcing. de Gruyter series in logic and its applications, vol 3, Walter de Gruyter, New York, 2000) to show that every β-model of MK^{**} can be extended to a minimal such model of MK^{**} with the same ordinals. A simpler version of the proof also provides a new and analogous minimality result for models of second-order arithmetic.

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1 Introduction

When considering forcing notions with respect to their size, there are three different types: the original version of forcing, where the forcing notion is a set, called set forcing; forcing in ZFC, where the forcing notion is a class, called definable class forcing and class forcing in Morse-Kelley class theory (MK). In this article we consider a fourth type which we call definable hyperclass forcing and give applications for this forcing in the context of Morse-Kelley class theory, where hyperclass forcing denotes a forcing with class conditions. We will define hyperclass forcing indirectly by using a correspondence between certain models of MK and models of a version of ZFC^- (minus PowerSet) and show that we can define definable hyperclass forcing by going to the related ZFC^- model and using definable class forcing there.

Two problems arise when considering definable class forcing in ZFC: the forcing relation might not be definable in the ground model and the extension might not preserve the axioms. As an example consider $Col(\omega, ORD)$ with conditions $p : n \to Ord$ for $n \in \omega$ which adds a cofinal sequence of length ω in the ordinals. Here Replacement fails.[1] These problems were addressed in a general way by the second author in [4] where class forcings are presented which are definable (with parameters) over a model $\langle M, A\rangle$ where M is a transitive model of ZFC, $A \subseteq M$ and Replacement holds in M for formulas mentioning A as a unary predicate. Two properties of the forcing notion are introduced, pretameness and tameness and it is shown that for a pretame forcing notion the Definability Lemma holds and Replacement is preserved and that tameness (which is a strengthening of pretameness) is equivalent to the preservation of the Power Set axiom. In this article we will adjust this approach to definable class forcing in ZFC^-. Pretameness is defined as follows:

Definition 1 A forcing notion P is pretame iff whenever $\langle D_i | i \in a\rangle$, $a \in M$, is an $\langle M, A\rangle$-definable sequence of dense classes and $p \in P$ then there is $q \leq p$ and $\langle d_i | i \in a\rangle \in M$ such that $d_i \subseteq D_i$ and d_i is predense $\leq q$ for each i.

For definable hyperclass forcing we will work in the context of Morse-Kelley class theory, by which we mean a theory with a two-sorted language, i.e. the object are sets and classes and we have corresponding quantifiers for each type of object. We denote the classes by upper case letters and sets by lower case letters, the same will hold for class-names and set-names and so on. Hence atomic formulas for the $\in$-relation are of the form "$x \in X$" where x is a set-variable and X is a set- or class-variable. The models $\mathcal{M}$ of MK are of the form $\langle M, \in, \mathcal{C}\rangle$, where M is a transitive model of ZFC, $\mathcal{C}$ the family of classes of $\mathcal{M}$ (i.e. every element of $\mathcal{C}$ is a subset of M) and $\in$ is the standard $\in$ relation (from now on we will omit mentioning this relation). We use the following axiomatization of MK:

[1] A detailed analyses on how even the Definability Lemma for class forcings can fail can be found in [7].