Algorithms and Combinatorics 26

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Editors

Topics in Discrete Mathematics

Dedicated to Jarik Nešetřil
on the Occasion of his 60th Birthday

With 62 Figures

Springer

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To our teacher, colleague and friend

Preface

The purpose of this book is twofold. We would like to offer our readers a collection of high quality papers in selected topics of Discrete Mathematics, and, at the same time, celebrate the 60th birthday of Jarik Nešetřil. Since our discipline has experienced an explosive growth during the last half century, it is impossible to cover all of its recent developments in one modest volume. Instead, we concentrate on six topics, those closest to Jarik's interests. We have invited leading experts and close friends of Jarik's to contribute to this endeavor, and the response has been overwhelmingly positive. We were fortunate to receive many outstanding contributions. They are divided into six parts.

Contents

The topics of the first part are rather diverse, including Algebra, Geometry, and Numbers and Games. Michael E. Adams and Aleš Pultr consider rigidity (lack of nontrivial homomorphisms) of algebraic structures, and they construct $2^{\aleph_0}$ rigid countable Heyting algebras. Vitaly Bergelson, Hillel Furstenberg, and Benjamin Weiss introduce a new notion of "large" sets of integers, piecewise-Bohr sets, and they show, in particular, that the sum of two sets of positive upper density is piecewise Bohr. Christopher Cunningham and Igor Kriz investigate a generalization of the Conway number games to more than two players and construct games with any given value. Miroslav Fiedler solves extremal geometric questions, namely, the shape of n-dimensional unit-volume simplices that maximize the length of a Hamilton cycle or path in their graph. The paper of Václav Koubek and Jiří Sichler in universal algebra concerns the relation of Q-universality and finite-to-finite universality of algebras. Christian Krattenthaler studies a simplicial complex associated to a colored root system, the generalized cluster complex, and proves a generalization of remarkable relations, discovered by Chapoton, concerning certain face counts.

Part II contains contributions in Ramsey theory. Ron Graham and József Solymosi give an elementary proof that an $n \times n$ integer grid colored by fewer than $\log\log n$ colors contains a monochromatic vertex set of an equilateral right triangle. András Gyárfás, Miklós Ruszinkó, Gábor N. Sárközy, and Endre Szemerédi use the regularity lemma for constructing coverings of edge-r-colored complete bipartite graphs by vertex-disjoint monochromatic cycles. Neil Hindman and Imre Leader consider a variant of partition-regularity of systems of linear equations, where they look for nonconstant solutions. Alexandr Kostochka and Naeem Sheikh construct infinitely many graphs for which the ratio of the induced Ramsey number to the weak induced Ramsey number is bounded away from 1, answering a question of Łuczak and Gorgol. Pavel Pudlák applies the recent Bourgain–Katz–Tao theorem on sums and products in finite fields to an explicit construction of 3-colorings of complete bipartite graphs with no large monochromatic complete bipartite subgraphs.

Topics in graph and hypergraph theory begin with Part III. József Balogh, Béla Bollobás, and Robert Morris consider the enumeration of ordered graphs not containing any ordered subgraph from a fixed (possibly infinite) set. The contribution of Zoltán Füredi, Kyung-Won Hwang, and Paul Weichsel is best described by its title: A proof and generalizations of the Erdős–Ko–Rado theorem using the method of linearly independent polynomials. Tomáš Kaiser, Daniel Král', and Serguei Norine prove that in any cubic bridgeless graph at least 60% of edges can be covered by two matchings, a result related to a conjecture of Berge and Fulkerson. Brendan Nagle, Vojtěch Rödl, and Mathias Schacht apply the hypergraph regularity method, a recent hypergraph generalization of the Szemerédi regularity lemma, to extremal problems for hypergraphs. Colin McDiarmid, Angelika Steger, and Dominic Welsh define addable graph classes, which include planar graphs and many other natural classes, and show that the probability of a random graph from such a class being connected is bounded away from both 0 and 1.

The papers in Part IV deal with graph homomorphisms. Noga Alon and Asaf Shapira survey the role of homomorphisms in recent results on constant-time probabilistic testing of graph properties. Christian Borgs, Jennifer Chayes, László Lovász, Vera T. Sós, and Katalin Vesztergombi look at the number of homomorphisms $G \to H$ from various perspectives such as graph isomorphism, reconstruction, probabilistic property testing, and statistical physics. Josep Díaz, Maria Serna, and Dimitrios Thilikos investigate an algorithmic problem, the fixed-parameter complexity of testing the existence of a homomorphism $G \to H$, where H is fixed, G is the input, and the number of preimages of certain vertices of H is restricted. Pavol Hell considers the Dichotomy Conjecture, stating that every class of constraint satisfaction problems specified by a fixed relational structure H is either polynomial-time solvable or NP-complete, establishes special cases, and connects the problem to graph colorings.

Part V is concerned mostly with generalized graph colorings. Those in the paper by Glenn Chappell, John Gimbel, and Chris Hartman are path-colorings of planar graphs. Dwight Duffus, Vojtěch Rödl, Bill Sands, and Norbert Sauer consider the minimum chromatic number of graphs and hypergraphs of large girth that cannot be homomorphically mapped to a specified graph or hypergraph, obtaining a new probabilistic hypergraph construction in the process. Mickaël Montassier, André Raspaud, and Weifan Wang prove acyclic 4-choosability of planar graphs with excluded cycles of certain lengths. Xuding Zhu presents an authoritative survey of the circular chromatic number, a parameter introduced by Vince in 1988 that carries more information than the chromatic number itself. The contribution of Claude Tardif sticks to the usual chromatic number and provides an algorithmic version of a special case of the celebrated Hedetniemi conjecture.

Part VI on graph embeddings opens with the paper by Hubert de Fraysseix and Patrice Ossona de Mendez, who consider embeddings of multigraphs in the k-dimensional Euclidean space such that automorphisms correspond to isometries and present an elegant characterization of such embeddings. Bojan Mohar extends an intriguing result of Youngs on quadrangulations of the projective plane, and constructs the first explicit family of infinitely many 5-critical graphs on a fixed surface. János Pach and Géza Tóth relate the torus crossing number of a graph to the planar crossing number. The survey of Jozef Širáň deals with the classification of regular maps (maps possessing the highest level of symmetry – their automorphism groups act transitively on the set of flags) and explains its intriguing connections to other branches of mathematics.

Presented in a part of its own comes the last article written by Jørgen Bang-Jensen, Bruce Reed, Mathias Schacht, Robert Šámal, Bjarne Toft and Ulrich Wagner about six problems posed by Jarik Nešetřil and their current status. This last paper is just a small example of the enormous influence Jarik has had on other researchers.

Dedication

Jarik Nešetřil is a scientist and an artist of extraordinary breadth and vision. His publication record and other achievements, including over a half-dozen textbooks and monographs, an honorary doctorate and an academy membership, speak for themselves. Equally important is Jarik's tireless work with students and younger colleagues. He founded the Prague Combinatorics Seminar, which helped shape the careers of several generations of Czech mathematicians and computer scientists. Among them, the present editors greatly benefited from Jarik's guidance, ideas, and endless enthusiasm. We would like to express our appreciation and wish him many more productive years filled with success and satisfaction.

Acknowledgement. Many people helped us with this volume. We are indebted to the referees, who generously gave their time and effort in order to improve the presentation of the contributions. In preparation of the final version we were greatly assisted by our technical editor Helena Nyklová, whose meticulous copyediting is warmly appreciated. We also thank Ms. J. Borkovcová for her kind permission to reprint the photograph of Jarik. Finally, we would like to thank the Institute for Theoretical Computer Science and Department of Applied Mathematics of Charles University for their support.[1]

Prague and Atlanta,
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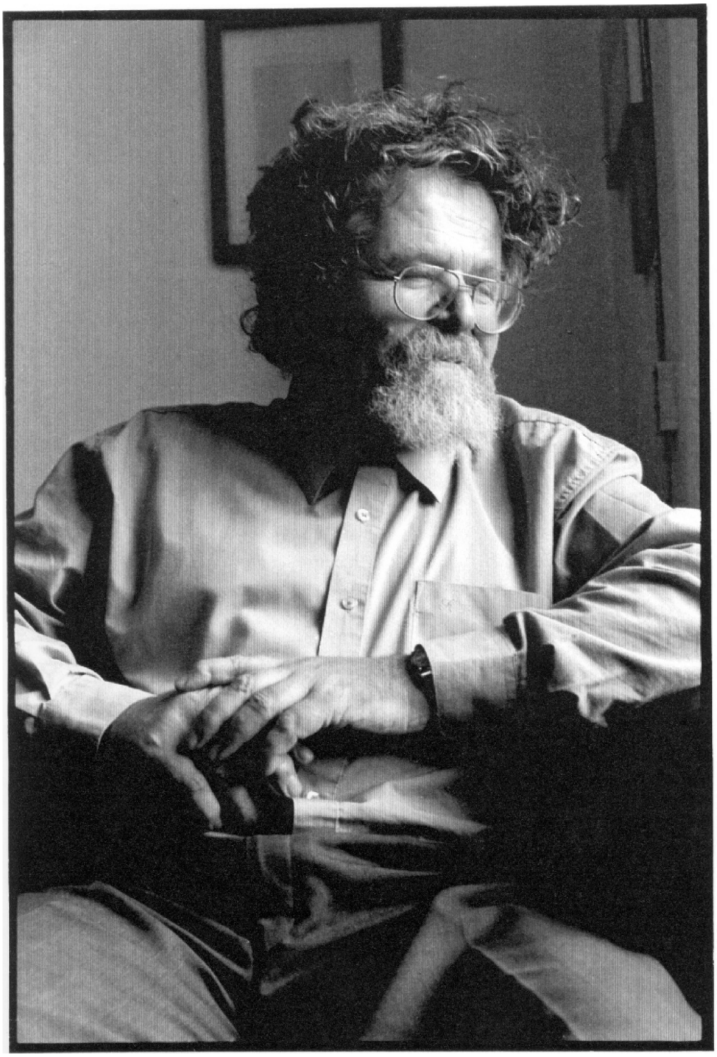

Photo of Jarik Nešetřil by Stanislav Tůma, © J. Borkovcová

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Part I

Algebra, Geometry, Numbers

Countable Almost Rigid Heyting Algebras

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Summary. For non-trivial Heyting algebras H_1, H_2 one always has at least one homomorphism $H_1 \to H_2$; if $H_1 = H_2$ there is at least one non-identical one. A Heyting algebra H is *almost rigid* if $|\,\mathsf{End}(H)| = 2$ and a system of almost rigid algebras $\mathcal{H}$ is said to be *discrete* if $|\,\mathsf{Hom}(H_1, H_2)| = 1$ for any two distinct $H_1, H_2 \in \mathcal{H}$. We show that there exists a discrete system of 2^ω countable almost rigid Heyting algebras.

AMS Subject Classification. 06D20, 18A20, 18B15.
Keywords. Heyting algebras, almost rigid, discrete system, Priestley duality.

Introduction

A *Heyting algebra*

$$(H; \vee, \wedge, \to, 0, 1)$$

is an algebra of type $(2, 2, 2, 0, 0)$ where $(H; \vee, \wedge, 0, 1)$ is a distributive $(0,1)$-lattice and the extra operation $x \to y$ satisfies the formula

$$z \leq x \to y \quad \text{iff} \quad x \wedge z \leq y.$$

That is to say, a Heyting algebra is a bounded relatively pseudocomplemented distributive lattice for which relative pseudocomplementation is taken to be a binary algebraic operation.

Since any two elements of a finite distributive lattice have a uniquely determined relative pseudocomplement, any finite distributive lattice can be viewed as a Heyting algebra. So too, any two elements of a Boolean algebra

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have a uniquely determined relative pseudocomplement $a \to b = \neg a \vee b$ which also provides an example of a Heyting algebra.

For an algebra A, let $\mathsf{Aut}(A)$ (resp $\mathsf{End}(A)$) denote the group of automorphisms (resp. the monoid of endomorphisms) of A under the operation of composition. An algebra is *automorphism rigid* provided $|\mathsf{Aut}(A)| = 1$, that is, the only automorphism of A is the identity.

Independently, Jónsson [Jón51], Katětov [Kat51], Kuratowski [Kur26], and Rieger [Rie51] have shown that there exists a proper class of non-isomorphic automorphism rigid Boolean algebras. Since, as observed above, every Boolean algebra is relatively pseudocomplemented, there exists a proper class of non-isomorphic automorphism rigid Heyting algebras as well.

In sharp contrast with respect to their automorphism groups, as independently shown by Magill [Mag72], Maxson [Max72], and Schein [Sch70], Boolean algebras are uniquely recoverable from their endomorphism monoids. That is, for Boolean algebras B and B', if $\mathsf{End}(B) \cong \mathsf{End}(B')$, then $B \cong B'$; or, by the result of Tsinakis ([Tsi79]), for bounded relative Stone lattices which are principal, $\mathsf{End}(S) \cong \mathsf{End}(S')$ implies $S \cong S'$ as well. However, this is far from the case for general Heyting algebras, where endomorphisms can be very few.

There are necessary non-identical homomorphisms, though. Every non-trivial Heyting algebra has at least one minimal prime ideal. Furthermore, for each minimal prime ideal I of a Heyting algebra H, and any other Heyting algebra H', $\varphi(x) = 0$ for $x \in I$ and 1 otherwise determines an homomorphism $\varphi : H \to H'$. Such homomorphisms will be referred to as

trivial homomorphisms.

In particular, if $|H| \geq 3$, then $|\mathsf{End}(H)| \geq 2$. Dismissing the necessary trivial endomorphisms one defines an *almost rigid* Heyting algebra H as such that $|\mathsf{End}(H)| = 2$. Thus, by the preceding remarks, H is almost rigid if and only if $|H| \geq 3$, if H has exactly one minimal prime ideal and the only endomorphism other than the identity is associated with the minimal prime as indicated.

In [AKS85] it was shown that there exists a proper class of non-isomorphic almost rigid Heyting algebras. All the almost rigid Heyting algebras from [AKS85] have cardinality $\geq 2^\omega$. Taking into account that for $|H| \geq 4$ there is no almost rigid finite Heyting algebra (for any such H either there are at least two minimal prime ideals I and I' or else a minimal prime ideal I and another prime ideal I' which is minimal with respect to properly containing I; in the former case obviously $|\mathsf{End}(H)| \geq 3$, in the latter case, for $a \in I' \setminus I$, $\psi(x) = 0$ for $x \in I$, a for $x \in I' \setminus I$, and 1 otherwise determines an endomorphism $\psi \in \mathsf{End}(H)$ distinct from φ associated with I, and $|\mathsf{End}(H)| \geq 3$ again) this begs the question whether there are countable almost rigid Heyting algebras. This is answered in the positive in this article. Moreover, we show that

there exists a system $\mathcal{H}$ of 2^ω countable almost rigid Heyting algebras such that

- *for each A in $\mathcal{H}$, $\mathsf{End}(H) = \{\mathrm{id}_A, \tau_{AA}\}$, and*

– *for any two distinct* A, B *in* $\mathcal{H}$ *there is exactly one homomorphism* $\tau_{AB}: A \to B$,

where the τ_{AB} *are unique trivial homomorphisms.*

For related background on Heyting or Boolean algebras see Balbes and Dwinger [BD74] or Koppelberg [Kop89].

1 Preliminaries

Let $(P, \leq)$ be a partially ordered set. For $Q \subseteq P$, set $\downarrow Q = \{x \in P \mid x \leq y \text{ for some } y \in Q\}$ and $\uparrow Q = \{x \in P \mid x \geq y \text{ for some } y \in Q\}$; for $Q = \{x\}$ we write just $\downarrow x$ and $\uparrow x$, respectively. A set Q is said to be *decreasing* or *increasing* if $Q = \downarrow Q$ or $Q = \uparrow Q$, respectively. For partially ordered sets P and P', a mapping $\varphi : P \longrightarrow P'$ is *order-preserving* providing $\varphi(x) \leq \varphi(y)$ whenever $x \leq y$.

A *Priestley space* $(P; \leq, \tau)$ is a partially ordered set $(P, \leq)$ endowed with a compact topology τ which is totally order-disconnected (namely, for any $x, y \in P$ such that $x \not\leq y$ there exists a clopen decreasing set $Q \subseteq P$ such that $x \notin Q$ and $y \in Q$).

As shown by Priestley [Pri70], [Pri72], the category of non-trivial distributive $(0,1)$-lattices together with all $(0,1)$-lattice homomorphisms is dually isomorphic to the category of Priestley spaces together with all continuous order-preserving maps. The equivalence functors are usually given as

$$\mathcal{P}(L) = \{x \mid L \neq x \text{ a prime ideal of } L\}, \quad \mathcal{P}(h)(x) = h^{-1}[x],$$
$$\mathcal{D}(X) = \{U \mid U = \downarrow U \subseteq X \text{ clopen }\}, \quad \mathcal{D}(f)(U) = f^{-1}[U];$$

$\mathcal{P}(L)$ is endowed with a suitable topology and ordered by inclusion.

Since every Heyting algebra is a distributive $(0,1)$-lattice, it is to be expected that the category of all non-trivial Heyting algebras is dually isomorphic to a well-defined subcategory of the category of all Priestley spaces. And indeed, the Priestley spaces X dual to Heyting algebras are precisely those with the additional property that $\uparrow U$ is clopen whenever U is clopen. Such Priestley spaces will be called

h-spaces,

and if L, M are Heyting algebras then the Heyting homomorphisms $h : L \to M$ correspond to the Priestley maps f such that, moreover,

$$f(\downarrow x) = \downarrow f(x).$$

Such maps will be referred to as

h-maps.

It is this dual equivalence that we will use in order to establish our result.

2 The Construction

The Posets

Set $X = \{n \in \mathbb{N} \mid n \geq 5\}$ and decompose this set as follows. Start with

$$X_1 = \{5\}, \quad \phi(2) = 5$$
$$X_2 = X_{2,1} = \{6, 7, 8, 9, 10\}, \quad \phi(3) = 10,$$

and further proceed inductively: if $X_k = \{\phi(k)+1, \phi(k)+2, \dots, \phi(k+1)\}$ is already defined (and, hence, $\phi(k)$ and $\phi(k+1)$ too), take, for each element $\phi(k)+j \in X_k$, a set $X_{k+1,j}$ determined as follows

$X_{k+1,1} = \{\phi(k+1)+1, \dots, \phi(k+1)+\phi(k)+1\}$, the first $\phi(k)+1$ natural numbers after $\phi(k+1)$,
$X_{k+1,2} = \{\phi(k+1)+\phi(k)+2, \phi(k+1)+\phi(k)+3, \dots, \phi(k+1)+2\phi(k)+3\}$, the next $\phi(k)+2$ natural numbers after $\phi(k+1)+\phi(k)+1$,
where, in general, for $1 \leq j \leq \phi(k+1)-\phi(k)$,
$X_{k+1,j} = \{\phi(k+1)+(j-1)\phi(k)+\binom{j}{2}+1, \dots, \phi(k+1)+j\phi(k)+\binom{j+1}{2}\}$,
the next $\phi(k)+j$ natural numbers after $\phi(k+1)+(j-1)\phi(k)+\binom{j}{2}$.

Then set

$$X_{k+1} = \{\phi(k+1)+1, \phi(k+1)+2, \dots, \phi(k+2)\} = \bigcup_{j=1}^{\phi(k+1)-\phi(k)} X_{k+1,j}.$$

For triples x, y, z of distinct elements belonging to the same X_k choose distinct

$$\tau(x, y, z) \notin X$$

and set

$$T = \{\tau(x, y, z) \mid x, y, z\}$$

and

$$Y = X \cup T \cup \{\omega\}$$

where ω is an element $\notin X \cup T$.

Now choose a countably infinite system $\mathbb{Q}$ of quadruples $\{x_1, x_2, x_3, x_4\}$ such that

- for every $q = \{x_1, x_2, x_3, x_4\}$, $q \subseteq X_k$ for some k, and
- if $p, q \in \mathbb{Q}$, $p \neq q$, then $p \cap q = \emptyset$.

For $A \subseteq \mathbb{Q}$ set

$$Z(A) = Y \cup A$$

and define an order $\sqsubseteq$ on $Z(A)$ by

$$\omega \sqsubseteq x \text{ for all } x \in Z(A),$$

and by transitivity from the successor relation $\prec$ where

$$x, y, z \prec \tau(x, y, z),$$
$$x_i \prec \{x_1, x_2, x_3, x_4\} \text{ for } \{x_1, x_2, x_3, x_4\} \in A,$$
$$\text{and for } x \in X_{k+1,j}, \quad x \prec \phi(k) + j.$$

Note that

$$\uparrow x \textit{ is finite for all } x \in Z(A) \setminus \{\omega\}.$$

To simplify the notation define the *degree*

$$d\,(n) = n \quad \text{for} \quad n \in X,$$
$$d\,(\tau(x, y, z)) = 3,$$
$$d\,(q) = 4 \text{ for } q \in A;$$

for ω the degree is undefined.

The Topology

$Z(A)$ is endowed with the topology in which
U is open iff either $\omega \notin U$ or $Z(A) \setminus U$ is finite.
Thus we have

Observation 2.1. *The clopen sets are precisely the finite M not containing ω and the complements of such sets,and hence the $\uparrow M$ with clopen M are clopen.*
If x is not $\sqsubseteq y$ then $y \notin \uparrow x$ and $\uparrow x$ is clopen. Thus, each $Z(A)$ with the order $\sqsubseteq$ and the topology just defined is an h-space.

From this we immediately obtain

Fact 2.2. *All the Heyting algebras $\mathcal{D}(Z(A))$ are countable.*

In the sequel, f will always be an h-map $Z(A) \to Z(B)$.

Lemma 2.3. *1. $f(\omega) = \omega$.*
2. If $f(M) = \{x\}$ for an infinite $M \subseteq Z(A)$ then $x = \omega$.

Proof. 1. $\{f(\omega)\} = f(\downarrow\omega) = \downarrow f(\omega)$; hence $\downarrow f(\omega)$ has just one element.
2. For an infinite M we have $\omega \in \overline{M}$. Thus, $\omega = f(\omega) \in f(\overline{M}) \subseteq \overline{f(M)} = \{x\}$. ◻

A *branch* of an element $x \in Z(A)$ is any $\downarrow y$ with $y \prec x$. Note that the degree $d(x)$ defined above is the number of distinct branches of x.

Lemma 2.4. *1. If $t = \tau(x_1, x_2, x_3)$ (resp. $q = \{x_1, x_2, x_3, x_4\} \in A$) and $f(x_i) \sqsubset f(t)$ (resp. $f(q)$) for all i then we cannot have $d(f(t)) \geq 4$ (resp. $d(f(q)) \geq 5$).*

2. *If $t = \tau(x_1, x_2, x_3)$ and $f(x_i) \sqsubset f(t)$ for all i then we cannot have two $f(x_i), f(x_j)$, $i \neq j$ in the same branch $\downarrow y$ of $f(t)$.*
3. *If $a_1, a_2, a_3 \prec a \in Z(A)$ are distinct, $f(a_1) \sqsubseteq f(a_2) \sqsubset f(a)$ and $f(a_3) \sqsubset f(a)$ then all the $f(a_i)$ are in the same branch of $f(a)$.*

Proof. 1. $\downarrow f(t) = f(\downarrow t) = \{f(t)\} \cup \downarrow f(x_1) \cup \downarrow f(x_2) \cup \downarrow f(x_3)$ (resp. $\{f(q)\} \cup \downarrow f(x_1) \cup \downarrow f(x_2) \cup \downarrow f(x_3) \cup \downarrow f(x_4)$) and hence $\downarrow f(t)$ cannot have more than three (resp. four) branches.
2. $\downarrow f(t)$ consists of at least three branches and hence it cannot be covered by $\downarrow y$ and just one more branch.
3. By 2, $a \in X \cup A$. Consider $t = \tau(a_1, a_2, a_3)$. By 2, $f(t) = f(a_2)$ or $f(t) = f(a_3)$. Then either $f(a_3) \sqsubseteq f(a_2)$ or $f(a_2) \sqsubseteq f(a_3)$. ◻

Observation 2.5. *The map*

$$const_\omega = const_\omega^{AB} : Z(A) \to Z(B)$$

defined by $const_\omega(x) = \omega$ for all $x \in Z(A)$ is an h-map.

(This is the Priestley image of the unique trivial homomorphism between the corresponding Heyting algebras, each of which has precisely one minimal prime ideal.)

3 The Result

Lemma 3.1. *For $a \neq \omega$ such that $f(a) \neq \omega$ one cannot have $d(f(a)) < d(a)$.*

Proof. If $f(b) \sqsubset f(a)$ for all $b \prec a$ we are led to a contradiction by 2.4.3: let C consist of all the $c \prec f(a)$. Then there have to be two distinct b_1, b_2 with $f(b_i) \sqsubseteq c$ for some $c \in C$ and we have an $x \sqsubset a$ such that $c = f(x)$. Now $x \sqsubseteq b$ for some $b \prec a$ and we have $f(b_1), f(b_2) \sqsubseteq f(b)$ (of course, b can be one of the b_i). Now by 2.4.3 all the $f(b)$ with $b \prec a$ are in the same branch of $f(a)$, a contradiction.

Thus, there is an $a_1 \prec a_0 = a$ such that $f(a_1) = f(a)$ and as $d(a_1) > d(a_0)$ we can repeat the procedure to obtain

$$a_0 \succ a_1 \succ a_2 \succ \cdots \quad \text{with } f(a_i) = f(a),$$

and by 2.3.2 we have $f(a) = \omega$ contradicting the assumption. ◻

Thus in particular

$$f[X \cup \{\omega\}] \subseteq X \cup \{\omega\}.$$

In the following four lemmas, f is, as before, an h-map $Z(A) \to Z(B)$, but since all the facts are relevant for the restriction $X \cup \{\omega\} \to X \cup \{\omega\}$ only (and since $\downarrow (X \cup \{\omega\}) = X \cup \{\omega\}$), we can use expressions such as $f(a) = a$, $f(f(a))$, or $f(\downarrow a) = \downarrow f(a)$.

Lemma 3.2. *If $a \in X$ and $f(a) \neq \omega$ then $f(a) \sqsubseteq a$ and $f(f(a)) = f(a)$. If $f(a) \sqsubset a$, we have an $a' \sqsubseteq a$ such that $a' \succ f(a) = f(a')$.*

Proof. We already know that $d(f(a)) \geq d(a)$, hence if $f(a) \neq \omega$ we have $f(a) \in X$. Suppose that $d(f(a)) > d(a)$. Then we cannot have $f(b) \sqsubset f(a)$ for all $b \prec a$ since in such a case

$$f(\downarrow a) = \{f(a)\} \cup \bigcup \{f(\downarrow b) \mid b \prec a\}$$

cannot cover $\downarrow f(a)$.

Hence for some $a_1 \prec a$ we have $f(a_1) = f(a)$. Now $d(a_1) > d(a)$. If we still have $d(f(a)) > d(a_1)$, we can repeat the argument and ultimately we obtain $a = a_0 \succ a_1 \succ \cdots \succ a_k$ with $f(a_i) = f(a)$, $d(a_i) < d(f(a))$ for $i < k$, and $d(a_k) = d(f(a))$ (by 3.1 we cannot have $d(a_k) > d(f(a))$). Since $d(a_k) \geq 5$, a_k and $f(a) = f(a_k)$ are in X and hence $a_k = f(a_k)$ by the equality of the degrees. ◻

Lemma 3.3. *If for an $a \in X$ one has $f(a) = a$ then f is identical on the whole of $\downarrow a$.*

Proof. Let $x \sqsubset a$ be an element with the shortest path $x \prec x_1 \prec \cdots \prec a$ such that $f(x) \sqsubset x$. As $\downarrow a = f(\downarrow a)$, $x = f(y) \sqsubset y$ for some $y \sqsubset a$. But then y is one of the x_i which is a contradiction, by Lemma 3.2. ◻

Lemma 3.4. *If there is an $x \in X$ such that $f(x) = b \sqsubset x$ then there is a $y \in X$ such that $f(y) = u \sqsubset y$ and b, u are incomparable.*

Proof. $f(b) = b$ and hence f is identical on $\downarrow b$. Choose $b_1 \neq b_2$, $b_i \prec b$; thus $f(b_i) = b_i$. Let $b_1, b_2 \in X_k$. Choose a $y \in X_k$ incomparable with b (since $b \sqsubset x \sqsubseteq 5$ there exist such incomparable elements). Set

$$t = \tau(b_1, b_2, y).$$

Now $b_i = f(b_i) \sqsubset f(t)$ (we cannot have an equality as b_i are incomparable) and hence $f(y) \neq \omega$ (else $f(\downarrow t) = \{f(t)\} \cup \downarrow b_1 \cup \downarrow b_2 \neq \downarrow f(t)$). Thus, $f(y) \sqsubseteq y$. We cannot have $f(y) = y$ for all such y: in such a case f would be identical on the whole of X_k which would fix all the elements above as well, including 5 and x, contradicting the assumption (if $f(a) = a$ for all $a \prec n \in X$ then $f(n) \sqsupseteq a$ for all $a \prec n$ and hence $f(n) \sqsupseteq n$; we cannot have $f(n) \sqsupset n$ though, since that would imply $d(f(n)) < d(n)$).

Thus there has to be some such y with $u = f(y) \sqsubset y$, and now u is incomparable with b. ◻

Lemma 3.5. *For $\omega \neq x \in X$ one cannot have $f(x) \neq \omega$ and $d(f(x)) > d(x)$.*

Proof. Suppose there is such an x. By 3.2 and 3.4 we can choose an instance of $b \prec a$ such that $f(a) = f(b) = b$ and that there exists a $u \in X$ incomparable with b such that $f(u) = u$ is in an X_l with $l \leq k$ where b is in X_k (this can be achieved but exchanging the b and u in 3.4 if necessary). Consider a $c \prec a$, $c \neq b$ and a general $z \neq b, c$ in X_k. Set $t = \tau(b, c, z)$. Since $f(c) \sqsubseteq f(a) = f(b)$ we cannot have (see 2.4.2)

$$f(b), f(c), f(z) \sqsubset f(t).$$

Now $f(t)$ cannot be equal to $f(z)$ and distinct from the others since then $b = f(b) \sqsubset f(t) = f(z)$ and hence, z being in the same X_k as b, $d(f(z)) < d(z)$ contradicting 3.1. Thus we have $f(t)$ equal to either $f(c)$ or $f(b)$ and hence $f(z) \sqsubseteq f(b) = b$.

Thus, $f(X_k) \subseteq \downarrow b$. Take a $v \sqsubseteq u$ in X_k. Then $f(v) = v$ by 3.3 and we have a contradiction: v cannot be in $\downarrow b$ since u and b are incomparable and the subposet $(X, \sqsubseteq)$ of $Z(A)$ is a tree. ◻

Lemma 3.6. *Let $f : Z(A) \to Z(B)$ be an h-map. Then either $f(X) = \{\omega\}$ or $f(n) = n$ for all $n \in X$.*

Proof. By 3.5 and 3.1, $f(5) = \omega$ or $f(5) = 5$. Hence $f(5) = 5$ and, by 3.3, f is identical on $X = \downarrow 5$. ◻

Theorem 3.7. *Let $f : Z(A) \to Z(B)$ be an h-map. Then either f is* const_ω *or it is the inclusion map $Z(A) \subseteq Z(B)$.*

On the other hand, any inclusion $A \subseteq B$, with $A, B \subseteq \mathbb{Q}$ can be extended to an inclusion h-map $Z(A) \subseteq Z(B)$.

Proof. If $f(5) = \omega$ then $f(X) = \{\omega\}$ by monotonicity. Now if for $y \in T$ or $y \in A$ one should have $f(y) = x \neq \omega$ we had $\downarrow x \neq f(\downarrow y) = \{f(y)\} \cup \{\omega\}$. Thus, also $f(y) = \omega$.

If $f(5) \neq \omega$ then f is identical on X. This also fixes T, since for $t = \tau(x, y, z)$, $x, y, z \sqsubseteq f(t)$ and equality is impossible since x, y, z are incomparable, and since any other element greater then all the x, y, z has too many branches to be covered by $f(\downarrow t)$. Finally for $q = \{x_1, x_2, x_3, x_4\} \in A$ one cannot have $f(q) \in T$ by 3.1. Since the x_i are incomparable, $f(q)$ coincides with none of the $f(x_i) = x_i$ and hence, by 2.4.1, $f(q) \notin X$. By monotonicity, $f(q) \neq \omega$ and hence $f(q) \in B$. But there is only one $p \in \mathbb{Q}$ such that $x_i \sqsubset p$ for $i = 1, 2, 3, 4$, namely q itself. The second statement is obvious. ◻

As above, denote by $\mathbb{N}$ the set of all natural numbers.

Theorem 3.8. *There exist countable almost rigid Heyting algebras $H(A)$ associated with the subsets $A \subseteq \mathbb{N}$ such that*

- *if $A \nsubseteq B$ there is no non-trivial homomorphism $H(A) \to H(B)$, and*
- *if $A \subseteq B$ there exists exactly one non-trivial homomorphism $H(A) \to H(B)$.*

Consequently there exist 2^ω countable almost rigid Heyting algebras such that the only homomorphism between any two distinct of them is the trivial one.

Proof. The first part immediately follows from 3.7 and 2.2.

For the second statement it suffices to observe that there are 2^ω many subsets of $\mathbb{N}$ such that no two of them are in inclusion.

For any $N \subseteq \mathbb{N}$ consider the set

$$\widetilde{N} = \{2n \mid n \in N\} \cup \{2n+1 \mid n \notin N\}.$$

Then $\widetilde{N}_1 \subseteq \widetilde{N}_2$ only if $N_1 = N_2$. $\square$

References

[AKS85] M. E. Adams, V. Koubek, and J. Sichler, *Endomorphisms and homomorphisms of Heyting algebras*, Algebra Universalis 20 (1985), 167–178.

[BD74] R. Balbes, P. Dwinger, *"Distributive Lattices"*, University of Missouri Press, Columbia, Missouri, 1974.

[Jón51] B. Jónsson, *A Boolean algebra without proper automorphisms*, Proc. Amer. Math. Soc. 2 (1951), 766–770.

[Kat51] M. Katětov, *Remarks on Boolean algebras*, Colloq. Math. 2 (1951), 229–235.

[Kop89] S. Koppelberg, *"Handbook of Boolean Algebras I"*, North-Holland, Amsterdam, 1989.

[Kur26] C. Kuratowski, *Sur la puissance de l'ensemble des "nombres de dimensions" de M. Fréchet*, Fund. Math. 8 (1926), 201–208.

[Mag72] K. D. Magill, *The semigroup of endomorphisms of a Boolean ring*, Semigroup Forum 4 (1972), 411–416.

[Max72] C. J. Maxson, *On semigroups of Boolean ring endomorphisms*, Semigroup Forum 4 (1972), 78–82.

[Pri70] H. A. Priestley, *Representation of distributive lattices by means of ordered Stone spaces*, Bull. London Math. Soc. 2 (1970), 186–190.

[Pri72] H. A. Priestley, *Ordered topological spaces and the representation of distributive lattices*, Proc. London Math. Soc. **324** (1972), 507–530.

[Rie51] L. Rieger, *Some remarks on automorphisms of Boolean algebras*, Fund. Math. 38 (1951), 209–216.

[Sch70] B. M. Schein, *Ordered sets, semilattices, distributive lattices, and Boolean algebras with homomorphic endomorphism semigroups*, Fund. Math. 68 (1970), 31–50.

[Tsi79] C. Tsinakis, *Brouwerian Semilattices Determined by their Endomorphism Semigroups*, Houston J. Math. 5 (1979), 427–436.

Piecewise-Bohr Sets of Integers and Combinatorial Number Theory

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Summary. We use ergodic-theoretical tools to study various notions of "large" sets of integers which naturally arise in theory of almost periodic functions, combinatorial number theory, and dynamics. Call a subset of **N** a Bohr set if it corresponds to an open subset in the Bohr compactification, and a piecewise Bohr set (PWB) if it contains arbitrarily large intervals of a fixed Bohr set. For example, we link the notion of PWB-sets to sets of the form A+B, where A and B are sets of integers having positive upper Banach density and obtain the following sharpening of a recent result of Renling Jin.

Theorem. If A and B are sets of integers having positive upper Banach density, the sum set A+B is PWB-set.

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1 Introduction to Some Large Sets of Integers

In combinatorial number theory, as well as in dynamics, various notions of "large" sets arise. Some familiar notions are those of sets of positive (upper) density, syndetic sets, thick sets (also called "replete"), return-time sets (in dynamics), sets of recurrence (also known as Poincaré sets), (finite or infinite) difference sets, and Bohr sets. We will here introduce the notion of "piecewise-Bohr" sets (or PWB-sets), as well as "piecewise-Bohr$_0$" sets (or PWB$_0$-sets), and we'll show how they arise in some combinatorial number-theoretic questions.

We begin with some basic definitions and elementary considerations. We'll say that a subset $A \subset \mathbb{Z}$ has *positive upper* (Banach) *density*, $d^*(A) > 0$,

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if for some $\delta > 0$, there exist arbitrarily large intervals of integers $J = \{a, a+1, \dots, a+l-1\}$ with $\frac{|J \cap A|}{|J|} \geq \delta$. (Here $|S|$ is the cardinality of the set S; $d^*(A) = \text{l.u.b.}\{\delta \text{ as above}\}$.) Syndetic sets are special cases of sets with positive upper density. Namely, A is *syndetic* if for some l, every interval J of integers with $|J| \geq l$ intersects A. Clearly $d^*(A) \geq 1/l$ in this case. We'll say a set A is *thick* if it contains arbitrarily long intervals; thus A is syndetic $\Leftrightarrow$ $\mathbb{Z} \backslash A$ is not thick $\Leftrightarrow$ $A \cap B \neq \emptyset$ for any thick set B. For any distinct r integers $\{a_1, a_2, \dots, a_r\}$ the set $\{a_j - a_i | 1 \leq i < j \leq r\}$ is called an r-*difference* set or a Δ_r-set. Every thick set contains some r-difference set for every r. This is obvious for $r = 2$, and inductively, if A is thick and if A contains the $(r-1)$-difference set formed from $\{a_1, \dots, a_{r-1}\}$, by choosing a_r in the middle of a large enough interval in A, we can complete this to an r-difference set. It follows that for any r, a set that meets every r-difference set is syndetic. An example of this is the set of (non-zero) differences $A - A = \{x - y : x, y \in A, x \neq y\}$ when A has positive upper density. For if $d^*(A) > 1/r$ and if the numbers $a_1, a_2, \dots, a_r$ are distinct, the sets $A + a_1, A + a_2, \dots, A + a_r$ cannot be disjoint; so, for some $1 \leq i < j \leq r$, $a_j - a_i \in A - A$. One conclusion which is behind much of our subsequent discussion is that if A has positive upper density, then $A - A$ is syndetic. We shall see in §3 that $d^*(A) > 0$ implies that $A - A$ is a *piecewise-Bohr* set.

Definition 1.1. *$S \subset \mathbb{Z}$ is a* Bohr *set if there exists a trigonometric polynomial $\psi(t) = \sum_{k=1}^{m} c_k e^{i\lambda_k t}$, with the λ_k real numbers, such that the set*

$$S' = \{n \in \mathbb{Z} : \operatorname{Re} \psi(n) > 0\}$$

is non-empty and $S \supset S'$. When $\psi(0) > 0$ we say S is a Bohr$_0$ set. (Compare with [Bilu97]).

The fact that a Bohr set is syndetic is a consequence of the almost periodicity of trigonometric polynomials. It is also a consequence of the "uniform recurrence" of the Kronecker dynamical system on the m-torus

$$(\theta_1, \theta_2, \dots \theta_m) \longrightarrow (\theta_1 + \lambda_1, \theta_2 + \lambda_2, \dots, \theta_m + \lambda_m).$$

Indeed, it is not hard to see that a set $S \subset \mathbb{Z}$ is Bohr if and only if there exist $m \in \mathbb{N}$, $\alpha \in \mathbb{T}^m$ and an open set $U \subset \mathbb{T}^m$ such that $S \supset \Big\{n \in \mathbb{Z} : n\alpha \in U\Big\}$.

Alternatively we can define Bohr sets and Bohr$_0$ sets in terms of the topology induced on the integers $\mathbb{Z}$ by imbedding $\mathbb{Z}$ in its Bohr compactification. Namely, a set in $\mathbb{Z}$ is Bohr if it contains an open set in the induced topology, and it is Bohr$_0$ if it contains a neighborhood of 0 in this topology.

We can apply the foregoing observations regarding $A - A$ to dynamical systems. We shall be concerned with *measure preserving systems* $(X, \mathcal{B}, \mu, T)$, where $(X, \mathcal{B}, \mu)$ is a probability space, $T \colon X \to X$ a measurable measure preserving transformation. We assume (for simplicity) that the system is *ergodic*

($T^{-1}A = A$ for $A \in \mathcal{B} \Rightarrow \mu(A)\mu(X\backslash A) = 0$). The ergodic theorem then ensures that for $A \in \mathcal{B}$ with $\mu(A) > 0$, the orbit $\{T^n x\}_{n\in\mathbb{Z}}$ of almost every x visits A along a set of times $V(x, A) = \{n : T^n x \in A\}$ of positive density. If we set $R_1(A) = \{n : A \cap T^{-n}A \neq \emptyset\}$ (the return time set of A), then for any x, $R_1(A) \supset V(x, A) - V(x, A)$. Hence $R_1(A)$ is syndetic. We can define a smaller set $R(A) = \{n : \mu(A \cap T^{-n}A) > 0\} = R(A')$ where $A' = A \setminus \bigcup\{(A \cap T^{-n}A) : \mu(A \cap T^{-n}A) = 0\}$, and it follows that $R(A)$ is also syndetic. This can be seen directly as well (and for arbitrary measure preserving systems), but the present argument illustrates the connection of dynamics to combinatorial properties of sets. We shall call sets containing sets of the form $R(A)$, where $\mu(A) > 0, RT$-sets (for return time). A set meeting every RT-set is called a Poincaré set since Poincaré's recurrence theorem gives content to the property by implying that $R(A)$ is never empty for $\mu(A) > 0$ even if T is not ergodic. These are also known in the literature as *intersective* sets. (See [Ruz82]). Much is known about these (see [Fur81], [B-M86], [BH96], [BFM96]). In particular $\{n^r; n = 1, 2, \ldots\}$ is a Poincaré set for each $r = 1, 2, 3, \ldots$

For a family $\mathcal{F}$ of subsets of $\mathbb{Z}$ it is customary to denote by $\mathcal{F}^*$ the dual family: $\mathcal{F}^* = \{S \subset \mathbb{Z} : \forall S' \in \mathcal{F}, S \cap S' \neq \emptyset\}$. Note that $\{\text{syndetic}\} = \{\text{thick}\}^*$, $\{\text{thick}\} = \{\text{syndetic}\}^*$ and $\{RT\} = \{\text{Poincaré}\}^*, \{\text{Poincaré}\} = \{\text{RT}\}^*$.

We have seen above that a Δ_r^*-set is necessarily syndetic. One of our objectives is to sharpen this statement.

We will need the notion of a "PW-$\mathcal{F}$" set for a family $\mathcal{F}$ of subsets of $\mathbb{Z}$. "PW" stands for "piecewise" and if $S \in \mathcal{F}$ and Q is a thick set then we shall say $S \cap Q$ is PW-$\mathcal{F}$ (or $S \cap Q \in$ PW-$\mathcal{F}$). Clearly this notion is useful only for families of syndetic sets. "PW-syndetic" is itself a useful notion. Van der Waerden's theorem [GRS80] implies that syndetic sets contain arbitrarily long arithmetic progressions. In fact this is true for PW-syndetic sets. Unlike the family of syndetic sets, the latter have the "divisibility" property: if S is PW-syndetic and $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is a finite partition, then some S_i is PW-syndetic, see [Bro71]. A recent result of Renling Jin [Jin02] is the following:

Theorem 1.2. *If $A, B \subset \mathbb{Z}$ and $d^*(A) > 0$, $d^*(B) > 0$, then $A + B$ is PW-syndetic.*

We will sharpen this to

Theorem I. *If $A, B \subset \mathbb{Z}$ and $d^*(A) > 0$, $d^*(B) > 0$, then $A + B$ is a PW-Bohr set (PWB-set).*

In particular $d^*(A) > 0$ will imply that $A - A$ is a PW-Bohr set. More precisely it is a PW-Bohr$_0$ (PWB$_0$)-set. This will also follow from our earlier observation that it is a Δ_r^*-set for sufficiently large r, and from

Theorem II. *For each $r \geq 2$, a Δ_r^*-set is PW-Bohr$_0$.*

It is not hard to see that the prefix "PW" is indispensable in these theorems. For example $A = \bigcup [10^n, 10^n + n]$ has $d^*(A) = 1$ but $A + A$ is not syndetic. Also since $x^3 + y^3 = z^3$ has no solution in non-zero integers, it follows that the set of non-cubes $S = \mathbb{Z} \backslash \{n^3; n = \pm 1, \pm 2, \pm 3, \ldots\}$ is a Δ_3^* set. But by Weyl's equidistribution theorem S is not a Bohr$_0$-set. (See Theorem 4.1 below for a stronger form of this observation.)

From Theorem I we shall deduce the following result which should be compared with a theorem due to Ruzsa ([Ruz82], Theorem 3) which states that if $d^*(A) > 0$, then $A + A - A$ is a Bohr set. (Both Ruzsa's theorem and our result can be viewed as improvements on a theorem of Bogoliouboff ([Bog39], [Føl54]) which implies that if $d^*(A) > 0$, then $A - A + A - A$ is a Bohr set.)

Corollary 1.3. *If A, B, C are three subsets of $\mathbb{Z}$ with positive upper density and one of them is syndetic, then $A + B + C$ is a Bohr set.*

2 Measure Preserving Systems, Time Series, and Generic Schemes

In this section we introduce a basic tool which will be needed repeatedly: the correspondence between data given on large intervals of time ("time series") and measure preserving dynamical systems. This tool has been used previously under the name "correspondence principle" (see e.g., [Ber96]) and here we present it in a more general form. We repeat the definition of a measure preserving system which was given informally in §1.

Definition 2.1. *A* measure preserving system *is a quadruple $(X, \mathcal{B}, \mu, T)$ where $(X, \mathcal{B}, \mu)$ is a probability space where we assume $\mathcal{B}$ is countably generated, and T is a measurable, invertible, and measure preserving map, $T: X \to X$. The system is* ergodic *if every measurable T-invariant set has measure* 0 *or* 1.

For a measurable function $f: X \to \mathbb{C}$ we denote by Tf the function $Tf(x) = f(Tx)$. We take note of the ergodic theorem (see, for example, [Kre85]):

Theorem 2.2. *If $(X, \mathcal{B}, \mu, T)$ is a measure preserving system and $f \in L^1(X, \mathcal{B}, \mu)$, then*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n f = \overline{f}$$

exists almost everywhere. If $f \in L^p(X, \mathcal{B}, \mu)$, $1 \leq p < \infty$, the convergence is in L^p as well. If the system is ergodic then $\overline{f} = \int f d\mu$ a.e., so that the average of the sequence $\{f(T^n x)\}$ equals a.e. the average of f over X.

Sequences of the form $\{f(T^n x)\}_{a\leq n\leq b}$ are referred to as "time series". In a certain sense the ergodic theorem enables one to reconstruct a dynamical system from "time series data". We shall make this precise in the notion of "generic schemes" which we proceed to define. In the next definitions the indices l and r range over the natural numbers.

Definition 2.3. *An* array *is a sequence $\{J_l\}$ of intervals of integers, $J_l = \{a_l, a_l+1, \ldots, b_l\}$ for which $|J_l| = b_l - a_l + 1 \to \infty$ as $l \to \infty$.*

Definition 2.4. *A* scheme *$(\{J_l\}, \{\xi_r^l\})$ is an array $\{J_l\}$ together with a doubly indexed set of complex-valued functions $\{\xi_r^l\}$ where, for each r, $\xi_r^l(n)$ is defined for $n \in J_l$ and, for each r, the functions $\{\xi_r^l; l = 1, 2, \ldots\}$ are uniformly bounded. For $n \notin J_l$ we take $\xi_r^l(n) = 0$. The $\{\xi_r^l\}$ will be referred to as* time series. *They are defined on all of $\mathbb{Z}$ but only the values on J_l have significance. The following notion relates closely to that of a "stationary stochastic process".*

Definition 2.5. *A* process *$(X, \mathcal{B}, \mu, T, \Phi)$ consists of a measure preserving system $(X, \mathcal{B}, \mu, T)$ together with an at most countable ordered set $\Phi = \{\varphi_1, \varphi_2, \ldots\}$ of L^∞-functions on X such that $\mathcal{B}$ is the σ-algebra generated by the functions of Φ and their translates under T. (When the φ_i are complex valued we assume Φ closed under conjugation). A process is ergodic if the underlying measure preserving system is ergodic.*

Finally we have

Definition 2.6. *A scheme $(\{J_l\}, \{\xi_r^l\})$ is* generic *for a process $(X, \mathcal{B}, \mu, T, \Phi)$ if for every m and for every choice of $i_1, i_2, \ldots, i_m$ and $j_1, j_2, \ldots, j_m$ (the indices here need not be distinct):*

$$\lim_{l\to\infty} \frac{1}{|J_l|} \sum_{n\in J_l} \xi_{i_1}^l(n+j_1)\xi_{i_2}^l(n+j_2)\cdots\xi_{i_m}^l(n+j_m) \tag{1}$$
$$= \int_X T^{j_1}\varphi_{i_1} T^{j_2}\varphi_{i_2}\cdots T^{j_m}\varphi_{i_m}\, d\mu$$

It will be convenient to introduce the countable family Φ^* consisting of the products appearing in (1):

$$\Phi^* = \{\psi = T^{j_1}\varphi_{i_1} T^{j_2}\varphi_{i_2}\cdots T^{j_m}\varphi_{i_m}\}$$

The corresponding time series have the form

$$\zeta^l(n) = \xi_{i_1}^l(n+j_1)\xi_{i_2}^l(n+j_2)\cdots\xi_{i_m}^l(n+j_m),$$

and when (1) holds, we say that $\{\zeta^l\}$ *represents* ψ.

It will be convenient in the sequel to regard Φ^* as the increasing union of finite sets, $\Phi^* = \bigcup_{h=1}^{\infty} \Phi_h^*$. The subscript h has no significance other than as an index with $\Phi_1^* \subset \Phi_2^* \subset \cdots \subset \Phi_h^* \subset \cdots$.

We note that the ergodic theorem implies that if $(X,\mathcal{B},\mu,T)$ is ergodic, then for almost every $x_0 \in X$, the scheme $(\{J_l\},\{\xi_r^l\}$ is generic for the process $(X,\mathcal{B},\mu,T,\Phi)$ with $J_l=[1,l]$ and $\xi_r^l(n)=\varphi_r(T^n x_0)$ independently of l.

The main result of this section goes in the opposite direction, and will attach to an arbitrary scheme an ergodic process. First we need the notions of *subarrays* and *subschemes.*

Definition 2.7. *An array $\{H_l\}$ is a* subarray *of $\{J_l\}$ if $l \to L_l$ is a monotone increasing function from $\mathbb{N}$ to $\mathbb{N}$ and H_l is a subinterval of J_{L_l}.*

Definition 2.8. *A scheme $(\{H_l\},\{\eta_r^l\})$ is a* subscheme *of $(\{J_l\},\{\xi_r^l\})$ if $\{H_l\}$ is a subarray of $\{J_l\}: H_l \subset J_{L_l}$, and η_r^l is the restriction of $\xi_r^{L_l}$ to H_l.*

Our main result in this section is

Theorem 2.9. *For any scheme $(\{J_l\},\{\xi_r^l\})$ there exists a subscheme and an ergodic process for which the subscheme is generic.*

Proof. First we will pass to a subscheme which is generic for a process $(X,\mathcal{B},\mu,T,\Phi)$ which is not necessarily ergodic. For each r, let $\Lambda_r \subset \mathbb{C}$ be a compact set with $\xi_r^l(n) \in \Lambda_r$ for all l and n. Let $\tilde{\Lambda} = \prod \Lambda_r$ and let $X = \tilde{\Lambda}^{\mathbb{Z}}$. We denote by ξ_r^l the point in $\Lambda_r^{\mathbb{Z}}$ with $\xi_r^l = (\ldots,\xi_r^l(-1),\xi_r^l(0),\xi_r^l(1),\ldots)$ and form $\tilde{\xi}^l = (\xi_1^l,\xi_2^l,\ldots) \in \tilde{\Lambda}^{\mathbb{Z}} = X$. X is a compact metrizable space and we form the measures

$$\nu_l = \frac{1}{|J_l|}\sum_{n\in J_l} \delta_{T^n\tilde{\xi}^l} \tag{2}$$

where $T\colon X \to X$ denotes the shift map $T\omega(n)=\omega(n+1)$. Since $|J_l| \to \infty$, any weak limit of a subsequence of ν_l is T-invariant, and we let ν be some such limit: $\nu = \lim \nu_{L_l}$. It is not hard to see that $(\{J_{L_l}\},\{\xi_r^{L_l}\})$ is generic for the process $(X,\mathcal{B},\nu,T,\Phi)$ where $\mathcal{B}$ is the Borel σ-algebra of sets in X and $\Phi=\{\varphi_1,\varphi_2,\ldots\}$ with φ_r the functions on $\tilde{\Lambda}^{\mathbb{Z}}$ given by $\varphi_r(\omega)=\omega(0)(r)$. By ergodic decomposition there will be an ergodic measure μ whose support is a subset of the support of ν. Any point in the support of μ is a limit of points of the form $T^n\tilde{\xi}^l$ with $n \in J_l$ and $l \to \infty$, by (2). Since μ is ergodic, almost every point ω in its support is *generic* for μ, in the sense that averages of a given bounded measurable function along the orbit of ω tend to the integral of the function. In particular for functions in Φ^* we have:

$$\frac{1}{N}\sum_{n=k}^{k+N-1} T^{j_i}\varphi_{i_1}T^{j_2}\varphi_{i_2}\cdots T^{j_m}\varphi_{i_m}(T^n\omega) \longrightarrow \int T^{j_1}\varphi_{i_1}T^{j_2}\varphi_{i_2}\cdots T^{j_m}\varphi_{i_m}\,d\mu \tag{3}$$

uniformly for $|k| \le N$.

We can find N sufficiently large that the difference of the two sides in (3) is $< \varepsilon$ for all $T^{j_1}\varphi_{i_1}\cdots T^{j_m}\varphi_{i_m} \in \Phi_h^*$. We then choose $T^n\tilde{\xi}^l$ close enough to ω, $n \in J_l$, so that the difference of the two sides of (3) remains $< \varepsilon$ with ω replaced by $T^n\tilde{\xi}^l$. Since $n \in J_l$, assuming l sufficiently large, we will have

$H_l = [n+k, n+k+N-1] \subset J_l$ for some k with $|k| \leq N$. We now let $\varepsilon \to 0$, $h \nearrow \infty$, and choose an appropriate subsequence of l; rescrambling the information in (3) we find a subscheme $(\{H_l\}, \{\xi_r^l\})$ which is generic for $(X, \mathcal{B}, \mu, T, \Phi)$. ◻

Scholium to Theorem 2.9. If for some r,

$$\limsup_{l\to\infty} \frac{1}{|J_l|} \Big| \sum_{n\in J_l} \xi_r^l(n) \Big| > 0,$$

we can add the condition that the corresponding φ_r does not vanish a.e. This follows from the fact that the measure ν satisfies $\int \varphi_r d\nu \neq 0$ and so ν must have an ergodic component with $\int \varphi_r d\mu \neq 0$.

We remark that in the case of ergodic processes, given a generic scheme, "many" subschemes will again be generic. This is made precise in the following: For any process $(X, \mathcal{B}, \mu, T, \Phi)$, Φ^* is countable and we fix an increasing family of finite sets $\Phi_h^* \subset \Phi^*$ increasing to Φ^*. Given a scheme $(\{J_l\}, \{\xi_r^l\})$ and fixing l, and letting $\varepsilon > 0$, we shall say that an interval $H \subset J_l$ is *ε-h-generic* for the process $(X, \mathcal{B}, \mu, T, \Phi)$ if (1) holds approximately; i.e, if for every $\psi \in \Phi_h^*$ and corresponding time series $\zeta^l(n)$.

$$\left| \frac{1}{|H|} \sum_{n\in H} \zeta^l(n) - \int \psi d\mu \right| < \varepsilon. \tag{4}$$

Assume now a process $(X, \mathcal{B}, \mu, T, \Phi)$ given with $\Phi^* = \bigcup \Phi_h^*$ as above, and let $(\{T_l\}, \{\xi_r^l\})$ be a generic scheme for the process.

Proposition 2.10. *If $(X, \mathcal{B}, \mu, T, \Phi)$ is an ergodic process, then for any $\varepsilon > 0$ and $h \in \mathbb{N}$ there exists $p_0 \in \mathbb{N}$ so that for any $p \geq p_0$ there exists a positive number $l_0(\varepsilon, h, p)$ so that for $l > l_0(\varepsilon, h, p)$, at least $(1-\varepsilon)(|J_l| - p + 1)$ of the $(|J_l| - p + 1)$ intervals of length p in J_l are ε-h-generic for the process.*

Letting p and l grow we see, according to the proposition, that the intervals J_l can be replaced by many choices of subintervals, and the scheme will remain generic. It is easy to see that this is not true for non-ergodic processes (where time series have different statistical behavior along different intervals of time).

Proof of Proposition 2.10. It suffices to treat a single function and the corresponding time series. For if for each of the $|\Phi_h^*|$ functions in Φ_h^* we have $(1-\varepsilon_1)(|J_l| - p + 1)$ "ε_1-generic" intervals with $\varepsilon_1 |\Phi_h^*| < \varepsilon$, the number of intervals common to all of these will not be less than $(1-\varepsilon)(|J_l| - p + 1)$, and these intervals are ε_1-h-generic, and so also ε-h-generic. So let $\psi \in \Phi^*$. Ergodicity assures that for p large, $\frac{1}{p} \sum_{q=0}^{p-1} T^q \psi$ is L^2-close to $\int \psi d\mu$, and so

$$\int \left(\frac{1}{p} \sum_{q=0}^{p-1} T^q \psi \right)^2 d\mu - \left(\int \psi d\mu \right)^2$$

is small. Fix p and set $\eta(n) = \frac{1}{p}\sum_{q=0}^{p-1}\zeta(n+q)$. η and ζ have the same long-term averages,

$$\begin{aligned}\frac{1}{|J_l|}\sum_{n\in J_l}\left(\eta(n) - \int\psi d\mu\right)^2 &= \frac{1}{J_l}\sum_{n\in J_l}\eta(n)^2 - 2\left(\frac{1}{|J_l|}\sum_{n\in J_l}\eta(n)\right)\left(\int\psi d\mu\right) \\ &\quad + \left(\int\psi d\mu\right)^2 \\ &\longrightarrow \int\left(\frac{1}{p}\sum_{q=0}^{p-1}T^q\psi\right)^2 d\mu - \left(\int\psi d\mu\right)^2\end{aligned}$$

which is small for large p. But this implies that most $\eta(n)$ are close to $\int\psi d\mu$ as asserted in the proposition. ◻

3 Some Examples of PW-Bohr Sets

3.1 Fourier Transforms

Our first example of PW-Bohr sets will lead to three more in the following subsections.

Theorem 3.1. *Let ω be a non-negative measure on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with a non-trivial discrete (atomic) component, and let $\hat{\omega}$ denote its Fourier transform: $\hat{\omega}(n) = \int_{\mathbb{T}} e^{2\pi itn}d\omega(t)$. If*

$$S = \{n : \mathrm{Re}\,\hat{\omega}(n) > 0\},$$

then S is a PW-Bohr$_0$ set.

Proof. Let ω_d denote the discrete component of ω: $\omega_d = \sum_{\lambda\in\Lambda}\omega(\{\lambda\})\delta_\lambda$ where Λ consists of all the atoms of ω . Let Λ_0 be a finite subset of Λ so that $\omega_d(\Lambda_0) > \frac{3}{4}\omega_d(\Lambda)$. Set

$$\psi(\tau) = \sum_{\lambda\in\Lambda_0}\omega_d(\lambda)e^{2\pi i\lambda\tau}$$

and let B_0 be the Bohr$_0$ set: $B_0 = \{n : \mathrm{Re}\,\psi(n) > \frac{2}{3}\omega_d(\Lambda_0)\}$. The measure $\omega - \omega_d$ is continuous and so by Wiener's theorem (see [Kre85], p.96)

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}\left|\hat{\omega}(n) - \hat{\omega}_d(n)\right|^2 = 0$$

It follows that $Q' = \left\{n : \left|\hat{\omega}(n) - \hat{\omega}_d(n)\right| \geq \frac{1}{3}\omega_d(\Lambda_0)\right\}$ has density 0 so that $Q = \mathbb{Z}\backslash Q'$ is a thick set.