Steven Givant · Hajnal Andréka

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Steven Givant
Department of Mathematics
Mills College
Oakland, CA, USA

Hajnal Andréka
Alfréd Rényi Institute of Mathematics
Institute of Mathematics
Hungarian Academy of Sciences
Budapest, Hungary

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To our spouses, Susan Wang and István Németi, without whose love and support this book would never have come into existence.

Preface

The theory of relation algebras is an abstract, finitely axiomatizable version of the calculus of relations, which in turn is an algebraic theory of binary relations dating back in origins to the second half of the nineteenth century. The theory was created by Alfred Tarski in 1941 and initially developed by Tarski and his students, J. C. C. McKinsey, Bjarni Jónsson, Roger Lyndon, and Donald Monk.

One of the important results of the theory is that every relation algebra is a subdirect product of relation algebras that are simple in the classic sense of the word, that is to say, they have exactly two ideals, the improper ideal and the trivial ideal. In analogy with the program of classifying the finite simple groups, the program suggests itself of attempting to classify the simple relation algebras, or at least the finite, simple relation algebras. The purpose of this book is to give an exposition of several different methods for constructing simple relation algebras—some of them older, but most of them new with this book—and, in particular, to demonstrate that these seemingly different methods are really all different aspects of one general approach to constructing simple relation algebras. Several different applications of the method are given. A broad sketch of the method and its applications is given in the Introduction.

Intended audience

The book will be of interest not only to mathematicians, in particular those interested in logic, algebraic logic, or universal algebra, but also to philosophers and theoretical computer scientists working in fields that use mathematics. For that reason, it has been written in a careful and detailed style so as to make the material accessible to as broad an audience as possible. In particular, the background in relation algebras that is needed to read this book is given in an appendix.

The book contains more than 400 exercises, some of them routine to help the reader grasp the material, others quite difficult. Hints and solutions to some of more challenging exercises are given in an appendix.

Acknowledgements

The first author learned the theory of relation algebras from Alfred Tarski in an inspiring course given in 1970 at the University of California at Berkeley, and through later collaboration with Tarski over a period of ten years. The work of Bjarni Jónsson, his great interest in relation algebras and, more generally, universal algebra, and his open and stimulating discussions with the authors have motivated a number of the results in this book. Most of the results in Chapter 3 (though not the particular development and presentation adopted there) are due to him, and discussions with him led the first author to the discovery of the results in Chapter 5. Roger Maddux's work has also had a significant impact on the development of some of the material in the book. In particular, one of his papers was the original stimulus for the research that culminated in the definitions and theorems in Chapter 8 and the first half of Chapter 9. It was Maddux who called our attention to the thesis of Mohamed El Bachraoui [13], and this thesis was the direct inspiration for the results in Chapter 6. The authors are deeply indebted to all these individuals.

We would also like to express our indebtedness to Loretta Bartolini and her entire production team at Springer for pulling out all stops, and doing the best possible job in the fastest possible way, to produce this volume. Loretta served as the editor of this volume, and her constant encouragement, help, and support made the process of publishing this book much, much easier than it might have otherwise been. Any errors or flaws that remain in the volumes are, of course, our own responsibility.

Finally, we would like to acknowledge the mutual interests, support, and years of devoted friendship that the authors have shared with each other and with István Németi. Without these, this book would not exist.

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Oakland, CA, USA Steven Givant
Budapest, Hungary Hajnal Andréka

Introduction

The calculus of relations is a mathematical theory of binary relations that is over one hundred and fifty years old, dating back to the middle of the nineteenth century. The first paper on the subject was written by De Morgan [11]. A proper foundation for the theory was formulated by Peirce in [45], after several earlier attempts. Schröder systematically developed Peirce's theory in the extensive monograph [50]. In 1903, Russell [48] could write

> The subject of symbolic logic is formed by three parts: the calculus of propositions, the calculus of classes, and the calculus of relations.

The objects of study of the calculus of relations are algebras of binary relations, consisting of a universe of binary relations on some base set, together with the operations of union, intersection, complement, relational composition, relational addition (the dual of relational composition), and relational inverse (or converse). There are also distinguished constants: the empty relation, the universal relation, the identity relation, and the diversity relation on the base set. (Some of these operations and constants—for instance, the operation of relational addition—are definable in terms of the others and may therefore not be included in the list of primitive notions.)

The calculus of relations was given an axiomatic foundation by Tarski, in the spirit of general algebra, starting with the 1941 paper [55]. The original axiomatization was not equational, but Tarski realized that an equational axiomatization was possible (see the remark at the top of p. 87 in *op. cit.*). Within a short time he produced a very elegant one (the one used in [58]) and succeeded in showing (sometime in the period 1942–44) that this simple, finitely axiomatized, equational theory provides a sufficiently rich formalism for the development of all of classical mathematics. (This theorem and its consequences were eventually published in [58].)

Tarski posed two fundamental questions in [55] regarding his foundation of the calculus of relations (or the theory of relation algebras, as it eventually came to be called). The first concerned *completeness*: is every true equation derivable from the set of axioms? The second concerned the existence of a general *representation theorem*: is every abstract model of the theory isomorphic to a concrete model of binary relations? Success eluded him in his pursuit of the answers to these questions, but along the way he and his collaborators McKinsey and Jónsson were able to establish some important algebraic properties of relation algebras. First of all, every relation algebra is *semi-simple* in the sense that it is a subdirect product of simple relation algebras (see [29] and Theorems 4.10 and 4.14 in [31] for statements of closely related results). Second, every relation algebra can be extended to a complete atomic relation algebra, and the extension is simple or integral just in case the original algebra is simple or integral (see Theorem 4.21 in [31]). (A relation algebra is said to be *integral* if the composition of two non-zero elements is always non-zero.) These two theorems provided the foundation for future algebraic investigations. They indicate clearly the important role that simple relation algebras, particularly those that are complete and atomic, play in the theory.

In 1950, Lyndon [33] proved that the answer to both of Tarski's questions is negative. He gave an example of a true equation that is not derivable from Tarski's axioms, and he gave an example of a finite relation algebra that is not representable as an algebra of binary relations. This original example of a non-representable relation algebra was not easy to understand. In [35], building on earlier work of Jónsson [26], Lyndon showed how to construct relation algebras as complex algebras (algebras of all subsets, or *complexes*) of arbitrary projective geometries, and proved that many of the algebras constructed from projective lines are not representable. Monk used these algebras in [41] to demonstrate that no finite set of postulates is sufficient to axiomatize the class of representable relation algebras. As a consequence, no finite set of postulates is strong enough to derive all true equations in the calculus of relations.

Before the negative results of Lyndon were discovered, Tarski established a partial representation theorem for the class of all relation algebras, and some full representation theorems for restricted classes of relation algebras. For instance, in Theorem 4.29 of [31] it is shown that every atomic relation algebra with functional atoms—atoms that satisfy a certain characteristic equational property of functions—is representable. A number of other such theorems have appeared over the years, for instance in [31, 58, 27, 37, 49, 14], and [12]. Tarski several times expressed the opinion that the negative results of Lyndon enhanced the interest of representation theorems for specialized classes of relation algebras.

Through the work of Tarski, Jónsson, Lyndon, Monk, and others (including a number of their students), there has gradually arisen a general algebraic theory of relation algebras, similar in spirit to group theory and the theory of Boolean algebras, with many applications to logic, computer science, and other domains of inquiry (see [24, 38], or [18] and [19]).

Simple relation algebras

One of the fundamental problems of any algebraic theory is the analysis of its models, and in particular the analysis of the basic algebras that form the building blocks for the models of the theory. In group theory, for instance, manifestations of this analysis include the classification of finite, simple groups and the classification of finite abelian groups. In Boolean algebra, one might mention the theorem that every Boolean algebra is a subdirect product of the two-element Boolean algebra (which is the only simple Boolean algebra), and every finite Boolean algebra is a direct product of the two-element Boolean algebra. The basic building blocks of relation algebras are simple relation algebras, particularly atomic ones. Several quite interesting classes of simple, atomic relation algebras have been studied over the years. For example, McKinsey showed that the complex algebra of a group is always a simple, atomic, representable relation algebra (see [29] and [31]). As was already mentioned, Lyndon showed that the complex algebra of a projective geometry is a simple, atomic relation algebra (see [26] and [35]). Maddux [36] proved that the complex algebra of a modular lattice with zero is a simple, atomic relation algebra.

In analogy with group theory, the program naturally suggests itself of classifying all simple relation algebras, or at least all finite, simple relation algebras. This program might be interpreted in several different ways, but the goal is to come up with a well-defined class of "basic" simple relation algebras, and a finite list of construction techniques such that every finite, simple relation algebra is obtainable from the basic examples by applying one or more of the construction techniques, perhaps in some specific order. As a concrete example of an interpretation of this program, consider the class of finite, integral relation algebras, that is to say, relation algebras of finite cardinality greater than one, such that the relative product of two elements is zero if and only if one of the elements is zero. All such algebras are known to be simple. One may ask if there is a list of finitely many construction techniques so that every finite, simple relation algebra can be constructed from the class of finite integral relation algebras using one or more of these techniques. In other words, the program may be viewed as reducing the problem of constructing and analyzing all finite, simple relation algebras to that of constructing and analyzing all finite, integral relation algebras.

This book can be viewed as a contribution to this program. It presents a fairly general method for developing tools to construct and analyze simple relation algebras, it gives some examples of how this general method can lead to constructions that are interesting, beautiful, and useful, and how these constructions can, in turn, be used to prove significant theorems. In each of these theorems, the method is indeed used to reduce the result for simple relation algebras to the corresponding result for integral relation algebras.

The intuitive idea behind the method is rather easy to understand from the perspective of analyzing simple relation algebras. Suppose the Boolean part of a simple relation algebra—the universe of the algebra together with the Boolean operations—has been decomposed into the direct product of *component* (or *factor*) Boolean algebras. The extra-Boolean operations of converse and relative multiplication are distributive, in each coordinate, over arbitrary Boolean sums. Therefore, their behavior on the whole universe is completely determined by their behavior on and between the individual components of the Boolean decomposition. Such a *semiproduct* decomposition—a Boolean direct product decomposition, together with completely distributive extra-Boolean operators (of converse and relative multiplication)—is useful when it is possible to describe the extra-Boolean operations in an illuminating manner in terms of the components of the decomposition. In other words, when the extra-Boolean operations are well behaved with respect to the Boolean components of the decomposition, it is possible to get a global description of the behavior of the operations (and of the elements) of the given simple relation algebra in terms of their local behavior on components.

Each such semiproduct construction involves four basic concepts. The first is a notion of a subalgebra system in a given simple relation algebra. Often, such a system consists of base algebras, together with some auxiliary elements, or sets of elements, or functions on the base algebras. The second notion is that of an isomorphism system between subalgebra systems. The third is the notion of a semiproduct system—the basic ingredients needed to construct a simple relation algebra of a given type. Again, such a system usually consists of base algebras, together with auxiliary elements, or sets of elements, or functions on the algebras. The last notion is that of the semiproduct of a semiproduct system—the simple algebra that is the result of the construction.

Parallel to the basic notions are a series of theorems and lemmas that are common to all semiproduct constructions. First, there is a subalgebra theorem that describes the elements and operations of the subalgebra generated by a subalgebra system. As a consequence, one may be able to show that the subalgebra generated by such a system is atomic, complete, finite, or integral just in case the base algebras of the system possess the same property. Second, there is an isomorphism theorem stating that an isomorphism system between two subalgebra systems can be extended in one and only one way to an isomorphism between the generated subalgebras. Its proof depends on the description of the elements and operations given in the subalgebra theorem.

Two lemmas connect the notion of a subalgebra system with that of a semiproduct system. The semiproduct-to-subalgebra (or "semi-to-sub") lemma says that every semiproduct system is, or gives rise to, a subalgebra system in its semiproduct, and this subalgebra system generates the semiproduct. The subalgebra-to-semiproduct (or "sub-to-semi") lemma says that every subalgebra system is, or gives rise to, a semiproduct system, and the semiproduct of that semiproduct system is just the subalgebra generated by the subalgebra system. The lemmas pave the way for three basic theorems about semiproducts. The uniqueness theorem states that a semiproduct of a semiproduct system is unique up to isomorphisms leaving the base algebras fixed. Thus, one may speak of *the* semiproduct of the system. The theorem is a rather direct consequence of the isomorphism theorem and the lemmas just mentioned. The existence theorem states that the semiproduct of a semiproduct system always exists. The proof usually involves the construction of an algebra out of the component pieces of the semiproduct system following the prescriptions laid down by the descriptions of elements and operations from the subalgebra theorem. There then follows a step-by-step verification that the relation algebraic postulates and a condition guaranteeing simplicity all hold in the constructed algebra. The decomposition theorem says that a simple relation algebra is decomposable into (or may be written as) the semiproduct of a semiproduct system just in case it satisfies certain conditions. The proof depends on the lemmas mentioned above and on the structural description given in the subalgebra theorem. There may also be representation theorems asserting that every semiproduct of a certain type is representable whenever the base algebras are representable. Indeed, it may be possible to describe all possible representations of the semiproduct in terms of representations of the base algebras. However, such representation theorems are highly dependent on the specific nature of the semiproduct construction under discussion.

Three applications of a theory of semiproducts are immediately apparent. The first is the investigation of specialized classes of relation algebras that have been studied in the literature, for instance in connection with representation theorems. The existence and decomposition theorems can be used to give a complete description of the algebras in the class, one that makes the structure of the algebras easy to visualize. In connection with representation theorems, this approach not only offers an alternate proof of representability but also gives a better feeling for the extent of the class to which the representation theorem applies. The second application is the construction of (classes of) new, interesting simple relation algebras using, as "pieces," component algebras that are comparatively better understood, for instance complex algebras of groups, projective geometries, or modular lattices. The third application is the establishment of representation and non-representation theorems for broader classes of relation algebras.

History of the method

The historical roots of semiproducts go back to the paper [27] of Jónsson. The paper was motivated by an earlier work of Olivier and Serrato, [44], in which the notion of a Schröder category was introduced in order to give certain results about relation algebras a category-theoretic and seemingly more general formulation. Jónsson showed that the apparent greater generality is illusory: every Schröder category can be used to build a simple relation algebra whose elements are systems of morphisms from the category (see Theorem 3.3 in [27]). Using this theorem, Jónsson went on to prove that, for every finite sequence of simple relation algebras (disjoint except for a common zero element), there is a unique simple relation algebra that contains the members of the sequence as relativizations (along the diagonal, covering the identity element) and is minimal in the sense that it is generated (as an algebra) by the union of the relativizations. (A relativization of a relation algebra is a localization of the algebra: a restriction of the universe and the operations to the set of elements below a given element e. When e has properties analogous to those of symmetry and transitivity for equivalence relations, the relativization of the relation algebra to e is again a relation algebra, one that is smaller, and hopefully easier to understand, than the original algebra.) Jónsson called his construction a *semiproduct*. He used it to analyze relation algebras generated by a single equivalence element, and in particular to prove that any such algebra is finite and representable.

Motivated by Jónsson's paper, Givant [14] attacked the more general problem of describing the subalgebra of a relation algebra generated by any relativization (or sequence of relativizations). The structure of the subalgebra was analyzed in terms of the structure of the relativization(s). On the basis of this analysis, it was shown that every relation algebra $\mathfrak{A}$ may actually be written as a relativization of some simple relation algebra, called a *simple closure* of $\mathfrak{A}$. Consequently, simple relation algebras are in reality just as complicated as arbitrary relation algebras. Simple closures are not unique, but the isomorphism type of a simple closure can be characterized by certain measure-theoretic invariants. If the original relation algebra is finite, atomic, integral, or representable, then so are its simple closures. These results were used to analyze relation algebras generated by trees of equivalence elements, and to show that such algebras are always representable and are finite whenever the tree is finite.

A separate line of development was motivated by the paper [37] of Maddux, in which a representation theorem was established for all *pair-dense* relation algebras, that is to say, all relation algebras in which the identity element is a sum of elements satisfying a certain equationally expressible property characteristic of relations with at most two pairs. An analysis of Maddux's theorem led Givant in 1988 to a generalization of McKinsey's group complex algebra construction: instead of a single group, one uses a system of groups, together with a family of "coordinating" isomorphisms between quotients of the groups (see Theorem 1 in [20]).

The final factor influencing the development of the notion of a semiproduct was the paper [12] of El Bachraoui, in which a representation theorem for (*strictly*) *elementary* relation algebras was given. An analysis of El Bachraoui's theorem led Givant to a common generalization of the representation theorems of Jónsson-Tarski, Maddux, and El Bachraoui. The description of the class of algebras to which the generalization applies involves the notion of a *semipower* of a relation algebra (treated in Part II of this work).

Eventually, it was realized that all four constructions—the construction of Jónsson, the simple closure construction, the construction of relation algebras from systems of groups and quotient isomorphisms, and the semipower construction—are special cases of a much broader phenomenon, and this gave rise to the general notion of a semiproduct.

The notions and theorems of Chapters 1, 4, and 6 were developed by Givant in January and February of 2002. He then prepared a first draft of the material now contained in Parts I and II. In mid-March, Hajnal Andréka read the draft and became interested in the work. She posed several problems, and there ensued a stimulating exchange of ideas between the two authors. The contributions of each of the authors are described in the various chapter introductions and also in the text itself. The notions and theorems of Chapters 7 and 8 are a result of this exchange, and date to March and April of 2002. Some of the theorems of Chapter 9 also date to this period. The remainder were found in July and August of 2002. The results of Chapter 11 are also a result of the exchange and were obtained in May of 2002. The results of Chapter 10 date to November and December of that year.

The structure of the book

The book consists of four parts. The first part of the book lays the groundwork for the subsequent parts. Two general types of subalgebra and semiproduct systems are discussed as examples: rectangular systems in Chapter 1 and equivalence systems in Chapter 2. In each case, the formulations of the basic notions and results substantially simplify the presentation in later parts of the book. For example, general necessary and sufficient conditions are given for verifying that a given system is a subalgebra system, an isomorphism system, or the semiproduct of a semiproduct system. Rectangular systems arise from families of disjoint rectangles with sides from a given partition of the identity element. Examples include the diagonal semiproduct systems of Chapter 3, the bijection semipower and semiproduct systems of Chapter 4, and the quotient semiproduct systems of Chapter 8. Equivalence systems arise from reflexive equivalence elements and their complements. (A reflexive equivalence element is an abstraction of the notion of an equivalence relation.) Examples include the simple closure systems of Chapter 5 and the insertion semiproduct systems of Chapter 10. Readers acquainted with the theory of Boolean algebras with operators (see [30] and [31]) will recognize that many of the results of Chapter 1 can be formulated and proved in that more general setting. We have refrained from doing so in order to keep the exposition as simple and direct as possible.

The second part of the book contains an exposition of the various notions of semiproducts that play a role in the formulation and proof of a representation theorem for quasi-bijective relation algebras in Chapter 6. Jónsson's construction from [27] is given in Chapter 3, under the name *diagonal semiproducts*. The presentation is in terms of the framework developed in Chapter 1, namely rectangular semiproduct systems. A diagonal semiproduct system is essentially a finite sequence of simple relation algebras—the *base algebras*—that are disjoint (except for a common zero element). The semiproduct of such a system is the smallest simple relation algebra that contains each of the base algebras as a relativization along the diagonal. Diagonal semiproducts provide a simple paradigm for later, more involved constructions that are discussed at the ends of Chapters 4 and 5. It is hoped that the presentation in Chapter 3 will make Jónsson's useful construction known to a broader audience of algebraists and logicians.

The *simple closure* construction from [14] is given in Chapter 5. The presentation is in terms of the framework developed in Chapter 2, namely equivalence semiproduct systems. A simple closure system consists of an arbitrary relation algebra—the base algebra—together with a four-valued measure on certain special elements—the *ideal elements*—of the base algebra. (Ideal elements are closely connected with the algebraic ideals in the base algebra.) The measure specifies an abstract "size" of each ideal element, and these "sizes" determine the isomorphism type of a semiproduct. The simple closure of the system—which is the name given to the semiproduct—is the smallest simple relation algebra $\mathfrak{A}$ that contains the base algebra as a relativization along the diagonal and such that the abstract measure of the ideal elements coincides with an intrinsically defined notion of the size of the ideal elements.

A (*bijection*) *semipower* construction is taken up in Chapter 4. A single, simple relation algebra—the base algebra—and a finite index set—the *power*—are given. A sequence of bijections (indexed by elements of the index set) is used to make copies of the base algebra in all of the (rectangular) components of the semiproduct of the system. A more general conception of a bijection semiproduct, discussed briefly at the end of the chapter, allows a finite sequence of simple base algebras, instead of a single base algebra, and a corresponding finite sequence of powers. The most general conception of a bijection semiproduct is discussed briefly near the end of Chapter 5. The requirement that the base algebras be simple is dropped. One first passes to a simple closure of each base algebra, and then forms the semiproduct discussed at the end of Chapter 4. Consequently, a four-valued measure on ideal elements must be associated with each base algebra of the semiproduct system. (This involves the notions and results of Chapter 5.)

Chapter 6 establishes a common generalization of several representation theorems from the literature. Call a relation algebra *quasi-bijective* if it is atomic, and if below each rectangle with atomic sides there is at most one non-bijective atom—at most one atom that does not satisfy a certain characteristic equational property of set-theoretic bijections. Examples of quasi-bijective relation algebras include atomic relation algebras with functional atoms, shown to be representable in [31], atomic pair-dense relation algebras (including all simple, pair-dense relation algebras), shown to be representable in [37], strictly elementary relation algebras, shown to be representable in [12], and elementary relation algebras, independently shown to be representable by Givant and El Bachraoui (see [13], where a different terminology is used). Chapter 6 gives a structural description of all quasi-bijective relation algebras. The formulation and proof of this structure theorem draws on the results from Chapters 3–5.

One consequence of this theorem is that every quasi-bijective relation algebra is completely representable. This gives the common generalization of the representation theorems cited above. Another consequence is a structural description of the classes of relation algebras to which the cited representation theorems apply. For example, atomic relation algebras with functional atoms are essentially just direct products of semipowers of complex algebras of groups. Atomic pair-dense relation algebras are essentially just direct products of diagonal semiproducts of semipowers of the complex algebras of one-element and two-element groups. Strictly elementary relation algebras are essentially just direct products of diagonal semiproducts of semipowers of minimal simple set relation algebras on one-element and three-element sets. Elementary relation algebras are essentially just direct products of diagonal semiproducts of semipowers of minimal simple set relation algebras on one-element, two-element, and three-element sets.

The third part of the book was motivated by the construction in [20] of simple relation algebras from systems of groups and quotient isomorphisms (see Theorem 1 of that paper, and see also Chapter 9 of the present monograph). There are two important auxiliary concepts, studied in Chapter 7, that underlie the construction. The first is a notion introduced in [39] of the quotient of a relation algebra by an equivalence element. This is *not* the same as the quotient of a relation algebra by a congruence relation (or an ideal). In other words, it is not the relation algebraic analogue of a quotient group or a quotient ring. Rather, the construction uses an equivalence element of the algebra (as opposed to a congruence relation or an ideal) to collapse, or glue, elements together. A quotient relation algebra (in this sense of the word) inherits many properties from its parent. For instance, if the parent algebra is simple, integral, finite, atomic, complete, or representable, then so are its non-degenerate quotients. *Normal* equivalence elements—equivalence elements that commute with every element of the parent algebra—play a particularly important role in the formation of quotients.

The second auxiliary concept that plays a critical role is that of an *equivalence bijection*, or an *equijection* for short. In set-theoretical contexts, an equijection is a binary relation that, roughly speaking, determines a bijection between the equivalence classes of two equivalence relations. Such relations were apparently first studied by Riguet in [46] under the name *difunctional relations*. Rather surprisingly, an abstract version of this notion can be defined by a very simple equation in the theory of relation algebras. Abstract equijections possess some of the important properties associated with bijections. For instance, relative multiplication by an equijection is distributive over Boolean products of certain types of elements. Most importantly, each equijection is associated with a domain and a range equivalence element, and induces an isomorphism between the corresponding quotient relation algebras. The isomorphism is similar in character to the inner automorphism of a group induced by a group element.

The notion of a quotient semiproduct is developed in Chapter 8 in a manner parallel to the development of the notion of a semipower in Chapter 4, and to the generalization of this notion to bijection semiproducts at the end of the chapter. Instead of using bijections to copy a single, simple base algebra to all components of a semiproduct, as is done in Chapter 4, the quotient semiproduct construction uses coordinating equijections (or the induced coordinating quotient isomorphisms) to copy *quotients* of a family of simple base algebras to components of a semiproduct. Actually, the semipower construction may be viewed as a special case of the quotient semiproduct construction, namely the case when the equivalence elements used to form the quotients are the identity elements of the base algebras, the equijections are actually bijections, and the base algebras are all isomorphic. Moreover, the diagonal semiproduct construction may also be viewed as a special kind of quotient semiproduct construction, at least in the atomic case. Generalizations of the quotient semiproduct construction, similar in spirit to the generalizations of the semipower construction (mentioned at the end of Chapters 4 and 5), are discussed near the end of Chapter 8.

Chapter 9, the final chapter in Part III, presents two extended examples of the quotient semiproduct construction. The base algebras in the first example are complex algebras of groups. It is shown that, in this case, a given system of group complex algebras with coordinating isomorphisms between the quotient relation algebras may always be replaced by a corresponding system of groups with coordinating isomorphisms between quotient groups. The principal theorem of the chapter says that every quotient semiproduct constructed with complex algebras of groups is representable.

The base algebras in the second example are complex algebras of projective geometries. Just as in the example with group complex algebras, a given system of geometric complex algebras with coordinating isomorphisms between quotient relation algebras may always be replaced by a system of projective geometries with coordinating isomorphisms between quotient geometries. However, it is no longer true that every quotient semiproduct with geometric complex algebras as base algebras is representable, and a concrete example of such a non-representable algebra is given. On the other hand, it is possible to characterize when a quotient semiproduct of geometric complex algebras is representable. The statement of this characterization is reminiscent of Lyndon's characterization of representability for geometric complex algebras (see [35]). Its proof requires a number of algebraic constructions from higher dimensional projective geometry. Because some readers may not be familiar with projective geometry, an introduction to the subject, with statements and proofs of the required results, is provided in an appendix.

The fourth part of the book is concerned with a semiproduct construction that uses both quotients and relativizations. Thus, some familiarity with the initial sections of Chapter 7 is helpful. The key idea is the following. Suppose a local part of a simple relation algebra $\mathfrak{B}$—that is to say, a relativization of $\mathfrak{B}$—looks like a collapsed (and hence simplified) version of a relation algebra $\mathfrak{C}$—that is to say, it looks like a quotient of $\mathfrak{C}$. The complicated structure of algebra $\mathfrak{C}$ may then be inserted into $\mathfrak{B}$ as a replacement for the simplified local part—the relativization of $\mathfrak{B}$—to create a more complex algebra. The resulting simple relation algebra, which is called an *insertion semiproduct*, is studied in Chapter 10. At the end of the chapter, more general versions of this construction are considered. A finite sequence $\mathfrak{B}_0$, ..., $\mathfrak{B}_{n-1}$ of local parts of $\mathfrak{B}$ are given that look like collapsed versions of a corresponding sequence $\mathfrak{C}_0, \ldots, \mathfrak{C}_{n-1}$ of more complicated relation algebras. For each index i, the complicated structure of $\mathfrak{C}_i$ may be inserted into $\mathfrak{B}$ as a replacement for the local part $\mathfrak{B}_i$.

The representation theorem in Chapter 6 for quasi-bijective relation algebras suggests a natural extension to 2-*quasi-bijective* relation algebras—atomic relation algebras in which every rectangle with atomic sides is above at most *two* non-bijective atoms. It turns out that such algebras are not always representable. A counterexample is given at the beginning of Chapter 11. (The example comes from [3], where it is used for other purposes.) If attention is restricted to the integral case, however, the situation changes. An integral relation algebra has just one non-zero rectangle, namely the Boolean unit. Therefore, such an algebra is 2-quasi-bijective just in case it is atomic with at most two non-bijective atoms in the whole algebra. These 2-*non-bijective* relation algebras, as they are called, form a narrow subclass of the 2-quasi-bijective relation algebras, and they are always representable. In fact, it is possible to give a complete structural description of the algebras in this class.

Chapter interdependence

Parts II–IV of the book are intended to be more or less independent of one another. In order to achieve this goal, definitions are occasionally repeated, as are a few constructions. A diagram illustrating the various chapter dependencies is given in Figure 1. All parts of the book require various bits and pieces of the relation algebraic background that is provided in Appendix A. After perusing the first two sections of that appendix to get sense of the basic definitions and laws that govern the theory, the reader may prefer to refer back to the appendix on an "as needed" basis. Parts II–IV also require a familiarity with those definitions and theorems in Part I that concern either rectangular systems or equivalence systems. For instance, the systems of Chapters 3, 4, and 8 are all rectangular systems, whereas those of Chapters 5 and 10 are equivalence systems.

Chapters 3, 4, and 5 in Part II are essentially independent of one another. Chapter 6 refers to the notions and results from Chapters 3–5. Chapter 7 can be read independently of all earlier chapters, while Chapter 8 depends only on Chapter 7 and the material on rectangular systems from Chapter 1. The results in Chapter 9 are intended as illustrations of the ideas and results presented in Chapters 7 and 8, so they require some familiarity with those two chapters. The second half of Chapter 9 also requires some knowledge of affine and projective geometry and the connections between the two. The necessary background is provided in Appendix B.

Part IV uses the notion of a quotient algebra, so it requires some information from the first third of Chapter 7. Otherwise, it is more or less independent of the earlier material. In particular, Chapter 10 makes no use of the material in Parts II and III (except for the material mentioned in Chapter 7). It does use the notions and results on equivalence systems from Chapter 2. Chapter 11 is based on the ideas of Chapter 10 and also contains occasional references to some results in Chapter 6.

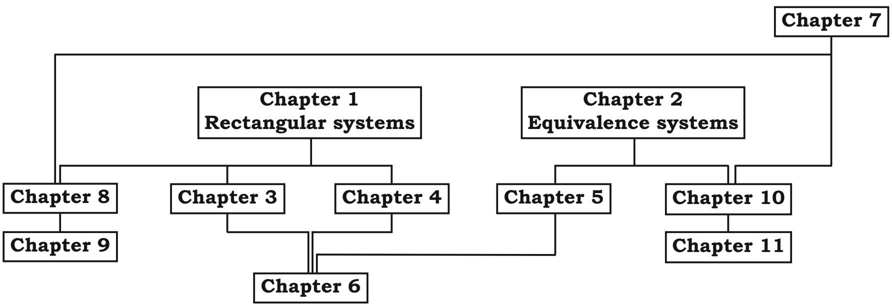

Fig. 1 Chapter dependence diagram.

Contents

Part I
Rectangular and Equivalence Semiproducts

Steven Givant

Chapter 1
Rectangular Semiproducts

An important technique for analyzing the structure of a subalgebra of a simple relation algebra is to break that structure into smaller pieces, analyze those pieces, and then describe how the overall structure of the algebra—its elements and operations—can be recovered from the pieces. This chapter provides a framework for a method of breaking the structure into smaller pieces using rectangles. A different method that uses equivalence elements is described in the next chapter.

1.1 Subalgebra systems

Consider a simple relation algebra $\mathfrak{S}$. Its universe is a non-empty set S, its operations are Boolean addition (join) $+$, complement $-$, relative multiplication $;$, and converse $^{\smile}$, and its distinguished element is the identity (that is to say, the identity element) 1'. Other distinguished elements such as zero 0, the unit 1, and diversity 0', and other operations such as Boolean multiplication (meet) $\cdot$ are defined in the usual manner. The discussion in this and the next section takes place inside of $\mathfrak{S}$.

The analysis of subalgebras of $\mathfrak{S}$ begins with a partition of the identity of $\mathfrak{S}$ into subelements called *local identities*. More precisely, a *partition of identity* (in $\mathfrak{S}$) is a system $(1'_i : i \in I)$ of non-zero, mutually disjoint elements that sum to the identity of $\mathfrak{S}$. The partition is said to be *finite* if the index set I is finite. A partition of identity induces a corresponding partition of the unit of $\mathfrak{S}$ into rectangles whose sides are the local identities. In more detail, the corresponding *partition of unity* is the system $(1_{ij} : i, j \in I)$ of rectangles, or *local units*, defined by

$$1_{ij} = 1'_i \,;\, 1 \,;\, 1'_j.$$

These rectangles are non-zero, mutually disjoint, and sum to the unit of $\mathfrak{S}$. The terms of these two partitions have a number of important properties that are summarized in the following lemma.

S. Givant, H. Andréka, *Simple Relation Algebras*,
DOI 10.1007/978-3-319-67696-8_1

Lemma 1.1. *Suppose* $(1'_i : i \in I)$ *is a partition of identity* (*in a simple relation algebra*).

(i) $1_{ij} \neq 0$.
(ii) $\sum_{ij} 1_{ij} = 1$.
(iii) $1_{ij} \cdot 1_{k\ell} = 0 \quad$ *if* $i \neq k$ *or* $j \neq \ell$.
(iv) $1'_i = 1' \cdot 1_{ii} \leq 1_{ii}$.
(v) $1' \cdot 1_{ij} = 0 \quad$ *if* $i \neq j$.
(vi) $1_{ij}^{\smile} = 1_{ji}$.
(vii) $1_{ij} \,;\, 1_{k\ell} = 0 \quad$ *if* $j \neq k$.
(viii) $1_{ij} \,;\, 1_{jk} = 1_{ik}$.
(ix) $1_{ij} \,;\, 1 \,;\, 1_{k\ell} = 1_{i\ell}$.
(x) $1'_i \,;\, 1_{ij} = 1_{ij} \,;\, 1'_j = 1_{ij}$.
(xi) $1_{ij} \,;\, 1 = 1'_i \,;\, 1$.

The preceding laws are all easy consequences of the laws about rectangles given in Rectangle Lemma A.7. The first three laws say that the system of local units really is a partition of unity in the sense described above. The fourth and fifth laws imply that a local identity $1'_i$ is either below or disjoint from a local unit $1_{k\ell}$, and it is below it just in case $i = k = \ell$. The sixth, seventh, eighth, and tenth laws say that the operations of converse and relative multiplication on the rectangles 1_{ij} and $1_{k\ell}$, and on the local identity elements $1'_i$ and $1'_j$, behave just as the set-theoretic operations of converse and relative multiplication behave on the singleton relations $\{(i, j)\}$ and $\{(k,\ell)\}$, and $\{(i,i)\}$ and $\{(j, j)\}$. Finally, the ninth and eleventh laws imply that the rectangles formed using the local units as sides coincide with the rectangles formed using the local identities as sides.

Figure 1.1 illustrates a partition of identity into three local identities, and the corresponding partition of unity into nine local units. (The *local diversity elements* $0'_i$ are defined by $0'_i = 1_{ii} - 1'_i$.)

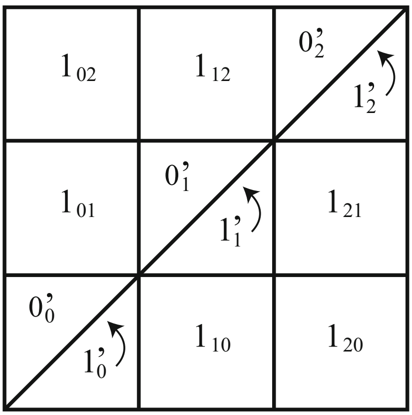

Fig. 1.1 A partition of identity and the induced partition of unity.

The pieces into which a subalgebra of $\mathfrak{S}$ is broken are defined in terms of the local units and local identities in the following manner.

Definition 1.2. A *rectangular subalgebra system* for a given finite partition of identity $(1'_i : i \in I)$ is a system

$$(A_{ij} : i, j \in I)$$

of subsets of (the universe of) $\mathfrak{S}$ with the following properties for all i, j, and k in I.

(i) The set A_{ij} is a subuniverse of the Boolean relativization of $\mathfrak{S}$ to 1_{ij}. In other words, the local unit 1_{ij} is in A_{ij}, each element in A_{ij} is below 1_{ij}, and if r and s are in A_{ij}, then so are $r+s$ and $1_{ij} - r$.
(ii) The local identity $1'_i$ is in A_{ii}.
(iii) If r is in A_{ij}, then $r^{\smile}$ is in A_{ji}.
(iv) If r is in A_{ij} and s is in A_{jk} , then $r\,;s$ is in A_{ik}.

The sets A_{ij} are called the *components* of the system, while the properties are called the *Boolean condition*, the *identity condition*, the *converse condition*, and the *product condition* respectively. □

Notice that the given partition of identity is required to be finite. Condition (i) says, in particular, that A_{ij} is a Boolean algebra. Conditions (i)–(iv) together imply that A_{ii} is a relation algebra, and in fact it is a subalgebra of the relation algebraic relativization of $\mathfrak{S}$ to the local unit 1_{ii}.

In order to avoid constantly repeating unwieldy phrases, we shall often use such abbreviations as “subalgebra system” or “rectangular system” to refer to a rectangular subalgebra system, and we employ similar terminology in other, related situations. When such abbreviations are employed, the context should make the intended meaning clear; for instance, that a given subalgebra system is intended to be rectangular, or that a given rectangular system is intended to be a subalgebra system.

Each subalgebra of $\mathfrak{S}$ that contains a given partition of identity (in the sense that it contains each term of the partition) induces a subalgebra system in a natural way.

Lemma 1.3. *If $\mathfrak{A}$ is any subalgebra of $\mathfrak{S}$ that contains a given finite partition of identity, then the sets*

$$A_{ij} = A(1_{ij}) = \{r \in A : r \leq 1_{ij}\}$$

form a rectangular subalgebra system for the partition of identity.

Proof. The proof of the lemma uses the laws formulated in Lemma 1.1. As examples, we verify the first part of condition (i), and conditions (iii) and (v), in the definition of a rectangular system. The subalgebra $\mathfrak{A}$ is assumed to contain the terms of the partition of identity. It contains the unit of $\mathfrak{S}$ and it is closed under relative multiplication, by the definition of a subalgebra, so it must contain the local units

$$1_{ij} = 1'_i\,;1\,;1'_j.$$

Consequently, the component A_{ij} contains the local unit 1_{ij}, and each element in this component is below this local unit, by the definition of the components. The local identity $1'_i$ is in $\mathfrak{A}$, by assumption, and it is below 1_{ii} by Lemma 1.1(iv), so it belongs to A_{ii}. Finally, if r is in A_{ij}, and s in A_{jk}, then both elements are in A, by definition, and

$$r\,;s \le 1_{ij}\,;1_{jk} = 1_{ik},$$

by the monotony law for relative multiplication and Lemma 1.1(viii). Consequently, $r\,;s$ belongs to the set A_{ik}, by the definition of that set. □

We shall refer to the subalgebra system defined in the preceding lemma as the subalgebra system *determined by*, or *corresponding to*, or *associated with*, the subalgebra $\mathfrak{A}$. In particular, the subalgebra system determined by $\mathfrak{S}$ itself is just $(S(1_{ij}) : i, j \in I)$.

There is a kind of converse to the preceding lemma: every rectangular system in $\mathfrak{S}$ generates a subalgebra of which it is the corresponding subalgebra system. To formulate this converse more precisely, it is helpful to introduce a bit of terminology. An *element system* in a subalgebra system $(A_{ij} : i, j \in I)$ is a system $(r_{ij} : i, j \in I)$, where r_{ij} is an element in A_{ij} for each i and j in I. In other words, element systems are just elements of the (set-theoretic) direct product $\prod_{ij} A_{ij}$ of the components.

Theorem 1.4 (Subalgebra Theorem). *Suppose $(A_{ij} : i, j \in I)$ is a rectangular subalgebra system for a finite partition of identity, and A is the set of sums of element systems:*

$$A = \{\textstyle\sum_{ij} r_{ij} : (r_{ij} : i, j \in I) \text{ is an element system}\}.$$

(i) *A is a subuniverse of $\mathfrak{S}$.*
(ii) *Every element in A can be written in just one way as the sum of an element system.*
(iii) *The distinguished constants and operations of A satisfy the following identities for all r, s, and t in A:*

$$1 = t, \quad \textit{where} \quad t_{ij} = 1_{ij} \quad \textit{for all} \quad i, j \in I; \tag{1}$$

$$0 = t, \quad \textit{where} \quad t_{ij} = 0 \quad \textit{for all} \quad i, j \in I; \tag{2}$$

$$1' = t, \quad \textit{where} \quad t_{ij} = \begin{cases} 1'_i & \textit{if } i = j, \\ 0 & \textit{if } i \ne j, \end{cases} \quad \textit{for all} \quad i, j \in I; \tag{3}$$

$$0' = t, \quad \textit{where} \quad t_{ij} = \begin{cases} 1_{ii} - 1'_i & \textit{if } i = j, \\ 1_{ij} & \textit{if } i \ne j, \end{cases} \quad \textit{for all} \quad i, j \in I; \tag{4}$$

$$r + s = t, \quad \textit{where} \quad t_{ij} = r_{ij} + s_{ij} \quad \textit{for all} \quad i, j \in I; \tag{5}$$

$$r \cdot s = t, \quad \textit{where} \quad t_{ij} = r_{ij} \cdot s_{ij} \quad \textit{for all} \quad i, j \in I; \tag{6}$$

$$-r = t, \quad \textit{where} \quad t_{ij} = 1_{ij} - r_{ij} \quad \textit{for all} \quad i, j \in I; \tag{7}$$

$$r\,;s = t, \quad \textit{where} \quad t_{ij} = \textstyle\sum_k r_{ik}\,;s_{kj} \quad \textit{for all} \quad i, j \in I; \tag{8}$$

$$r^{\smile} = t, \quad \textit{where} \quad t_{ij} = r^{\smile}_{ji} \quad \textit{for all} \quad i, j \in I. \tag{9}$$

(iv) *$A_{ij} = A(1_{ij})$ for all $i, j \in I$.*
(v) *The union of the subalgebra system generates A.*

Proof. The proof of (ii) is straightforward. Suppose

$$r = \sum_{k\ell} r_{k\ell} = \sum_{k\ell} s_{k\ell},$$

where $r_{k\ell}$ and $s_{k\ell}$ are in $A_{k\ell}$. If $k \neq i$ or $\ell \neq j$, then

$$r_{k\ell} \cdot 1_{ij} \leq 1_{k\ell} \cdot 1_{ij} = 0, \qquad \text{and} \qquad r_{ij} \cdot 1_{ij} = r_{ij},$$

by the monotony law for Boolean multiplications, Lemma 1.1(iii), and Definition 1.2(i). Therefore,

$$r \cdot 1_{ij} = (\sum_{k\ell} r_{k\ell}) \cdot 1_{ij} = \sum_{k\ell}(r_{k\ell} \cdot 1_{ij}) = r_{ij}.$$

A similar argument shows that $r \cdot 1_{ij} = s_{ij}$, so $r_{ij} = s_{ij}$ for all i and j.

Turn now to the proofs of (i) and (iii). Notice first that if a term t_{ij} is determined by the right side of one of (1)–(9), then t_{ij} belongs to A_{ij}, by the definition of a subalgebra system. Consider, for a concrete example, the case when

$$t_{ij} = \sum_k r_{ik} ; s_{kj}.$$

The elements r_{ik} and s_{kj} belong to A_{ik} and A_{kj} respectively, for each k, by assumption. Consequently, $r_{ik} ; s_{kj}$ is in A_{ij}, by Definition 1.2(iv). Since A_{ij} is closed under finite sums, by Definition 1.2(i), it follows that t_{ij} must belong to A_{ij}, as claimed.

If, now, r and s are elements in A, say,

$$r = \sum_{ij} r_{ij} \qquad \text{and} \qquad s = \sum_{ij} s_{ij},$$

then

$$\begin{aligned} r ; s &= (\sum_{ij} r_{ij}) ; (\sum_{ij} s_{ij}) \\ &= \sum_{i,j,k,\ell}(r_{ij} ; s_{k\ell}) \\ &= \sum_{i,k,\ell}(r_{ik} ; s_{k\ell}) \\ &= \sum_{i,j,k}(r_{ik} ; s_{kj}). \end{aligned}$$

The second equality uses the distributive law for relative multiplication over addition. The third equality depends on the fact that $r_{ij} ; s_{k\ell}$ is zero when $j \neq k$, since in this case

$$r_{ij} ; s_{k\ell} \leq 1_{ij} ; 1_{k\ell} = 0,$$

by the monotony law for relative multiplication and Lemma 1.1(vii). The fourth sum is just a reindexing of the third sum, so the final equality is trivial. The equality of the first and last terms immediately implies (8). The proofs of (1)–(7) and (9) are similar, but easier.

The term $t_{ij} = \sum_k r_{ik} ; s_{kj}$ belongs to A_{ij}, by the argument in the second paragraph of the proof, so the sum $t = \sum_{ij} t_{ij}$ belongs to A, by the definition of A. Since this sum coincides with $r ; s$, by (8), it follows that A is closed under relative multiplication. The proofs that A contains the distinguished constants and is closed under the

Boolean operations and under converse are entirely analogous. This completes the proofs of parts (i) and (iii) of the theorem.

Turn now to the proof of (iv). If r is any element in A_{ij}, then r is below 1_{ij} and $r = \sum_{k\ell} r_{k\ell}$, where

$$r_{k\ell} = \begin{cases} r & \text{if} \quad k = i \text{ and } \ell = j\,, \\ 0 & \text{if} \quad k \neq i \text{ or } \ell \neq j\,, \end{cases}$$

so r is in A, by the definition of A, and therefore also in $A(1_{ij})$. Thus, A_{ij} is included in $A(1_{ij})$. To establish the reverse inclusion, suppose r belongs to $A(1_{ij})$. Since r belongs to A, it has the form $r = \sum_{k\ell} r_{k\ell}$, where $r_{k\ell}$ is in $A_{k\ell}$ for all k and ℓ. In particular, $r_{k\ell}$ is below $1_{k\ell}$, by Definition 1.2(i). Also, r is below 1_{ij}, by assumption, and therefore so is $r_{k\ell}$. It follows that if $k \neq i$ or $\ell \neq j$, then

$$r_{k\ell} \leq 1_{ij} \cdot 1_{k\ell} = 0,$$

by monotony and Lemma 1.1(iii), so that $r_{k\ell} = 0$. Consequently, $r = r_{ij}$. The element r_{ij} belongs to A_{ij}, by assumption, so r belongs to A_{ij}, as desired.

The assertion in part (v) of the theorem is an easy consequence of the definition of A and the assumption that the index set I is finite. Every component A_{ij} is included in A, by part (iv), so the subuniverse generated by the union of these components is certainly included in A, by part (i). On the other hand, every element in A is, by definition, a finite sum of elements from the various components, and therefore belongs to the subuniverse generated by the union of the components. Thus, A coincides with the subuniverse generated by the subalgebra system. □

In view of the preceding theorem, it is reasonable to say that the subalgebra system in the theorem *generates*, or *determines*, or *corresponds to*, the subalgebra with universe A.

Corollary 1.5. *If $\mathfrak{A}$ is a subalgebra (of $\mathfrak{S}$) that contains a given finite partition of identity, and if*

$$(A_{ij} : i, j \in I) \tag{i}$$

is the rectangular system determined by $\mathfrak{A}$, then the subalgebra determined by (i) is just $\mathfrak{A}$. Conversely, if (i) is a rectangular system for the partition of identity, and if $\mathfrak{A}$ is the subalgebra determined by (i), then the rectangular system determined by $\mathfrak{A}$ is just (i).

Proof. Start with a subalgebra $\mathfrak{A}$ of $\mathfrak{S}$ that contains the given partition of identity. The rectangular system (i) determined by $\mathfrak{A}$ is defined by

$$A_{ij} = A(1_{ij}) \tag{1}$$

for all i and j in I (see Lemma 1.3). Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{S}$ generated by this system. Since A_{ij} is included in A for each i and j, by (1), the set of generators of B is included in A, and therefore B is included in A. On the other hand, for any element r in A, write

$$r_{ij} = r \cdot 1_{ij}.$$

The element r_{ij} belongs to A_{ij}, by (1), and

$$\textstyle\sum_{ij} r_{ij} = \sum_{ij}(r \cdot 1_{ij}) = r \cdot \sum_{ij} 1_{ij} = r \cdot 1 = r,$$

by the definition of r_{ij}, the distributive law for Boolean multiplication over addition, and Lemma 1.1(ii), so r is generated by elements from the components. In other words, r is in B. It follows that B is included in A, so the subalgebras $\mathfrak{A}$ and $\mathfrak{B}$ are equal.

Now consider an arbitrary rectangular system (i) for the given partition of identity, and let $\mathfrak{A}$ be the subalgebra of $\mathfrak{S}$ that it generates. Equation (1) holds, by part (iv) of the Subalgebra Theorem, so (i) is the rectangular system determined by $\mathfrak{A}$, by the definition of that system. □

One consequence of the corollary is that, for a given partition of identity, the correspondence that takes each rectangular system for that partition to the subalgebra it generates is a bijection from the set of rectangular systems for the partition to the set of subalgebras of $\mathfrak{S}$ that contain the partition. Another way of phrasing the corollary is as follows.

Corollary 1.6. *If $\mathfrak{A}$ is a subalgebra that contains a given finite partition of identity, then a rectangular system $(A_{ij} : i, j \in I)$ for the partition generates $\mathfrak{A}$ if and only if $A(1_{ij}) = A_{ij}$ for all i and j.*

In the setting of arbitrary relation algebras, the structure of the subalgebra generated by a set X is extremely complicated. One defines inductively a sequence $(X_n : n \in N)$ of subsets of the universe, indexed by the set N of natural numbers: $X_0 = X \cup \{1'\}$, and if X_k is defined for each $k < n$, then X_n is defined to be the set of values

$$x + y \quad , \quad -x \quad , \quad x ; y \quad , \quad x^{\smile},$$

where x and y belong to the union of the sets X_k for $k < n$. The subalgebra generated by X is the union of the sets X_k for k in N. This level-by-level construction of the generated subalgebra is analogous to the level-by-level construction of first-order formulas from the set of atomic formulas, using the operators associated with conjunction, disjunction, negation, and existential and universal quantification. In the case of subalgebra systems, the level-by-level construction of the generated subalgebra is reduced to an especially simple form of Boolean generation: it is the set of sums of element systems. Thus, the Subalgebra Theorem may be viewed as an algebraic analogue of a quantifier elimination theorem in logic.

In the setting of arbitrary relation algebras, questions regarding a generated subalgebra, such as its finiteness, or atomicity, or completeness, are quite difficult to answer. In the setting of subalgebra systems, these questions become much easier, since the analysis of the generated subalgebra reduces to the analysis of the corresponding subalgebra system, by parts (i) and (ii) of the Subalgebra Theorem.