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TEXTS AND READINGS IN MATHEMATICS 69

Atiyah-Singer Index Theorem

An Introduction

Amiya Mukherjee

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To Soham

Introduction

Alles sollte so einfach wie möglich gemacht sein, aber nicht einfacher	Everything should be done as simply as possible, but not more simply

——— Albert Einstein

This monograph is primarily written with the intention of presenting a systematic and comprehensive account of the Atiyah-Singer index theorem for beginners. It is influenced by the lectures and seminars at Mathematical Institute, Oxford, during mid sixties by Professor Michael Atiyah and others, when I was a beginner there.

The index theorem is a remarkable result which relates the solution space of an elliptic differential operator on a smooth compact manifold in terms of the symbol of the operator and purely topological information on the manifold. The importance of the theorem may be seen from the Abel Prize citation for Sir Michael Atiyah and Isadore Singer in 2004, which reads "The Atiyah-Singer index theorem is one of the great landmarks of twentieth-century mathematics, influencing profoundly many of the most important later developments in topology, differential geometry and quantum field theory". Indeed, the index theorem has entered into the threshold of the physics of elementary particles in problems related to the gauge theories, and inspired physicists in presenting experimental proofs of some of their predictions, for example, the discovery of neutrius's diffusion reactions on matter, and the detection of charmed particles.

Let X be a smooth compact manifold without boundary. Let E and F be smooth complex vector bundles over X. Let $\Gamma(E)$ and $\Gamma(F)$ be the spaces of smooth sections of these bundles, and $P : \Gamma(E) \to \Gamma(F)$ a differential operator. Let $\pi : T^*X \to X$ denote the cotangent bundle of X, and π^*E and π^*F be the pull-back bundles over T^*X. The principal symbol $\sigma(P)$ of P is defined in terms of the coefficients of the highest order terms of the operator P leaving out all lower order terms. It provides a bundle homomorphism $\sigma(P) : \pi^*E \to \pi^*F$ over T^*X. The differential operator P is called elliptic if its principal symbol $\sigma(P)$ is such that, for each $(x, \xi) \in T^*X$, $\sigma(P)(x, \xi)$ is an isomorphism of the fibres over (x, ξ) if ξ is a non-zero vector of the cotangent space T^*_xX, that is, outside the zero section of T^*X. In this case, $\operatorname{Ker} P$ and $\operatorname{Coker} P = \Gamma(F)/\operatorname{Im} P$

are finite dimensional vector spaces, and the analytic index of P, denoted by $\operatorname{ind} P$, is defined to be the integer $\dim \operatorname{Ker} P - \dim \operatorname{Coker} P$.

On the other hand, the topological index of P, denoted by t-ind P, is a number $(\alpha \cup \beta)[X]$, which is obtained by evaluating the cup product of certain cohomology classes α and β of X on the fundamental homology class $[X]$ of X. The class α depends on P, while the classes β and $[X]$ are independent of P. The class β is actually Hirzebruch's $\widehat{A}$-genus of X, $\widehat{A}(X)$, which is a polynomial with rational coefficients in Pontrjagin cohomology classes of X. The index theorem states that $\operatorname{ind} P =$ t-ind P. The essence of the theorem is that $\operatorname{ind} P$ is given in terms of purely topological data of X.

This remarkable theorem took its shape after some experiments. In 1962, Atiyah and Singer introduced the concept of the Dirac operator on a Riemannian manifold, generalizing Dirac's equation for a spinning electron. This is an elliptic operator on a Clifford bundle with connection over a Riemannian spin manifold. They conjectured that the index of the Dirac operator is the $\widehat{A}$-genus of the spin manifold, and finally proved the conjecture using a method based on Hirzebruch's proof of the signature theorem. This answers a question of Atiyah with which he initiated this research. The answer is that the $\widehat{A}$-genus of a spin manifold is an integer. Of course the answer was established earlier by A. Borel and F. Hirzebruch, however, the formulation of the problem and its proof by Atiyah and Singer are more elegant and have been highly influential.

Subsequently, Atiyah and Singer followed up their ideas to study a general elliptic operator on a smooth manifold, and in 1963 they announced the index theorem with a sketch of its proof which is an extension of Hirzebruch's arguments using Thom's cobordism theory. They never published the proof in full form, and the proof was published by Palais in 1965 as an outcome of a seminar run by him at Princeton University.

The idea of this proof may be described roughly as follows. Consider the cobordism ring generated by equivalence classes of pairs (X, V), where X is a smooth compact oriented manifold and V is a smooth vector bundle on it, and the ring operations are disjoint union and product of manifolds with obvious operations on the vector bundles. This is same as the cobordism ring of compact oriented manifolds, except that manifolds have vector bundles on them. One checks that the analytic and topological indices are homomorphisms of this ring, and they are the same on a particular set of special generators, provided by Thom's cobordism theory. Therefore the indices are equal.

In 1969 Atiyah and Singer published a second proof of the theorem, where the cobordism theory is replaced by K-theory making it more direct and susceptible to further generalizations. Although the proof is very difficult, the clever strategy of the proof can be described in simple language. For an embedding $i : X \to Y$ of smooth compact manifolds, one constructs a "push-forward map" i_* from the space of elliptic operators on X to the space of elliptic operators on Y such that $\operatorname{ind} P = \operatorname{ind} i_*(P)$ for an elliptic operator P on X. Then taking Y

as some sphere, where X embeds in, the index problem reduces to the case of a sphere. Next taking X to be a point in a sphere Y, the problem can further be reduced to that of a point, where the solution is trivial.

This book is an attempt to describe the second proof of Atiyah and Singer and some of its applications, with a view to providing a clear understanding of the index theorem and the ideas surrounding it. This is the most powerful index theorem whose elegance lies in its simplicity and generality. In this volume we have not treated the alternative heat equation approach to the index theorem in local geometrical terms by Atiyah, Bott and Patodi, and postponed the topic and its further simplifications, for a future second volume. The local form of the index theory is important for manifolds with boundary and non-compact manifolds.

The materials are organized into nine chapters, the brief descriptions of which would run as follows.

Chapter 1 deals with K-theory of complex vector bundles giving all elementary concepts required for understanding the subsequent chapters. Chapter 2 introduces Fredholm operators between separable complex Hilbert spaces, gradually going into the realm of K-theory. Here we prove the Atiyah-Jänich theorem, which identifies the K-group $K(X)$ with the set of homotopy classes of families of Fredholm operators on X. We then prove the subsidiary Kuiper theorem on contractibility of the group of invertible elements in certain Banach space of operators.

Chapter 3 gives the first flavour of the index theorem for a Toeplitz operator L_f on the Hardy's space for unit circle defined by a complex valued non-zero smooth function f on the unit circle, which says that $\operatorname{ind} L_f$ is the negative of the winding number of f about the centre of the unit circle, or the degree of f, which is a topological invariant. We then discuss the family of Toeplitz operators which leads to the index bundle. We then prove the Bott periodicity theorem for K-theory, and use this to prove the Thom isomorphism theorem for a complex vector bundle over a compact base space, and then over a locally compact base space.

Chapter 4 starts with brief reviews of Sobolev spaces, pseudo-differential operators, and Fourier integral operators on Euclidean spaces. Then we transfer these concepts to compact Riemannian manifolds using partition of unity arguments. We discuss spectral theory of self-adjoint elliptic pseudo-differential operators. Here we also consider heat operator with the heat kernel and the index. We have made this chapter self-contained assuming only basic analysis.

Chapter 5 is on the theory of characteristic classes. We first prove the existence and uniqueness of Chern classes in general, then pass on to the differential-geometric derivation of the Chern classes of a smooth vector bundle with a connection over a smooth manifold, using the Chern-Weil construction. In Chapter 6 we introduce Clifford algebra which is necessary for the definition of spin structure on a manifold and Dirac operator on a bundle of Clifford modules with a connection. In Chapter 7 we present elementary equivariant K theory, and corresponding Bott periodicity theorem and Thom isomorphism theorem. We then discuss the localization theory.

Finally in Chapter 8 we prove the K-theoretic index theorem. Chapter 9 gives the cohomological formulation of the index theorem and some applications. The applications include signature theorem, Riemann-Roch-Hirzebruch theorem, Atiyah-Segal-Singer fixed point theorem, etc.

The prerequisites for reading this book are as follows. We presume a basic knowledge of algebraic topology, and a knowledge of fibre bundles with obstruction theory, differential geometry with differential forms and connection on vector bundles. In algebra we assume linear algebra, exterior product and tensor product, also basic representation theory of finite groups and compact groups. In analysis we need basic knowledge of Banach spaces and Hilbert spaces, Haar integration over compact Lie groups.

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Contents

CHAPTER 1

K-Theory

K-theory is a cohomology theory for vector bundles. It studies a functor K from the category of compact topological spaces to the category of abelian groups. If X is such a space, and Vect(X) is the semigroup of isomorphism classes of vector bundles over X, then $K(X)$ is the Grothendieck group completion of Vect(X). The functor K satisfies all the Eilenberg and Steenrod axioms for a cohomology theory, except the dimension axiom which specifies the cohomology of a one-point space.

We begin by reviewing some essential features about vector bundles. A more detailed account may be found in Atiyah [**4**]. The notion of general fibre bundles may be obtained from these by leaving out the role of linear algebra from the picture, replacing vector spaces and linear maps whenever they appear by topological spaces and continuous maps. The proofs of the facts about fibre bundles, which we do not discuss here, are standard, and may be found in Steenrod [**60**], and Husemoller [**34**].

1.1. Vector bundles

A vector bundle (complex, unless it is stated otherwise) of rank k, or a k-plane bundle, E over a topological space X is a locally trivial family of vector spaces indexed by X with the help of a continuous surjective map $\pi : E \longrightarrow X$ so that for each $x \in X$ the fibre $E_x = \pi^{-1}(x)$ is a complex vector space of dimension k. The terms 'locally trivial family' signify that each $x \in X$ has an open neighbourhood U such that $\pi^{-1}(U)$ is homeomorphic onto $U \times \mathbb{C}^k$ so that the fibre E_y is mapped linearly and isomorphically onto $\{y\} \times \mathbb{C}^k$ for each $y \in U$. The space E is called the total space, X the base space, and π the projection of the bundle.

The local triviality condition assures that there is an open covering $\{U_i\}$ of X and homeomorphisms $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^k$ such that the homeomorphism $\phi_j \circ \phi_i^{-1}$ of $(U_i \cap U_j) \times \mathbb{C}^k$ defines a map from $U_i \cap U_j$ to the group $GL_k(\mathbb{C})$ of linear automorphisms of $\mathbb{C}^k$. It follows that $\dim(E_x)$ is a locally constant function on X, and therefore it is constant on connected components of X.

The projection $\pi : X \times \mathbb{C}^k \to X$ onto the first factor X is a vector bundle. This is called the product k-plane bundle, and is denoted by $\mathcal{E}^k$. A bundle of rank one is called a line bundle.

A homomorphism from a bundle $\pi : E \longrightarrow X$ to a bundle $\pi' : E' \longrightarrow X$ is a continuous map $\phi : E \longrightarrow E'$ such that $\pi' \circ \phi = \pi$, and $\phi_x = \phi|E_x : E_x \longrightarrow E'_x$ is a linear map for each $x \in X$. Moreover, if ϕ is a bijection and ϕ^{-1} is also a continuous map, then ϕ is called an isomorphism or a bundle map. In this case we say that E and E' are equivalent, and write $E \cong E'$. In fact, any bijective homomorphism is a homeomorphism, that is, a bundle map. A bundle E of rank k is called trivial if $E \cong \mathcal{E}^k$.

In general, a morphism $(\phi, f) : E \longrightarrow E'$ between vector bundles E and E' over different base spaces X and X' consists of a pair of continuous maps $\phi : E \longrightarrow E'$ and $f : X \longrightarrow X'$ such that $\pi' \circ \phi = f \circ \pi$, and for each $x \in X$ the map $\phi_x = \phi|E_x : E_x \longrightarrow E'_{f(x)}$ is linear.

The direct sum or Whitney sum of two bundles $\pi_1 : E_1 \longrightarrow X$ and $\pi_2 : E_2 \longrightarrow X$ is a bundle $\pi : E_1 \oplus E_2 \longrightarrow X$, where $\pi^{-1}(x) = \pi_1^{-1}(x) \oplus \pi_2^{-1}(x)$. Similarly we can define the tensor product $E_1 \otimes E_2$, the bundle of homomorphisms $\mathrm{Hom}(E_1, E_2)$, etc. The fibres of the bundles $E_1 \otimes E_2$ and $\mathrm{Hom}(E_1, E_2)$ over $x \in X$ are respectively $\pi_1^{-1}(x) \otimes \pi_2^{-1}(x)$ and $\mathrm{Hom}(\pi_1^{-1}(x), \pi_2^{-1}(x))$. The local triviality these bundles may be seen easily from the local triviality of the bundles E_1 and E_2.

Clearly, the tensor product is commutative and associative, in the sense that there are canonical isomorphisms

$$E_1 \otimes E_2 \cong E_2 \otimes E_1, \quad (E_1 \otimes E_2) \otimes E_3 \cong E_1 \otimes (E_2 \otimes E_3).$$

Similarly properties also hold for the direct sum.

The bundle $\mathrm{Hom}(E, \mathcal{E}^1)$ is called the dual bundle of E, and is denoted by E^*. There is a canonical isomorphism $(E^*)^* \cong E$.

A section of a bundle $\pi : E \longrightarrow X$ is a continuous map $s : X \longrightarrow E$ such that $\pi \circ s = \mathrm{Id}_X$. Note that a homomorphism $\phi : E_1 \to E_2$ of bundles over X is a section of the bundle $\mathrm{Hom}(E_1, E_2) \to X$. Also the space of sections of a trivial bundle $X \times \mathbb{C}^k \to X$ can be identified with the space of continuous maps $X \to \mathbb{C}^k$ with the compact-open topology..

An n-plane bundle $\pi : E \to X$ is trivial, that is, $E \cong X \times \mathbb{C}^n$, if and only if it admits n sections $s_1, \ldots, s_n : X \to E$ such that the vectors $s_1(x), \ldots, s_n(x)$ are linearly independent in the fibre $\pi^{-1}(x)$ for each $x \in X$.

If $f : X \longrightarrow Y$ is a continuous map between topological spaces, and $\pi : E \longrightarrow Y$ is a bundle, then the pull-back of E by f is the bundle

$$\pi' : f^*(E) \longrightarrow X,$$

where $f^*(E)$ is the subspace of $X \times E$ consisting of pairs (x, v) such that $f(x) = \pi(v)$, and $\pi'(x, v) = x$. There is a morphism $(\widetilde{f}, f) : f^*(E) \longrightarrow E$ given by $\widetilde{f}(x, v) = v$ such that each $\widetilde{f}_x$ is a linear isomorphism. The morphism $\widetilde{f}$ is called the canonical morphism of the pull-back. Any bundle morphism $(\phi, f) : E_1 \longrightarrow E_2$ can be factored as $\phi = \widetilde{f} \circ \phi_1$, where $\phi_1 : E_1 \longrightarrow f^*(E_2)$ is the bundle homomorphism given by $\phi_1(v) = (\pi_1(v), \phi(v))$.

Pull-backs verify the following properties :

(1) $\mathrm{Id}^*(E) \cong E$,

(2) $(g \circ f)^*(E) \cong f^*(g^*(E))$,

(3) $f^*(E_1 \oplus E_2) \cong f^*(E_1) \oplus f^*(E_2)$,

(4) $f^*(E_1 \otimes E_2) \cong f^*(E_1) \otimes f^*(E_2)$.

Similarly, we can define the pull-back of a bundle homomorphism. Let E_1, E_2 be bundles over Y, and $\phi : E_1 \to E_2$ be a bundle homomorphism. Let $f : X \to Y$ be a continuous map. Then the pull-back of ϕ by f is the bundle homomorphism

$$f^*\phi : f^*E_1 \to f^*E_2,$$

which is defined by $(f^*\phi)(x, v_1) = (x, \phi(v_1))$, where $(x, v_1) \in f^*E_1 \subset X \times E_1$.

If $E \to X$ is a bundle over X, and $A \subset X$ is a subspace of X with inclusion $i : A \to X$, then the pull-back $i^*E \to A$ is called the restriction of E over A, and is denoted by $E|A$.

The set of isomorphism classes of bundles over X is denoted by $\mathrm{Vect}(X)$. It is a commutative semiring under the operations $\oplus$ and $\otimes$, with the zero element given by $\mathcal{E}^0$, and the multiplicative identity by $\mathcal{E}^1$.

A continuous map $f : X \longrightarrow Y$ induces a homomorphism of the semirings

$$f^* : \mathrm{Vect}(Y) \longrightarrow \mathrm{Vect}(X).$$

by pulling back vector bundles over Y to vector bundles over X. In fact, we have a contravariant functor from the category of topological spaces to the category of commutative semirings. The restriction of this functor to the subcategory of paracompact Hausdorff spaces is a homotopy invariant, because of the following homotopy property of the pull-back, as described in part (c) of the following lemma (parts (a) and (b) are used to prove part (c)).

Lemma 1.1.1. *Let E and F be vector bundles over a paracompact Hausdorff space X, and A be a closed subspace of X. Then the following facts are true.*

(a) Any section s' of the vector bundle $E|A$ can be extended to a section s of the vector bundle E.

(b) Any homomorphism $\phi' : E|A \to F|A$ can be extended to a homomorphism $\phi : E \to F$. Moreover, if ϕ' is an isomorphism, then there is a neighbourhood U of A such that $\phi|U : E|U \to F|U$ is an isomorphism.

(c) If Y is another space and B is a vector bundle over it, and if $f_t : X \to Y$ is a homotopy, $0 \leq t \leq 1$. then $f_0^*B \cong f_1^*B$.

Proof. (a) Cover X by a locally finite open covering $\{U_i\}$ such that each $E|U_i$ is trivial. Let $\{\lambda_i\}$ be a partition of unity subordinate to this covering. Extend $s'_i = s'|A \cap U_i : A \cap U_i \to \mathbb{C}^k$ to $s_i : U_i \to \mathbb{C}^k$ ($k = \mathrm{rank}\, E$), by the Tietze-Urysohn extension theorem, which says that a continuous map from a closed subspace of a normal space U into a vector space can be extended to a continuous map over U. Then $s = \sum_i \lambda_i s_i$ is a section of E extending s'.

(b) Extend the section ϕ' of the vector bundle $\mathrm{Hom}(E,F)|A$ to a section of the vector bundle $\mathrm{Hom}(E,F)$. If ϕ' is an isomorphism, then by continuity there is a neighbourhood U of A such that $\phi|U$ is an isomorphism. Note that the subspace of isomorphisms $\mathrm{Iso}(E,F)$ is open in $\mathrm{Hom}(E,F)$.

(c) Let I denote the unit interval $[0,1]$. If $f : X \times I \to Y$ is the homotopy defined by $f(x,t) = f_t(x)$, and $p : X \times I \to X$ is the projection, then the bundles f^*B and $p^* f_t^* B$ are isomorphic when restricted to the closed subspace $X = X \times \{t\}$ of $X \times I$, and so they are isomorphic on a neighbourhood $X \times (t-\delta, t+\delta)$ for some $\delta > 0$. Therefore the isomorphism class of $f_t^* B$ is locally constant, and hence a constant function of t, showing that $f_0^* B \cong f_1^* B$. □

The lemma implies that if $f : X \longrightarrow Y$ is a homotopy equivalence, then

$$f^* : \mathrm{Vect}(Y) \longrightarrow \mathrm{Vect}(X)$$

is an isomorphism of semirings. This follows, because if g is a homotopy inverse of f, then $g^* f^* = (\mathrm{Id}_Y)^* = \mathrm{Id}$, and $f^* g^* = (\mathrm{Id}_X)^* = \mathrm{Id}$. In particular, if X is contractible, that is, if the identity map Id_X is homotopic to a constant map, then every bundle over X is trivial, and $\mathrm{Vect}(X)$ is isomorphic to the semiring of non-negative integers.

The lemma has another simple application. Let S^n denote the unit n-sphere, and $GL_k(\mathbb{C})$ the general linear group of linear automorphisms of $\mathbb{C}^k$. Let $[S^{n-1}, GL_k(\mathbb{C})]$ be the set of homotopy classes of continuous maps from S^{n-1} to $GL_k(\mathbb{C})$, and $\mathrm{Vect}_k(S^n)$ the set of isomorphism classes of k-plane bundles over S^n.

Lemma 1.1.2. *There is a bijection* $[S^{n-1}, GL_k(\mathbb{C})] \longrightarrow \mathrm{Vect}_k(S^n)$.

PROOF. Write $S^n = D^n_+ \cup D^n_-$, where D^n_+ and D^n_- are the upper and lower hemispheres of S^n

$$D^n_+ = \{(x_1, x_2, \cdots, x_{n+1}) \in S^n : 0 \le x_{n+1} \le 1\}.$$

$$D^n_- = \{(x_1, x_2, \cdots, x_{n+1}) \in S^n : -1 \le x_{n+1} \le 0\},$$

so that $D^n_+ \cap D^n_- = S^{n-1}$. Given a map

$$f : S^{n-1} \longrightarrow GL_k(\mathbb{C}),$$

let E_f be the quotient of the disjoint union $D^n_+ \times \mathbb{C}^k \cup D^n_- \times \mathbb{C}^k$ by the identification of $(x,v) \in \partial D^n_+ \times \mathbb{C}^k$ with $(x, f(x)(v)) \in \partial D^n_- \times \mathbb{C}^k$. Then $\pi : E_f \longrightarrow S^n$, where $\pi([x,v]) = x$, is a k-plane bundle (we shall prove this fact in a more general context in §1.4). The bundle E_f is said to be obtained by gluing or clutching construction; f is called a clutching function for E_f. Two homotopic clutching functions $f, g : S^{n-1} \longrightarrow GL_k(\mathbb{C})$ produce isomorphic bundles $E_f \cong E_g$, because a homotopy $h : S^{n-1} \times I \longrightarrow GL_k(\mathbb{C})$ between f and g gives a bundle $E_h \longrightarrow S^n \times I$ by clutching construction such that $E_h|S^n \times \{0\} \cong E_f$ and $E_h|S^n \times \{1\} \cong E_g$. Therefore we have a map

$$\phi : [S^{n-1}, GL_k(\mathbb{C})] \longrightarrow \mathrm{Vect}_k(S^n).$$

On the other hand, if $E \longrightarrow S^n$ is a k-plane bundle, then $E|D^n_+$ and $E|D^n_-$ are trivial bundles, since D^n_+ and D^n_- are contractible. If $f_\pm : E|D^n_\pm \longrightarrow D^n_\pm \times \mathbb{C}^k$ are isomorphisms, then the map $f_- \circ (f_+)^{-1}$ over S^{n-1} gives a bundle map $S^{n-1} \times \mathbb{C}^k \longrightarrow S^{n-1} \times \mathbb{C}^k$, and thus defines a map $f : S^{n-1} \longrightarrow GL_k(\mathbb{C})$. The homotopy class of f is well-defined, because the homotopy classes of f_+ and f_- are well-defined. The proof uses the facts that D^n_+ and D^n_- are contractible, and $GL_k(\mathbb{C})$ is path-connected. Therefore we have a map

$$\psi : \mathrm{Vect}_k(S^n) \longrightarrow [S^{n-1}, GL_k(\mathbb{C})].$$

Clearly, ϕ and ψ are inverses of each other. □

Remark 1.1.3. We may replace $GL_k(\mathbb{C})$ by the group $U(k)$ of unitary matrices A, because $U(k)$ is a deformation retract of $GL_k(\mathbb{C})$. Note that, by matrix polar decomposition, $GL_k(\mathbb{C}) = U(k) \times H(k)$, where $H(k)$ is the set of positive definite Hermitian matrices. Since $H(k)$ is a convex set in the real vector space of Hermitian matrices, it is contractible (see Steenrod [**60**], §12.13).

Remark 1.1.4. Lemma 1.1.2 is not true for real vector bundles, because the corresponding general linear group $GL_k(\mathbb{R})$ of linear automorphisms of $\mathbb{R}^k$ is not path connected. However, if we consider the path connected subgroup $GL^+_k(\mathbb{R})$ of $GL_k(\mathbb{R})$ consisting of matrices of positive determinant, and consider the semiring $\mathrm{Vect}^+_k(S^n)$ of equivalence classes of real oriented vector bundles over S^n, then the same proof will show that there is a bijection

$$[S^{n-1}, GL^+_k(\mathbb{R})] \to \mathrm{Vect}^+_k(S^n).$$

1.2. Classification of bundles

Let $G_k(\mathbb{C}^n)$ be the complex Grassmann manifold of k-planes in $\mathbb{C}^n$. The Stiefel manifold $V_k(\mathbb{C}^n)$ of orthonormal k-frames in $\mathbb{C}^n$ is a closed (and hence compact) subset of the product of spheres

$$S^{2n-1} \times \cdots \times S^{2n-1} \quad (k \text{ times}).$$

There is a natural projection $p : V_k(\mathbb{C}^n) \longrightarrow G_k(\mathbb{C}^n)$ mapping a k-frame onto the k-plane spanned by it. The space $G_k(\mathbb{C}^n)$ is given the quotient topology so that p becomes a continuous map, and $G_k(\mathbb{C}^n)$ a compact space. The 'tautological' k-plane bundle $\pi : \gamma^k_n \longrightarrow G_k(\mathbb{C}^n)$ is defined by

$$\gamma^k_n = \{(\alpha, v) \in G_k(\mathbb{C}^n) \times \mathbb{C}^n : v \in \alpha\} \text{ and } \pi(\alpha, v) = \alpha.$$

Clearly π is continuous, and $\pi^{-1}(\alpha) \cong \mathbb{C}^k$ for $\alpha \in G_k(\mathbb{C}^n)$.

Keeping k fixed, we have an increasing sequence of Grassmann manifolds

$$G_k(\mathbb{C}^k) \subset G_k(\mathbb{C}^{k+1}) \subset G_k(\mathbb{C}^{k+2}) \subset \cdots .$$

induced by natural inclusions $\mathbb{C}^n \subset \mathbb{C}^{n+1}$. The direct limit $\mathbb{C}^\infty = \cup_n \mathbb{C}^n$ is the space of complex sequences for which all but a finite number of terms are non-zero with the direct limit topology, where $U \subset \mathbb{C}^\infty$ is open if and only if $U \cap \mathbb{C}^n$ is open in $\mathbb{C}^n$ for each n..

The direct limit of the sequence of Grassmann manifolds is the infinite Grassmann manifold

$$G_k(\mathbb{C}^\infty) = \cup_{n \geq k} G_k(\mathbb{C}^n)$$

with the direct limit topology. This space is called the classifying space of k-plane bundles, and is also denoted by $BU(k)$.

The universal k-plane bundle $\pi : \gamma^k \longrightarrow BU(k)$ is defined by

$$\gamma^k = \{(\alpha, v) \in BU(k) \times \mathbb{C}^\infty \ : \ v \in \alpha\} \text{ and } \pi(\alpha, v) = \alpha.$$

Then $\gamma^k | G_k(\mathbb{C}^n) = \gamma^k_n$. In fact,

$$\gamma^k = \lim_{n \to \infty} \gamma^k_n = \cup_n \gamma^k_n.$$

Lemma 1.2.1. *The projections $\pi : \gamma^k_n \longrightarrow G_k(\mathbb{C}^n)$, and $\pi : \gamma^k \longrightarrow BU(k)$ are k-plane bundles.*

PROOF. We have to verify the local triviality. Suppose n is finite, and fix a Hermitian metric in $\mathbb{C}^n$. Then for each $\alpha \in G_k(\mathbb{C}^n)$, we have an orthogonal decomposition $\mathbb{C}^n = \alpha \oplus \alpha^\perp$, and an orthogonal projection $\pi_\alpha : \mathbb{C}^n \longrightarrow \alpha$. Define

$$U_\alpha = \{\beta \in G_k(\mathbb{C}^n) \ : \ \pi_\alpha(\beta) = \alpha, \quad \text{or} \quad \beta \cap \alpha^\perp = \{0\}\}.$$

This is an open neighbourhood of α in $G_k(\mathbb{C}^n)$, since

$$p^{-1}(U_\alpha) = \{(v_1, \ldots, v_k) \in V_k(\mathbb{C}^n) \ : \ (\pi_\alpha(v_1), \cdots, \pi_\alpha(v_k)) \text{ is a basis of } \alpha\}$$

is the open set $V_k(\mathbb{C}^n) \cap \pi_\alpha^{-1}(\alpha)$ in $V_k(\mathbb{C}^n)$.

Let $(v_1, \cdots, v_k, v_{k+1}, \cdots, v_n)$ be an orthonormal basis of $\mathbb{C}^n$ so that $(v_1, \ldots, v_k)$ is a basis of α, and $(v_{k+1}, \cdots, v_n)$ is a basis of $\alpha^\perp$. Each $\beta \in U_\alpha$ determines a unique orthonormal basis $(w_1, \ldots, w_k)$ of itself such that $\pi_\alpha(w_i) = v_i$. This gives a section $U_\alpha \longrightarrow p^{-1}(U_\alpha)$, which is clearly continuous. Define a linear isomorphism

$$T_\beta : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

by $T_\beta(v) = \pi_\alpha(v)$ if $v \in \beta$, and $T_\beta(v) = v$ if $v \in \alpha^\perp$. Clearly the map

$$T : U_\alpha \longrightarrow GL(\mathbb{C}^n),$$

which maps β to T_β, is continuous.

Then the map $\phi_\alpha : U_\alpha \times \alpha \longrightarrow p^{-1}(U_\alpha)$ defined by $\phi_\alpha(\beta, w) = (\beta, {T_\beta}^{-1}(w))$ is continuous, and it has a continuous inverse given by $\phi_\alpha^{-1}(\beta, v) = (\beta, \pi_\alpha(v))$. This proves the local triviality for the bundle γ^k_n, since $U_\alpha \times \alpha \cong U_\alpha \times \mathbb{C}^k$.

For the infinite case, suppose $\alpha \in BU(k)$, and $\pi_\alpha : \mathbb{C}^\infty \longrightarrow \alpha$ be the orthogonal projection. Define $U = \{\beta \in BU(k) \ : \ \pi_\alpha(\beta) = \alpha\}$. This is an open set of the direct limit topology in $BU(k)$, because $U \cap G_k(\mathbb{C}^n) = U_\alpha$ is open in $G_k(\mathbb{C}^n)$ for each n. Define $\phi : U \times \alpha \longrightarrow p^{-1}(U)$ as in the case of finite n. Then ϕ is continuous, because $\phi_\alpha = \phi|U_\alpha$ is continuous for each n. Similarly ϕ^{-1} is continuous. Thus ϕ is a homeomorphism. □

The classification theorem of k-plane bundles is

Theorem 1.2.2. *If X is a paracompact Hausdorff space, then the correspondence $[f] \mapsto [f^*\gamma^k]$ sets up a natural bijection*

$$\Phi : [X, BU(k)] \longrightarrow \mathrm{Vect}_k(X).$$

PROOF. First note that if $\pi : E \longrightarrow X$ is a k-plane bundle, then $E \cong f^*\gamma^k$ for some $f : X \to BU(k)$ if and only if there is a map $g : E \longrightarrow \mathbb{C}^\infty$ which is a linear monomorphism on each fibre of E. The map g is called a Gauss map for E. The assertion follows easily. On one hand, if $\phi : E \longrightarrow f^*\gamma^k$ is a bundle equivalence, and $p : \gamma^k \longrightarrow \mathbb{C}^\infty$ is the projection $p(\alpha, v) = v$, then $p\widetilde{f}\phi$ is a Gauss map for E, where $\widetilde{f} : f^*\gamma^k \longrightarrow \gamma^k$ is the canonical bundle morphism of the pull-back. On the other hand, if $g : E \longrightarrow \mathbb{C}^\infty$ is a Gauss map for a k-plane bundle E, then there is a bundle morphism $(\phi, f) : E \longrightarrow \gamma^k$ which is isomorphic on each fibre, where $f : X \longrightarrow BU(k)$ is defined by $f(x) = g(\pi^{-1}(x))$, and $\phi : E \longrightarrow \gamma^k$ is defined by $\phi(v) = (f\pi(v), g(v))$. Then ϕ factors into a bundle equivalence $E \cong f^*\gamma^k$, as defined earlier in §1.1 in connection with pull-back.

Next construct a Gauss map g for a k-plane bundle E in the following way. Take a locally finite open covering $\{U_i\}$ of X such that each $E|U_i$ is trivial. Let $\phi_i : E|U_i \longrightarrow U_i \times \mathbb{C}^k$ be the isomorphisms, and $p_i : U_i \times \mathbb{C}^k \longrightarrow \mathbb{C}^k$ be the projections. Let $\{\lambda_i\}$ be a partition of unity subordinate to the covering $\{U_i\}$, and $g_i : E|U_i \longrightarrow \mathbb{C}^k$ be the map $v \mapsto \lambda_i(\pi(v)) \cdot p_i\phi_i(v)$. Then $g = \sum_i g_i : E \longrightarrow C^\infty$ is the required Gauss map. This proves that the map Φ is surjective.

The proof of the injectivity proceeds as follows. Let j_+, $j_- : \mathbb{C}^n \longrightarrow \mathbb{C}^\infty$ be the linear monomorphisms given by

$$j_+(z) = (0, z_1, 0, z_2, \ldots, 0, z_n, 0, 0, \ldots),$$
$$j_-(z) = (z_1, 0, z_2, 0, \ldots, z_n, 0, 0, 0, \ldots),$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Then $\mathrm{Id} \simeq j_+$, and $\mathrm{Id} \simeq j_-$ by the homotopies $h_\pm : C^n \times [0,1] \to \mathbb{C}^\infty$ given by

$$h_\pm(z,t) = (1-t) \cdot z + t \cdot j_\pm(z),$$

which are linear monomorphisms for each t.

Suppose that for a bundle $\pi : E \longrightarrow X$ and two maps $f_0, f_1 : X \longrightarrow BU(k)$, we have isomorphisms $E \cong f_0^*\gamma^k$ and $E \cong f_1^*\gamma^k$. Let $g_0, g_1 : E \longrightarrow \mathbb{C}^\infty$ be the corresponding Gauss maps so that $f_0(x) = g_0(\pi^{-1}(x)$ and $f_1(x) = g_1(\pi^{-1}(x))$, $x \in X$. Then j_+g_0 and j_-g_1 are homotopic through linear monomorphisms $g_t = (1-t)j_+g_0 + tj_-g_1$. Then $f_t = g_t(\pi^{-1}(x))$ is a homotopy between f_0 and f_1. This proves the injectivity of Φ. □

A simple consequence of the classification theorem is the following result.

Lemma 1.2.3. *For any bundle E over X, there is a bundle E' over X such that $E \oplus E'$ is trivial (E' is called a complementary bundle of E).*

PROOF. Let $f : X \longrightarrow G_k(\mathbb{C}^n)$ be the classifying map for E, so that $E \cong f^*(\gamma_n^k)$. Consider the complementary bundle $\overline{\pi} : \overline{\gamma}_n^k \longrightarrow G_k(\mathbb{C}^n)$, where

$$\overline{\gamma}_n^k = \{(W, x) \in G_k(\mathbb{C}^n) \times \mathbb{C}^n \ : \ x \perp W\} \text{ and } \overline{\pi}(W, x) = W.$$

We have a bundle equivalence $\phi : \gamma_n^k \oplus \overline{\gamma}_n^k \longrightarrow \mathcal{E}^n$ over $G_k(\mathbb{C}^n)$, where the isomorphism ϕ is given by $\phi((W, x), (W, x')) = (W, x + x')$. Let $E' = f^*(\overline{\gamma}_n^k)$. Then

$$E \oplus E' \cong f^*(\gamma_n^k) \oplus f^*(\overline{\gamma}_n^k) \cong f^*(\gamma_n^k \oplus \overline{\gamma}_n^k) \cong \mathcal{E}^n.$$

This establishes the assertion. □

1.3. The functors K and $\widetilde{K}$

The ring completion of a semiring S is a pair $(K(S), \alpha)$, where $K(S)$ is a ring and $\alpha : S \longrightarrow K(S)$ is a semiring homomorphism such that if $\beta : S \longrightarrow R$ is any semiring homomorphism into a ring R, then there is a unique ring homomorphism $\gamma : K(S) \longrightarrow R$ with $\gamma \circ \alpha = \beta$. The ring $K(S)$ satisfying this universal property is called the Grothendieck ring of S. It follows that if $K(S)$ exists, then it must be unique up to isomorphism.

The existence of $K(S)$ may be seen by the following construction. Consider the product $S \times S$ and an equivalence relation $\sim$ on it defined by $(a_1, b_1) \sim (a_2, b_2)$ if and only if there exists $c \in S$ such that $a_1 + b_2 + c = a_2 + b_1 + c$. Then $K(S)$ is the quotient $S \times S/\sim$. This is a ring where the ring operations are defined by $[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$ and $[a_1, b_1] \cdot [a_2, b_2] = [a_1a_2 + b_1b_2, a_1b_2 + b_1a_2]$. The negative of $[a, b]$ is $[b, a]$, and the zero is $[0, 0]$. The semiring homomorphism $\alpha : S \longrightarrow K(S)$ is given by $\alpha(a) = [a, 0]$. Note that this is exactly the method of constructing the ring of integers $\mathbb{Z}$ from the semiring of natural numbers $\mathbb{N}$.

Alternatively, we may define $K(S)$ in the following way. Let $F(S)$ be the ring generated by the elements of S as a free abelian group, together with multiplication extended from that of S. Then $K(S)$ is the quotient of $F(S)$ modulo the ideal generated by the elements of the form $(a+b) - a - b$, $a, b \in S$. Here the homomorphism α is the restriction to S of the canonical projection $F(S) \longrightarrow K(S)$ of the quotient ring.

These constructions give isomorphic $K(S)$, by the universal property. In each case, we may represent an element of $K(S)$ as a difference $\alpha(a) - \alpha(b)$ where a, $b \in S$.

For a compact space X, we define $K(X)$ as the ring completion of the commutative semiring $\mathrm{Vect}(X)$ of equivalence classes of complex vector bundles over X. The multiplicative identity 1 in $K(X)$ is represented by $\mathcal{E}^1$.

An element $[E]-[F]$ of $K(X)$ is called a virtual bundle. It can also be written as $[H]-[\mathcal{E}^n]$ for some bundle H and integer n. For, we can find a bundle F' by Lemma 1.2.3 such that $F\oplus F'\cong\mathcal{E}^n$, and therefore, if $E\oplus F'=H$, then

$$[E]-[F]=([E]+[F'])-([F]+[F'])=[E\oplus F']-[F\oplus F']=[H]-[\mathcal{E}^n].$$

A continuous map $X\longrightarrow Y$ induces a ring homomorphism $K(f):K(Y)\longrightarrow K(X)$ given by $K(f)([E]-[F])=[f^*E]-[f^*F]$. The usual practice is to denote $K(f)$ by f^*. By Lemma 1.1.1, the homomorphism f^* depends only on the homotopy class of f. We have the functorial properties $\mathrm{Id}^*=\mathrm{Id}$ and $(g\circ f)^*=f^*\circ g^*$ coming from the corresponding properties of pull-back.

We denote the rank of a bundle E by $\mathrm{rk}(E)$. This gives a semiring homomorphism $\mathrm{rk}:\mathrm{Vect}(X)\longrightarrow\mathbb{Z}$, and this can be extended to a homomorphism $\mathrm{rk}:K(X)\longrightarrow\mathbb{Z}$ given by $\mathrm{rk}([E]-[F])=\mathrm{rk}(E)-\mathrm{rk}(F)$, by the universal property. The integer $\mathrm{rk}(E)-\mathrm{rk}(F)$ is called the virtual dimension of $[E]-[F]$. This may be positive, negative, or zero. Since for the multiplicative identity $1\in K(X)$, $\mathrm{rk}(\mathcal{E}^1)=1$, we have a ring homomorphism $\theta:\mathbb{Z}\longrightarrow K(X)$ such that $\theta(n)=[\mathcal{E}^n]$, for $n\geq 0$.

We define the reduced contravariant functor $\widetilde{K}$ by setting

$$\widetilde{K}(X)=\ker(\mathrm{rk}:K(X)\longrightarrow\mathbb{Z}), \tag{1.3.1}$$

and, for a continuous map $f:X\longrightarrow Y$, $\widetilde{K}(f)=K(f)|\widetilde{K}(X)$, or $\widetilde{f}^*=f^*|\widetilde{K}(X)$. We have an exact sequence

$$0\longrightarrow\widetilde{K}(X)\longrightarrow K(X)\xrightarrow{\mathrm{rk}}\mathbb{Z}\longrightarrow 0$$

which splits as $K(X)=\widetilde{K}(X)\oplus\mathbb{Z}$, since $(\mathrm{rk})\circ\theta=\mathrm{Id}_{\mathbb{Z}}$.

Definition 1.3.1. Two bundles E and F over X are called s-equivalent (or stably equivalent), written $E\overset{s}{\sim}F$, if $E\oplus\mathcal{E}^j\cong F\oplus\mathcal{E}^k$ for some j,k.

This gives an equivalence relation on $\mathrm{Vect}(X)$. Note that isomorphic bundles are s-equivalent, however s-equivalence does not imply equivalence. For example, any two trivial bundles are stably equivalent, but they are not equivalent unless they have the same rank.

Another example is as follows. If $\tau(S^n)$ and $\nu(S^n)$ are respectively the tangent and normal bundle of the n-sphere S^n, then $\tau(S^n)\oplus\nu(S^n)\cong\mathcal{E}^{n+1}$ and $\nu(S^n)\cong\mathcal{E}^1$, and so $\tau(S^n)$ is stably trivial, however, $\tau(S^n)$ is not trivial unless $n=1,3,\mathit{or}\,7$. This follows from a result of Adams [**1**]. Here $\tau(S^n)$ and $\nu(S^n)$ are subbundles of the real trivial bundle $S^n\times\mathbb{R}^{n+1}$ over S^n, where $\tau(S^n)=\{(x,v)\in S^n\times\mathbb{R}^{n+1}:x\perp v\}$, and $\nu(S^n)=\{(x,tx)\in S^n\times\mathbb{R}^{n+1}:t\in\mathbb{R}\}$. There is a bundle map $\tau(S^n)\oplus\nu(S^n)\to S^n\times\mathbb{R}^{n+1}$ given by $((x,v),(x,tx))\mapsto(x,v+tx)$. The line bundle $\nu(S^n)$ is trivial, since it admits a non-zero section $S^n\to\nu(S^n)$ given by $x\mapsto(x,tx)$ for a fixed $t\in\mathbb{R}$, $t\neq 0$.

Lemma 1.3.2. *There is an isomorphism α from the set of stable equivalence classes in* $\mathrm{Vect}(X)$ *onto $\widetilde{K}(X)$, obtained by sending the stable class of an n-plane bundle E to the element $[E] - [\mathcal{E}^n] \in \widetilde{K}(X)$.*

PROOF. Clearly, if $\alpha([E]) = \alpha([F])$, then $E \overset{s}{\sim} F$. Also α is surjective. To see this, take $[E] - [F] \in \widetilde{K}(X)$. Then $\mathrm{rk}E = \mathrm{rk}F$. Find F' and n such that $F \oplus F' \cong \mathcal{E}^n$. Then in $\widetilde{K}(X)$ we have

$$[E] - [F] = [E] + [F'] - ([F] + [F']) = [E \oplus F'] - [\mathcal{E}^n],$$

and $\mathrm{rk}(E \oplus F') = \mathrm{rk}(F \oplus F') = n$. Therefore $\alpha(E \oplus F') = [E] - [F]$. □

Lemma 1.3.3. (a) *Let E be a k-plane bundle over a CW-complex X of dimension n. Let m be a non-negative integer $\leq k$, and $n \leq 2(k - m)$. Then E is isomorphic to $F \oplus \mathcal{E}^m$ for some $(k - m)$-plane bundle F over X.*

(b) *If E_1, E_2 are k-plane bundles over a CW-complex X of dimension n such that $E_1 \oplus \mathcal{E}^m \cong E_2 \oplus \mathcal{E}^m$ for some $m > 0$ satisfying the inequalities in part* (a) *above, then $E_1 \cong E_2$.*

Note that the case $m = k$ says that a vector bundle over a point is trivial.

In the proof of the lemma, we will encounter a fibre bundle. As mentioned earlier, its notion is obtained from the definition of a vector bundle by dropping all references to linear algebra. Thus the fibre is a topological space, linear maps are continuous maps, and linear isomorphisms are homeomorphisms.

The proof uses following results from obstruction theory (Steenrod [**60**], §§ 35 – 37).

Let (X, A) be a relative CW-complex with $\dim(X - A) = n$. Let F be a fibre bundle over (X, A) with fibre S^{2k-1}, and f_0 a section of $F|A$. Then the primary obstructions to a section f of F on X which agrees with f_0 on A lie in the cohomology groups $H^i(X, A; \pi_{i-1}(S^{2k-1}))$, $i = 0, 1, \ldots, n$, and F admits such a section f if and only if all these obstructions are zero. Moreover, if F admits such a section, then the set of homotopy classes rel A of such sections corresponds bijectively with the elements of the group $H^n(X, A; \pi_n(S^{2k-1}))$.

PROOF. (a) First suppose that $m = 1$ and $n < 2k - 1$. Let $E_0 \longrightarrow X$ be the fibre bundle of non-zero vectors of E. Its fibre is $\mathbb{C}^k - \{0\}$, which has the homotopy type of S^{2k-1}. Therefore, the primary obstructions to a section of E_0 lie in the cohomology groups

$$H^i(X, \pi_{i-1}(S^{2k-1})), \quad i = 0, 1, \ldots, n.$$

Since S^{2k-1} is $(2k-2)$-connected, these obstructions are all zero if $n < 2k - 1$, so E_0 has a non-zero section s, and any two sections are homotopic. Define a monomorphism $\phi : \mathcal{E}^1 \longrightarrow E$ by $\phi(x, \lambda) = \lambda s(x)$, $x \in X$, $\lambda \in \mathbb{C}$ (note that the section s is nowhere zero). Let F be the quotient bundle of ϕ which is $\mathrm{Coker}\,\phi = E/\phi(\mathcal{E}^1)$. Now E can be given a Hermitian metric, and so $E \cong \phi(\mathcal{E}^1) \oplus (\phi(\mathcal{E}^1))^\perp$ and we have another quotient bundle $(\phi(\mathcal{E}^1))^\perp$ of ϕ. Therefore $E \cong \mathcal{E}^1 \oplus F$, because any two quotient bundles of a monomorphism are isomorphic, by the five lemma. This proves the lemma in the special case $m = 1$. The general case for other values of m may be obtained by repeating these arguments.

(b) As before, it is sufficient to prove the result for $m = 1$. The general case will then follow by induction. First, we assert that if $\phi_1 : \mathcal{E}^1 \longrightarrow E$ and $\phi_2 : \mathcal{E}^1 \longrightarrow E$ are monomorphisms, then $\operatorname{Coker}\phi_1 \cong \operatorname{Coker}\phi_2$. To see this, note that any monomorphism $\phi : \mathcal{E}^1 \longrightarrow E$ is completely determined by a section s of the bundle $\pi : E_0 \longrightarrow X$, where s determines ϕ by $\phi(x,\lambda) = \lambda s(x)$, $\lambda \in \mathbb{R}$, and ϕ determines s by $s(x) = \phi(x,1)$. Let $s_1 : X \longrightarrow E_0$ and $s_2 : X \longrightarrow E_0$ be sections corresponding to ϕ_1 and ϕ_2. Then s_1 and s_2 define a section s' of the bundle $\pi \times \mathrm{Id} : E_0 \times I \longrightarrow X \times I$ over $X \times \{0\} \cup X \times \{1\}$ such that $s'|X \times \{0\} = s_1$ and $s'|X \times \{1\} = s_2$. The partial section s' extends to a full section s of $E_0 \times I$ over $X \times I$ by the obstruction theory, since $\dim(X \times I) = n + 1 \leq 2k - 1$. Then the section s defines a monomorphism $\psi : \mathcal{E}^1 \longrightarrow E \times I$ so that $\operatorname{Coker}\psi$ is a bundle over $X \times I$ with $(\operatorname{Coker}\psi)|X \times \{0\} \cong \operatorname{Coker}\phi_1$ and $(\operatorname{Coker}\psi)|X \times \{1\} \cong \operatorname{Coker}\phi_2$. Clearly, this means that $\operatorname{Coker}\phi_1 \cong \operatorname{Coker}\phi_2$, by an argument used in the proof of Lemma 1.1.1 (c). This proves the assertion.

Therefore, if $E_1 \oplus \mathcal{E}^1 \cong E_2 \oplus \mathcal{E}^1$, then the natural monomorphisms

$$\mathcal{E}^1 \longrightarrow E_1 \oplus \mathcal{E}^1, \quad \text{and} \quad \mathcal{E}^1 \longrightarrow E_2 \oplus \mathcal{E}^1$$

have isomorphic cokernels, that is, $E_1 \cong E_2$. □

Thus $E \cong F \oplus \mathcal{E}^{k-m}$ for some m-plane bundle F if $k > m$ and $n \leq 2m$, and so E and F give the same element in $\widetilde{K}(X)$. A k-plane bundle E over a CW complex X of dimension n is said to be in the stable range if $k > m$, and $n \leq 2m$ for some $m > 0$. In this case, E is stably equivalent to a bundle of lower rank m. On the other hand, if the rank k of E is smaller than m, then E is stably equivalent to $E \oplus \mathcal{E}^{m-k}$ whose rank is m. Also if two k-plane bundles in the stable range are stably equivalent, then they are equivalent; the converse is true in any way.

Therefore if m is an integer $\geq n/2$, then

$$\widetilde{K}(X) \cong \mathrm{Vect}_m(X).$$

In particular, if $X = S^n$, then

$$\widetilde{K}(S^n) \cong \mathrm{Vect}_m(S^n), \text{ for } m \geq n/2.$$

By Lemma 1.1.2 and Remark 1.1.3, we have $\mathrm{Vect}_m(S^n) \cong \pi_{n-1}(U(m))$. Therefore $\widetilde{K}(S^n) = \pi_{n-1}(U(m))$.

The homotopy groups of $U(n)$ are given as follows :

$$\begin{aligned} \pi_i(U(1)) = \pi_i(S^1) &= \mathbb{Z}, \quad i = 1 \\ &= 0, \quad i \neq 1, \end{aligned}$$

and if $i < 2n$, then

$$\pi_i(U(n)) \cong \pi_i(U(n+1)) \cong \pi_i(U(n+2)) \cong \cdots \cong \pi_i(U),$$

where U is the direct limit group

$$U = \lim_{n \to \infty} U(n).$$

The first of the sequence of above isomorphisms follows from the exact homotopy sequence of the principal bundle $U(n) \to U(n+1) \to S^{2n+1}$, since S^{2n+1} is $2n$-connected. The subsequent isomorphisms follow from this, and the last isomorphism is obtained by passing to the direct limit.

We shall prove in Theorem 3.1.11 the Bott periodicity theorem for unitary groups, which says that if $i \geq 1$, then $\pi_{i-1}(U(n)) \cong \pi_{i+1}(U(n))$. Thus

$$\pi_0(U) = \pi_0(U(1)) = 0, \quad \pi_1(U) = \pi_1(U(1)) \cong \mathbb{Z}, \text{ and}$$

$$\pi_0(U) \cong \pi_2(U) \cong \pi_4(U) \cong \cdots = 0,$$
$$\pi_1(U) \cong \pi_3(U) \cong \pi_5(U) \cong \cdots = \mathbb{Z}.$$

These are called the stable homotopy groups of U. They are periodic of period 2. Then we have the theorem

Theorem 1.3.4.

$$\widetilde{K}(S^n) = \mathbb{Z}, \text{ if } n \text{ is } 0 \text{ or even},$$
$$= 0, \text{ if } n \text{ is odd}.$$

We shall explore the theorem still further in the next section.

1.4. Clutching construction

The clutching construction described in the proof of Lemma 1.1.2 can be generalized. Let X_1 and X_2 be compact spaces. and $X = X_1 \cup X_2$, $A = X_1 \cap X_2$. Let $\pi_i : E_i \longrightarrow X_i$, $i = 1, 2$, be bundles, and $\alpha : E_1|A \longrightarrow E_2|A$ a bundle isomorphism over A. Then there is a bundle $\pi : E \longrightarrow X$, where E is the quotient of the disjoint union $E_1 \cup E_2$ by the identification $v \equiv \alpha(v)$ for $v \in E_1|A$, and π is the natural projection obtained from π_1 and π_2. We write $E = E_1 \cup_\alpha E_2$. We call (E_i, α) a clutching data on X_i. and α a clutching function.

The local triviality of E may be seen as follows. This is clear at a point $x \in X - A$. Therefore, let $a \in A$. Let V_1 be a closed neighbourhood of a in X_1 so that $E_1|V_1$ is trivial by an isomorphism $\phi_1 : E_1|V_1 \to V_1 \times \mathbb{C}^n$. We have then an isomorphism by restriction $\phi_1' : E_1|V_1 \cap A \to (V_1 \cap A) \times \mathbb{C}^n$, and an isomorphism $\phi_2' = \phi_1' \circ \alpha^{-1} : E_2|V_1 \cap A \to (V_1 \cap A) \times \mathbb{C}^n$. Extend ϕ_2' to an isomorphism $\phi_2 : E_2|V_2 \to V_2 \times \mathbb{C}^n$, where V_2 is a neighbourhood of $V_1 \cap A$ in X_2, by Lemma 1.1.1(b). Then ϕ_1 and ϕ_2 give an isomorphism $\phi_1 \cup_\alpha \phi_2 : E_1 \cup_\alpha E_2|V_1 \cup V_2 \to (V_1 \cup V_2) \times \mathbb{C}^n$ in an obvious way. Thus E is locally trivial.

Next, the isomorphism class of $E_1 \cup_\alpha E_2$ depends only on the homotopy class of the isomorphism $\alpha : E_1|A \to E_2|A$. To see this, consider a homotopy of isomorphisms $E_1|A \to E_2|A$, that is, an isomorphism

$$\Phi : \pi^* E_1|A \times I \to \pi^* E_2|A \times I,$$

where $\pi : X \times I \to X$ is the projection, $I = [0,1]$. Let $f_t : X \to X \times I$ be the map $f_t(x) = x \times \{t\}$, and $\phi_t : E_1|A \to E_2|A$ be the isomorphism which is the pull-back of Φ by $f_t|A$. Then

$$E_1 \cup_{\phi_t} E_2 \cong f_t^*(\pi^* E_1 \cup_\Phi \pi^* E_2).$$

Since f_0 and f_1 are homotopic, it follows from Lemma 1.1.1(c) that

$$E_1 \cup_{\phi_0} E_2 \cong E_1 \cup_{\phi_1} E_2.$$

This proves the assertion.

The following facts may also be proved easily.

(1) If (E_i, α) and (E_i', α') are clutching data on X_i, and $\phi_i : E_i \to E_i'$ are isomorphisms, $i = 1, 2$, such that $\alpha' \phi_1 = \phi_2 \alpha$, then

$$E_1 \cup_\alpha E_2 \cong E_1' \cup_{\alpha'} E_2'.$$

(2) If (E_i, α) and (E_i', α') are clutching data on X_i, $i = 1, 2$, then

$$(E_1 \cup_\alpha E_2) \oplus (E_1' \cup_{\alpha'} E_2') \cong (E_1 \oplus E_1') \cup_{\alpha \oplus \alpha'} (E_2 \oplus E_2'),$$
$$(E_1 \cup_\alpha E_2) \otimes (E_1' \cup_{\alpha'} E_2') \cong (E_1 \otimes E_1') \cup_{\alpha \otimes \alpha'} (E_2 \otimes E_2'),$$
$$(E_1 \cup_\alpha E_2)^* \cong E_1^* \cup_{(\alpha^*)^{-1}} E_2^*.$$

In particular, if $\pi : E \longrightarrow X$ is a bundle, and $\pi \times \mathrm{Id} : E \times S^2 \longrightarrow X \times S^2$ is the product bundle with $E_1 = (E \times D_+^2)|X \times D_+^2$ and $E_2 = (E \times D_-^2)|X \times D_-^2$, then a bundle isomorphism $\alpha : E \times S^1 \longrightarrow E \times S^1$ produces a bundle

$$E_1 \cup_\alpha E_2 \longrightarrow X \times S^2$$

by clutching construction. We denote this bundle by $[E, \alpha]$.

A homotopy of bundles equivalences $\alpha_t : E \times S^1 \longrightarrow E \times S^1$ gives an isomorphism $[E, \alpha_0] \cong [E, \alpha_1]$, because the bundle equivalence

$$E \times S^1 \times I \longrightarrow E \times S^1 \times I,$$

defined by $((x, y), t) \mapsto (\alpha_t(x, y), t))$, produces a bundle over $X \times S^2 \times I$ (by clutching construction) which restricts to $[E, \alpha_0]$ and $[E, \alpha_1]$ over $X \times S^2 \times \{0\}$ and $X \times S^2 \times \{1\}$ respectively.

Every bundle $E' \longrightarrow X \times S^2$ is isomorphic to $[E, \alpha]$ for some E and α. To see this, suppose that $E_+' = E'|(X \times D_+^2)$, $E_-' = E'|(X \times D_-^2)$, and $E = E'|(X \times \{1\}) \subset E_+' \cap E_-'$. The projection $p_\pm : X \times D_\pm^2 \longrightarrow X \times \{1\} \hookrightarrow X \times D_\pm^2$ is homotopic to the identity map on $X \times D_\pm^2$. Also $E_\pm'|X \times \{1\} = E$, and $p_\pm^* E = E \times D_\pm^2$. Therefore $E_\pm' \cong p_\pm^* E = E \times D_\pm^2$. If $\phi_\pm : E_\pm' \longrightarrow E \times D_\pm^2$ are the isomorphisms, let $\alpha = \phi_- \circ \phi_+^{-1}|E \times S^1 : E \times S^1 \to E \times S^1$. Clearly, $E' \cong [E, \alpha]$.

We shall apply the last construction for finding the clutching function α when E' is a line bundle over S^2 and X is a point so that E is the fibre of E' over $1 \in S^1$.

We denote the tautological line bundle γ_2^1 over $G_1(\mathbb{C}^2)$ by H^* (it is the dual of a bundle H called the Hopf bundle; we shall give a more specific description of the bundle H after Lemma 1.4.1). The bundle H^* may be obtained by a clutching function, as described in Lemma 1.1.2. The space $G_1(\mathbb{C}^2)$ is the complex projective space $P(\mathbb{C}^2)$, the space of all lines through 0 in $\mathbb{C}^2$. A line is an equivalence class of $\mathbb{C}^2 - \{0\}$ by the equivalence relation $(z_1, z_2) \sim \lambda(z_1, z_2)$ for $\lambda \in \mathbb{C} - \{0\}$. So a line $[z_1, z_2]$ may be represented by the ratio $z = z_1/z_2 \in \mathbb{C} \cup \{\infty\} \cong S^2$. The points of the lower hemisphere D^2_- (resp. the upper hemisphere D^2_+) of S^2 are represented uniquely by $z \in \mathbb{C}$ with $|z| \leq 1$ (resp. with $|z| \geq 1$), that is, by $[z, 1] \in P(\mathbb{C}^2)$ with $|z| \leq 1$ (resp. by $[1, z^{-1}] \in P(\mathbb{C}^2)$ with $|z^{-1}| \leq 1$). A section of the tautological line bundle H^* over D^2_- (resp. D^2_+) is given by $[z, 1] \mapsto (z, 1)$ (resp. $[1, z^{-1}] \mapsto (1, z^{-1})$). These sections may be glued together over S^1 by a clutching function $S^1 \longrightarrow \mathrm{GL}_1(\mathbb{C})$ which sends $z \in S^1$ to the linear transformation $\lambda \mapsto z\lambda$, $\lambda \in \mathbb{C}$ (multiplication by z). We shall denote this clutching function simply by z. Note that if we had interchange D^2_+ and D^2_-, then the clutching function would have been z^{-1}.

Lemma 1.4.1. *If $f, g : S^1 \longrightarrow GL_1(\mathbb{C})$ are clutching functions for line bundles E_f, E_g over $P(\mathbb{C}^2)$, then the bundles $E_f \oplus E_g$ and $E_f \otimes E_g \oplus \mathcal{E}^1$ over $P(\mathbb{C}^2)$ are isomorphic.*

PROOF. One can see that the clutching functions for the bundles $E_f \oplus E_g$, and $E_f \otimes E_g$ over $P(\mathbb{C}^2)$ are given respectively by the functions

$$f \oplus g : S^1 \longrightarrow GL_2(\mathbb{C}) \quad \text{and} \quad f \cdot g : S^1 \longrightarrow GL_1(\mathbb{C}),$$

where

$$f \oplus g(z) = \begin{pmatrix} f(z) & 0 \\ 0 & g(z) \end{pmatrix} \in GL_2(\mathbb{C}), \text{ and } (f \cdot g)(z) = f(z) \cdot g(z) \in GL_1(\mathbb{C}).$$

Also, the constant function $z \mapsto 1 \in GL_1(\mathbb{C})$ is a clutching function of the trivial line bundle $\mathcal{E}^1$ over $P(\mathbb{C}^2)$. The functions $f \oplus g$, $f \cdot g \oplus 1 : S^1 \longrightarrow GL_2(\mathbb{C})$ are homotopic, since $GL_2(\mathbb{C})$ is path connected. Explicitly, if $\sigma : [0, 1] \longrightarrow GL_2(\mathbb{C})$ is a path with

$$\sigma(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } \sigma(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the matrix product $(f \oplus 1) \cdot \sigma(t) \cdot (1 \oplus g) \cdot \sigma(t)$ is a homotopy from $f \oplus g$ to $f \cdot g \oplus 1$. This means $E_f \oplus E_g \cong E_f \otimes E_g \oplus \mathcal{E}^1$. □

The Hopf bundle H is given by $H^* = \mathrm{Hom}(H, \mathcal{E}^1)$, and we have $H \otimes H^* \cong \mathcal{E}^1$ (the isomorphism is given by $v_z \otimes \phi_z \mapsto \phi_z(v_z) \in \mathbb{C}$, where v_z and ϕ_z belong to the fibres of H and H^* over $z \in S^2$). Therefore the clutching function for H is z^{-1}, and we have by Lemma 1.4.1 the relation $H \oplus H = H \otimes H \oplus \mathcal{E}^1$, or $([H] - [1])^2 = 0$, in $K(S^2)$.

Referring to the clutching construction $[E, \alpha]$, if $X = \text{pt}$ and $E = \mathcal{E}^1$, then $[\mathcal{E}^1, z]$ is the tautological line bundle H^* over $S^2 = P(\mathbb{C}^2)$, and $[\mathcal{E}^1, z^{-m}]$ is the m-fold tensor product $H^m = H \otimes \cdots \otimes H$, $H^0 = \mathcal{E}^1$. The element $[H] - [H^0] \in \widetilde{K}(S^2)$ is called the Bott class, and is denoted by b. We have

Lemma 1.4.2. *The Bott class b in $\widetilde{K}(S^2)$ satisfies the relation $b^2 = 0$.*

It is clear from the above discussion that there is a ring homomorphism

$$\mathbb{Z}[H]/(H-1)^2 \to K(S^2)$$

whose domain is the quotient of the polynomial ring $\mathbb{Z}[H]$ by the ideal generated by $(H-1)^2$. An additive basis of the ring $\mathbb{Z}[H]/(H-1)^2$ is $\{1, H\}$. We shall prove in Corollary 3.1.6 that the above ring homomorphism is an isomorphism.

1.5. Relative K-theory and long exact sequence

Let $\mathcal{C}$ denote the category of compact spaces, $\mathcal{C}^+$ the category of compact spaces with basepoints, and $\mathcal{C}^2$ the category of pairs of compact spaces. We have a functor $\mathcal{C} \longrightarrow \mathcal{C}^2$ given by $X \mapsto (X, \emptyset)$, and a functor $\mathcal{C}^2 \longrightarrow \mathcal{C}^+$ given by $(X, A) \mapsto (X/A, *)$, where $* = A/A$. If $A = \emptyset$, then we define $X/\emptyset = X^+ = X \cup \{\text{pt}\}$, where $\{\text{pt}\}$ denotes a point not in X. Thus we have a functor $\mathcal{C} \to \mathcal{C}^+$ given by $X \mapsto X^+$. We identify $(X, \emptyset) = X$, and $X/x_0 = X$, where x_0 is a point in X.

The contravariant functor $\widetilde{K}$ on $\mathcal{C}^+$ may also be defined by

$$\widetilde{K}(X) = \ker[i^* : K(X) \longrightarrow K(x_0)],$$

where $i : x_0 \longrightarrow X$ is the inclusion map. If $j : X \longrightarrow x_0$ is the constant map, then $j \circ i = \text{Id}$, and so there is a splitting $K(X) \cong \widetilde{K}(X) \oplus K(x_0)$. Clearly, $\widetilde{K}(x_0) = 0$.

The definition of $\widetilde{K}(X)$ is equal to the previous definition $\widetilde{K}(X) = \text{Ker rk}$ (given in (1.3.1)), since $\text{rk} : K(x_0) \longrightarrow \mathbb{Z}$ is an isomorphism.

We define the contravariant functor K on $\mathcal{C}^2$ by

$$K(X, A) = \widetilde{K}(X/A).$$

In particular, $K(X) = K(X, \emptyset) = \widetilde{K}(X^+)$ if X is without base point, and $K(X, x_0) = \widetilde{K}(X)$ if $x_0 \in X$.

Lemma 1.5.1. *If $(X, A) \in \mathcal{C}^2$, $i : A \longrightarrow X$ is the inclusion map, and $j : X \longrightarrow X/A$ is the collapsing map, then the we have an exact sequence*

$$\widetilde{K}(X/A) \xrightarrow{j^*} \widetilde{K}(X) \xrightarrow{i^*} \widetilde{K}(A).$$

Moreover, if A is contractible, then j^* is an isomorphism.

PROOF. The map $i^* \circ j^* = 0$, since $j \circ i$ is a constant map, and therefore $\mathrm{Im}\, j^* \subset \mathrm{Ker}\, i^*$. To see the reverse inclusion, first note that a trivialization of an n-plane bundle $E \longrightarrow X$ over A is an isomorphism $\alpha : E|A \longrightarrow A \times \mathbb{C}^n$. This makes $E|A$ a trivial bundle, and gives a bundle E/α over X/A by collapsing all the fibres at points of A to the single fibre at $* \in X/A$. In fact, E/α is the quotient of E by an equivalence relation $\sim$, where, for $(x, v), (x', v') \in E$, $(x, v) \sim (x', v')$ if $x, x' \in A$ and $\alpha(v) = \alpha(v')$, or if $x = x' \in X - A$ and $v = v'$. The local triviality of E/α follows, because the trivialization α extends to a trivialization α' of E over a neighbourhood U of A in X, and α' induces a trivialization of E/α over U/A.

Now, take an element $\xi \in \mathrm{Ker}\, i^*$, and write $\xi = [E] - [\mathcal{E}^n]$ where E and $\mathcal{E}^n$ are bundles over X. Since $i^*\xi = 0$, we have $E|A \overset{s}{\sim} \mathcal{E}^n|A$ (stable equivalence). Therefore there is an isomorphism $E \oplus \mathcal{E}^m \longrightarrow \mathcal{E}^{n+m}$ over A, for some m. Composing this with the projection $\mathcal{E}^{n+m} \longrightarrow \mathbb{C}^{n+m}$, we get a trivialization $\alpha : E \oplus \mathcal{E}^m \longrightarrow \mathbb{C}^{n+m}$ over A, and hence a bundle $(E \oplus \mathcal{E}^m)/\alpha$ over X/A. Therefore we have an element $\eta = [(E \oplus \mathcal{E}^m)/\alpha] - [\mathcal{E}^{n+m}] \in \widetilde{K}(X/A)$, and $j^*(\eta) = [E \oplus \mathcal{E}^m] - [\mathcal{E}^{n+m}] = [E] - [\mathcal{E}^n] = \xi$. Thus $\mathrm{Ker}\, i^* \subset \mathrm{Im}\, j^*$, and the first part of the lemma is proved.

For the second part, note that j^* is an epimorphism, since A is contractible. Next, any two trivializations $\alpha_1, \alpha_2 : E|A \longrightarrow A \times \mathbb{C}^n$ of an n-plane bundle $E \longrightarrow X$ over A are homotopic over A, because they differ by a map $A \longrightarrow GL_n(\mathbb{C})$ which is homotopic to a constant map, A being contractible and $GL_n(\mathbb{C})$ connected. Therefore a trivialization $\alpha : E|A \to A \times \mathbb{C}^n$ is unique up to homotopy, and so the isomorphism class $[E/\alpha]$ is uniquely determined by $[E]$. This means that j^* is also a monomorphism. □

Corollary 1.5.2. *If $(X, A) \in \mathcal{C}^2$, then the following sequence is exact :*

$$K(X, A) \xrightarrow{j^*} K(X) \xrightarrow{i^*} K(A).$$

Here $i : A \longrightarrow X$ and $j : (X, \emptyset) \longrightarrow (X, A)$ are the inclusions.

Moreover, if A is a retract of X, then there is a splitting

$$K(X) \cong K(X, A) \oplus K(A).$$

PROOF. The exactness follows, because $K(X, A) = \widetilde{K}(X/A)$, and the inclusion map $i : A \longrightarrow X$ induces homomorphisms $K(X) \longrightarrow K(A)$ and $\widetilde{K}(X) \longrightarrow \widetilde{K}(A)$ with the same kernel. The second part follows, because the exact sequence splits, if A is a retract of X. □

In the next theorem, we shall extend this exact sequence to a long exact sequence.

Recall that the wedge product $X \vee Y$, and the smash product $X \wedge Y$ of two topological spaces X and Y with basepoints x_0 and y_0 respectively are defined