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TEXTS AND READINGS IN MATHEMATICS 6

Basic Ergodic Theory

Third Edition

M. G. Nadkarni

HINDUSTAN BOOK AGENCY

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Published by

Hindustan Book Agency (India)
P 19 Green Park Extension
New Delhi 110 016
India

email: info@hindbook.com
www.hindbook.com

ISBN 978-93-80250-43-4 ISBN 978-93-86279-53-8 (eBook)
DOI 10.1007/978-93-86279-53-8

Preface

This book treats mainly some basic topics of ergodic theory in a revised form, bringing into focus its interactions with classical descriptive set theory more than is normally the practice. The presentation has a slow pace and can be read by anyone with a background in measure theory and point set topology. In particular, the first two chapters, the core of ergodic theory, can form a course of four to six lectures at third year B.Sc., M.Sc., or M.Phil. level in Indian Universities. I have borrowed freely from existing texts (with acknowledgements) but the overall theme of the book falls in the complement of these.

G. W. Mackey has emphasised the need to look at group actions also from a purely descriptive standpoint. This helps clarify ideas and leads to sharper theorems even for the case of a single transformation. With this in view, basic topics of ergodic theory such as the Poincaré recurrence lemma, induced automorphisms and Kakutani towers, compressibility and Hopf's theorem, the Ambrose representation of flows etc. are treated at the descriptive level before appearing in their measure theoretic or topological versions. In addition, topics centering around the Glimm-Effros theorem are discussed. These topics have so far not found a place in texts on ergodic theory. Dye's theorem, proved at the measure theoretic level in Chapter 11, when combined with some descriptive results of earlier chapters, becomes a very neat theorem of descriptive set theory.

A more advanced treatment of these topics is so far available only in the form of unpublished "Lectures on Definable Group Actions and Equivalence Relations", by A. Kechris (California Institute of Technology, Pasadena).

Professor Henry Helson has kindly edited the entire manuscript and suggested a number of corrections, greatly improving the language and the exposition. I am deeply indebted to him for this and many other acts of encouragement over the past several years.

It is a pleasure to acknowledge the consideration shown and help given by Dr. Mehroo Bengalee. She made the sabbatical leave available for this project

during her tenure as the Vice Chancellor of University of Bombay. Finally, my sincere thanks go to V.Nandagopal for making his expertise with computers available in the preparation of this book.

M. G. Nadkarni

Preface to the Second Edition

In this edition a section on rank one automorphisms has been added to Chapter 7 and a brief discussion on the ergodic theorem due to Wiener and Wintner appears in Chapter 2. Typographical and other errors that were noticed or were brought to my notice have been corrected and the language has been changed in some places. The unpublished lectures of A. Kechris mentioned in the preface to the first edition have since appeared as "The Descriptive Set Theory of Polish Group Actions", H. Baker and A. Kechris, London Math. Soc. Lecture Note Series, 232, Cambridge University Press.

M. G. Nadkarni

Preface to the Third Edition

In this edition a chapter entitled 'Additional Topics' has been added. It gives Liouville's Theorem on the existence of invariant measure, entropy theory leading up to the Kolmogorov-Sinai Theorem, and the topological dynamics proof of van der Waerden's theorem on arithmetical progressions. It is a pleasure to acknowledge the help given by B. V. Rao and Joseph Mathew in this. These new topics are within the reach of interested undergraduates and beginning graduate students. Ankush Goswami pointed out some mathematical and typographical errors in the earlier edition. These and some other errors which were noticed have been corrected. I hope the new edition will be found useful.

I am much indebted to D. K. Jain of Hindustan Book Agency for suggesting a new edition of this book, and for monitoring its progress through timely emails and encouraging telephone calls. My sincere thanks also go to Vijesh Antony for quickly resolving my difficulties with the computer whenever I sought his help. Finally, thanks are due to Indian Institute of Technology, Indore, for visiting appointments during which this edition was prepared.

M. G. Nadkarni

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Borel Spaces

1.1. Let X be a non-empty set. A σ-algebra $\mathcal{B}$ on X is a non-empty collection of subsets of X which is closed under countable unions and complements. A set together with a σ-algebra $\mathcal{B}$ is called a Borel space or a Borel structure $(X, \mathcal{B})$.

1.2. The intersection of any family of σ-algebras on X is again a σ-algebra. If $\mathcal{A}$ is any collection of subsets of X, then the intersection of all the σ-algebras on X which contain the collection is again a σ-algebra. It is the smallest σ-algebra containing $\mathcal{A}$. It is called the σ-algebra generated by $\mathcal{A}$.

1.3. If $(X, \mathcal{B})$ is a Borel space then a subcollection $\mathcal{N} \subseteq \mathcal{B}$ is called a σ-ideal if

1. $\mathcal{N}$ is closed under countable unions
2. $B \in \mathcal{B}$ and $N \in \mathcal{N}$ implies that $B \cap N \in \mathcal{N}$.

For example if m is a countably additive measure on $\mathcal{B}$, then the collection of sets in $\mathcal{B}$ of m measure zero forms a σ-ideal. We will come across other σ-ideals later. If $\mathcal{E} \subseteq \mathcal{B}$ is any collection then there is a smallest σ-ideal containing $\mathcal{E}$, namely, the intersection of all the σ-ideals containing $\mathcal{E}$. We call it the σ-ideal generated by $\mathcal{E}$. It is formed by taking all sets of the form $B \cap E$, $B \in \mathcal{B}$, $E \in \mathcal{E}$ and taking countable unions of such sets. If $\mathcal{N} \subseteq \mathcal{B}$ is a σ-ideal and A, B belong to $\mathcal{B}$, then we write $A = B(\text{mod } \mathcal{N})$ if $A \triangle B = (A - B) \cup (B - A) \in \mathcal{N}$.

1.4. An interesting σ-algebra on a complete separable metric space X is the σ-algebra of sets with the property of Baire. A set A is said to have the property of Baire if A can be expressed in the form $A = G \triangle P$ where G is open and P is of first category.

1.5. Theorem. *A set A in a complete separable metric space X has the property of Baire if and only if it can be expressed in the form $A = F \triangle Q$ where F is closed and Q is of the first category.*

Proof. If $A = G \triangle P$, G open, P of first category, then $N = \overline{G} - G$ is a nowhere dense closed set, and $Q = N \triangle P$ is of first category. Let $F = \overline{G}$. Then $A = G \triangle P = (\overline{G} \triangle N) \triangle P = \overline{G} \triangle (N \triangle P) = F \triangle Q$. Conversely if $A = F \triangle Q$, where F is closed and Q is of first category, let G be the interior of F. Then $N = F - G$ is nowhere dense, $P = N \triangle Q$ is of first category, and $A = F \triangle Q = (G \triangle N) \triangle Q = G \triangle (N \triangle Q) = G \triangle P$. This proves the theorem.

1.6. Corollary. *If A has the property of Baire then so does its complement.*
Proof. For any two sets A and B, $(A \triangle B)^c = A^c \triangle B$. Hence if $A = G \triangle P$, G open, P of first category, then $A^c = G^c \triangle P$ which again has the property of Baire by the above theorem.

1.7. Theorem. *The class of sets having the property of Baire is a σ-algebra. It is the σ-algebra generated by open sets together with the sets of first category.*

Proof. Let $A_i = G_i \triangle P_i$, $(i = 1, 2, 3, ...)$ be any sequence of sets having the property of Baire. Put $G = \cup G_i$, $P = \cup P_i$, $A = \cup A_i$. Then G is open, P is of first category, and $G - P \subseteq A \subseteq G \cup P$. Hence $G \triangle A \subseteq P$ is of first category, and $A = G \triangle (G \triangle A)$ has the property of Baire. This result and the corollary above show that the class in question is a σ-algebra. It is evidently the smallest σ-algebra that includes all open sets and all sets of first category. This proves the theorem.

Note that the first category sets form a σ-ideal in the σ-algebra of sets with the property of Baire.

1.8. Two Borel spaces $(X_1, \mathcal{B}_1)$, $(X_2, \mathcal{B}_2)$ are said to be isomorphic if there a one-one map ϕ of X_1 onto X_2 such that $\phi(\mathcal{B}_1) = \mathcal{B}_2$. The map ϕ is called a Borel isomorphism between the two Borel spaces.

1.9. If $(X, \mathcal{B})$ is Borel structure and A is a non-empty subset of X, then the collection of sets of the form $A \cap B$ with $B \in \mathcal{B}$ is a σ-algebra on A called the induced σ-algebra on A and denoted by $A \cap \mathcal{B}$ or $\mathcal{B} \mid_A$. Two sets A, $B \subseteq X$ are said to be Borel isomorphic if there is a one-one map ϕ of A onto B such that $\phi(\mathcal{B} \mid_A) = \mathcal{B} \mid_B$, i.e., the Borel structures $(A, \mathcal{B} \mid_A)$, $(B, \mathcal{B} \mid_B)$ are isomorphic.

Standard Borel Spaces

1.10. Let X be a complete separable metric space and $\mathcal{B}_X$ the σ-algebra generated by the collection of open sets in X. $\mathcal{B}_X$ is called the Borel σ-

algebra of X. The following results are known from descriptive set theory. (see K.R.Parthasarathy [5])

1. A set in $\mathcal{B}_X$ is either countable or has the cardinality c of the continuum.
2. If A and B in $\mathcal{B}_X$ are of the same cardinality, then A and B are Borel isomorphic.
3. If Y is another complete separable metric space of the same cardinality as X and $\mathcal{B}_Y$ its Borel σ-algebra, then $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ are Borel isomorphic.
4. It follows from (1), (2) and (3) that if $A \in \mathcal{B}_X$ and $B \in \mathcal{B}_Y$ have the same cardinality, then the Borel spaces $(A, A \cap \mathcal{B}_X)$ and $(B, B \cap \mathcal{B}_Y)$ are Borel isomorphic.

1.11. A Borel space isomorphic to the Borel space of an uncountable complete separable metric space is called standard. Such a space is, in view of the results above, isomorphic to the Borel space of the unit interval equipped with the σ-algebra generated by its usual topology. If a standard Borel space is equipped with a finite or a σ-finite measure m then the resulting measure space is called a standard measure space. In particular if $m(X) = 1$ then such a measure space is called a standard probability space.

1.12. We know that the forward image of a measurable set under a measurable map need not be measurable, in general. However a theorem of Lusin in classical descriptive set theory states that if f is a measurable function on a standard Borel space into another such space and if f is countable to one in the sense that the inverse image of every singleton is at most countable, then the forward image under f of any Borel set is Borel. In particular if such an f is one-one and onto then it is a Borel isomorphism.

1.13. In a complete separable metric space every Borel set has the property of Baire since the σ-algebra of sets with the property of Baire includes the Borel σ-algebra. It should be noted that if X is a complete separable metric space and if A is a Borel set in X, then A can be expressed as $G \triangle P$ where G is open and P is not only of first category but also a Borel set in X. This is because the class of sets of the form $G \triangle P$, G open, P a Borel set of first category forms a σ-algebra which coincides with the Borel σ-algebra of X.

Borel Automorphisms

1.14. A one-one measurable map τ of a Borel space $(X, \mathcal{B})$ onto itself such that τ^{-1} is also measurable is called a Borel automorphism of X. If $(X, \mathcal{B})$ is a standard Borel space then a measurable one-one map of X onto X is a Borel automorphism in view of **1.12**. In what follows, we will carry out an elementary analysis of a Borel automorphism on a standard Borel space.

1.15. Let τ be a Borel automorphism of the unit interval $X = [0, 1]$ equipped with its Borel σ-algebra. For any $x \in X$, the set $\{\tau^n x \mid n \in \mathbb{Z}\}$ is called the orbit of x under τ and denoted by orb (x, τ). A point $x \in X$ is said to be periodic if there is an integer n such that $\tau^n x = x$, and the smallest such positive integer is called the period of x under τ. If n is the period of x under τ, then the set $x, \tau x, \tau^2 x, ..., \tau^{n-1} x$ consists of distinct points of $[0, 1]$. Let

$$\begin{aligned} E_1 &= \{x \mid \tau x = x\} \\ E_2 &= \{x \mid \tau x \neq x,\ \tau^2 x = x\} \\ &\vdots \\ E_\infty &= \{x \mid \tau^n x \neq x \text{ for all integers } n\} \end{aligned}$$

The set E_n for $n < \infty$ is made up of precisely those points in X which have period n. Each E_n is in $\mathcal{B}$, $E_m \cap E_n = \emptyset$ if $m \neq n$, and the union of all the E_i, $i = \infty, 1, 2, 3, \ldots$ is X. A set in X is said to be invariant under τ or τ-invariant if $\tau A = A$. It is clear that all the sets E_n are τ-invariant.

1.16. Let us consider E_n for $n < \infty$. If x is in E_n, then $x, \tau x, \ldots \tau^{n-1} x$ are all distinct and if $y \in \{x, \tau x, \ldots, \tau^{n-1} x\}$ then $\{x, \tau x, \ldots, \tau^{n-1} x\} = \{y, \tau y, \ldots, \tau^{n-1} y\}$. If further $y = \min\{x, \tau x, \ldots, \tau^{n-1} x\}$, then $y < \tau y$, $y < \tau^2 y, \ldots, y < \tau^{n-1} y$, $\tau^n y = y$. We put $B_n = \{y \in E_n \mid y < \tau y, y < \tau^2 y, \ldots, y < \tau^{n-1} y\}$. B_n is a measurable subset of E_n which contains exactly one point of the orbit of each x in E_n. We may view the restriction of τ to E_n pictorially as in Figure 1.1.

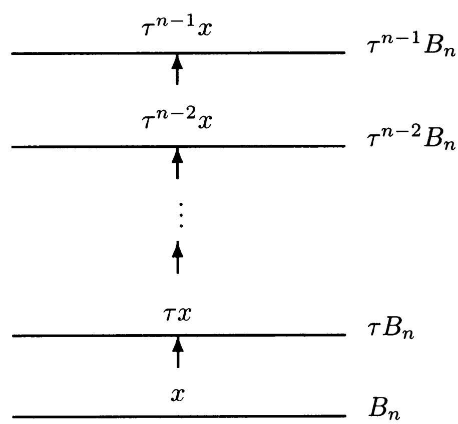

Figure 1.1

E_n is viewed as the union of n horizontal lines $B_n, \tau B_n, \ldots, \tau^{n-1}B_n$. A point $x \in B_n$ moves one step up with each application of τ until it reaches $\tau^{n-1}x \in \tau^{n-1}B_n$. One more application of τ brings it back to x. Now,

$$\begin{aligned} X - E_\infty &= \cup E_n \text{ where the union is over } n < \infty \\ &= \cup_{n=1}^{\infty} \cup_{k=0}^{n-1} \tau^k B_n \end{aligned}$$

and we may view τ on $X - E_\infty$ pictorially as in Figure 1.2.

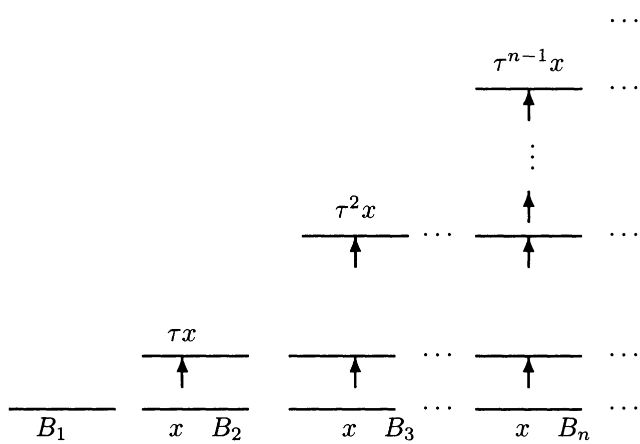

Figure 1.2

As before, a point $x \in B_n$ moves one step up with each application of τ with $\tau^n(x) = x$. The set $B = \bigcup_{k=1}^{\infty} B_k$ is a Borel set and has the property that orbit of any point in $X - E_\infty$ intersects B in exactly one point.

1.17. Let us now consider τ on E_∞. In this case there is, in general, no neat way in which we can find a measurable set B_∞ which intersects the orbit of each x in exactly one point. The set $\{\tau^n x \mid n \in \mathbb{Z}\}$ being infinite in this case, we can no longer conclude that $\inf\{\tau^n x \mid n \in \mathbb{Z}\}$ is in the set $\{\tau^n x \mid n \in \mathbb{Z}\}$. Indeed it can happen that $\inf\{\tau^n x \mid n \in \mathbb{Z}\} = 0$ for all $x \in E_\infty$. (For example, the orbit of every point in E_∞ may be dense in $[0,1]$.) We can use the axiom of choice to select one point from each orbit and thus form a set B_∞ which intersects the orbit of each $x \in E_\infty$ in exactly one point. But such a B_∞ may not be measurable, and we are not interested in sets which are not measurable. We give below two examples. In the first example $X = E_\infty$ and there is a B_∞ which is measurable. In the second example, the so called irrational rotation of the circle, $E_\infty = X$ and there is no B_∞ which is measurable.

1.18. Example 1. $X = \mathbb{R}$ and $\tau x = x+1$. In this case the Borel set $B_\infty = [0,1)$ has the property that the orbit of every point in $\mathbb{R}$ intersects $[0,1)$ in exactly one point. Moreover, $\tau^n x \neq x$ for any n and the union of $\tau^n[0,1)$ over n in $\mathbb{Z}$ is $\mathbb{R}$.

1.19. Example 2. $X =$ the unit circle $= \{e^{i\vartheta} \mid 0 \leq \vartheta < 2\pi\}$. Let α be an irrational number and $\beta = e^{2\pi i\alpha}$. Define τ by setting $\tau e^{i\vartheta} = \beta e^{i\vartheta}$. Now $\tau^n e^{i\vartheta} = \beta^n e^{i\vartheta}$ cannot be equal to $e^{i\vartheta}$ for any $n \neq 0$, for if $\tau^n e^{i\vartheta} = e^{i\vartheta}$ for some integer $n \neq 0$, then $\beta^n = e^{2\pi i\alpha n} = 1$, i.e., $n\alpha$ is an integer which contradicts the irrationality of α. Thus τ has no periodic points. Next we show that τ admits no measurable B_∞. Suppose $B_\infty \subseteq X$ is a set which intersects every orbit in exactly one point. Then $\tau^n B_\infty$, $n \in \mathbb{Z}$, are pairwise disjoint with union X. Let ℓ denote the Lebesgue measure on X. If B_∞ were measurable then $\ell(\tau^n B_\infty) = \ell(B_\infty)$ for all n, in view of the invariance of ℓ under rotation. Since $\tau^n B_\infty$, $n \in \mathbb{Z}$, are pairwise disjoint with union X we have $\ell(X) = 2\pi = \sum \ell(\tau^n B_\infty)$, which is a contradiction because $\ell(\tau^n B_\infty)$ are all the same $= \ell(B_\infty)$. Thus this τ admits no measurable B_∞ which intersects each orbit in exactly one point.

1.20. Let us return to the consideration of a general Borel automorphism τ on a standard Borel space X which we may assume to be $[0,1]$ without loss of generality. Let $c_n(\tau)$ denote the cardinality of the set of orbits of points in $E_n =$ cardinality of B_n, $n < \infty$. Let $c_\infty(\tau)$ denote the cardinality of the set of orbits of points in E_∞. The sequence of integers $\{c_\infty(\tau), c_1(\tau), c_2(\tau), \ldots\}$ is called the cardinality sequence associated to τ.

1.21. Definition. A Borel automorphism τ is said to be an elementary Borel automorphism if there exists a measurable set B_∞ which intersects the orbit of each point in E_∞ in exactly one point, equivalently, τ is elementary if and only if there exists a set B which is measurable and intersects each orbit in exactly one point. The equivalence of the two formulations above is obvious because if a measurable B_∞ exists, then we can take $B = \bigcup B_k$, where the union is taken over $1 \leq k \leq \infty$. On the other hand if B as postulated in the definition above exists, then we can take $B_\infty = B - \bigcup E_k$, where the union is taken over $1 \leq k < \infty$.

Orbit Equivalence and Isomorphism

1.22. Definition. Two Borel automorphisms τ_1 and τ_2 on Borel spaces $(X_1, \mathcal{B}_1)$, $(X_2, \mathcal{B}_2)$ respectively are said to be isomorphic if there is a Borel isomorphism $\phi : X_1 \to X_2$ such that $\phi\tau_1\phi^{-1} = \tau_2$.

Definition. We say that τ_1 and τ_2 are weakly equivalent or orbit equivalent if there is a Borel isomorphism $\phi : X_1 \to X_2$ such that for all $x \in X_1$, $\phi(\text{orb } (x, \tau_1)) = \text{orb } (\phi(x), \tau_2)$.

It is clear that if two Borel automorphisms are isomorphic then they are also orbit equivalent. However the converse is not true in general as will emerge in the sequel.

1.23. Exercise 1. If τ_1 and τ_2 are orbit equivalent then the associated cardinality sequences (see **1.20.**) are the same.

Exercise 2. If τ_1 and τ_2 are elementary and the associated cardinality sequences are the same, then τ_1 and τ_2 are isomorphic, hence also orbit equivalent.

1.24. Given a Borel automorphism τ on a standard Borel space $(X, \mathcal{B})$ we say that the orbit space of τ admits a Borel cross-section (or simply that τ admits a Borel cross-section) if there is a Borel set B which intersects each orbit in exactly one point. Clearly, the statements "τ is elementary" and "τ admits a Borel cross-section" are equivalent. In contrast to exercise 2 above, the question "when are two non-elementary Borel automorphisms isomorphic?" has so far not found a simple answer although there are some deep theorems in ergodic theory dealing with this question in a measure theoretic setting. The question "when are two Borel automorphisms orbit equivalent?" has now a complete solution and will be discussed in chapter 11.

Poincare Recurrence Lemma

1.25. The study of Borel automorphisms which do not admit Borel cross-sections is intimately connected with both topological and measure theoretic ergodic theory. However some of the basic concepts such as recurrence, induced automorphisms, dissipative and conservative automorphisms etc., are in essence set theoretic in nature and can be explained without any reference to measure or topology. One of the very first and very basic results is the Poincaré recurrence lemma which we give below in a rather distilled form (see J.C.Oxtoby [6]).

1.26. Definition. A measurable set W is said to be wandering with respect to a Borel automorphism τ if $\tau^n W$, $n \in \mathbb{Z}$, are pairwise disjoint. The σ-ideal generated by all wandering sets in $\mathcal{B}$ will be denoted by $\mathcal{W}_\tau$ and called the Shelah-Weiss ideal of τ.

It is clear that if W is a wandering set then W intersects the orbit of any point in at most one point. Moreover W never intersects the orbit of a periodic point. If a measurable B_∞ (see **1.17**) exists then it is a wandering set. A subset A of orb (x, τ) is called bounded below (bounded above) if the set of integers n such that $\tau^n x \in A$ is bounded below (bounded above). Bounded and unbounded subsets of orb (x, τ) are defined similarly. If $A \subseteq$ orb (x, τ) is bounded below then there is a smallest integer n such that $\tau^n x \in A$; we call such $\tau^n x$ the smallest element of A. Similarly we define the largest element of A in case A is bounded above. A very useful sufficient condition for a set $N \in \mathcal{B}$ to belong to $\mathcal{W}_\tau$ is that for all $x \in X$, orb $(x, \tau) \cap N$ is either bounded below or above. For then, firstly, we can prescribe a well defined procedure for choosing a point from each orbit which has a non-empty intersection with N : choose the least element of orb $(x, \tau) \cap N$ if there is one, otherwise choose the largest element; secondly, the set W of points thus chosen from each orbit having non-empty intersection with N is a wandering Borel set whose powers under τ cover N so that N belongs to the Shelah-Weiss ideal of τ. For any set A in $\mathcal{B}$ the set M of points x in A such that orb $(x, \tau) \cap A$ is bounded below or above belongs to the Shelah-Weiss ideal of τ and for any $x \in A - M$, the points $\tau^n x$ return to $A - M$ for infinitely many positive and infinitely many negative n. This is, in short, the Poincaré recurrence lemma which we formally prove below.(The Borel nature of M needs to be proved).

1.27. Poincaré Recurrence Lemma. *Let τ be a Borel automorphism of a standard Borel space $(X, \mathcal{B})$. Then given $A \in \mathcal{B}$ there exists $N \in \mathcal{W}_\tau$ such that for each $x \in A - N$ the points $\tau^n x$ return to A for infinitely many positive n and also for infinitely many negative n.*

Proof. Consider

$$\begin{aligned} W &= \{x \in A \mid \tau^k x \notin A \text{ for all } k \geq 1\} \\ &= A - \bigcup_{k=1}^{\infty} \tau^{-k} A \\ W_1 &= \{x \in A \mid \tau^k x \notin A \text{ for all } k \leq -1\} \\ &= A - \bigcup_{k=1}^{\infty} \tau^{k} A \end{aligned}$$

Now W is a wandering set for if $\tau^k W \cap \tau^l W \neq \emptyset$, for some $k < l$, then $W \cap \tau^{l-k} W \neq \emptyset$, so that there is an $x \in W \subseteq A$ such that $\tau^{l-k} x \in W \subseteq A$, which is a contradiction since $l - k$ is positive. Similarly we can prove that W_1 is a wandering set. The set $\bigcup_{-\infty}^{\infty} \tau^k (W \cup W_1) = N$ (say) thus belongs to $\mathcal{W}_\tau$. We show that if $x \in A - N$, then $\tau^n x \in A$ for infinitely many positive n and also for infinitely many negative n. Indeed if $\tau^n x$ does not return to A for infinitely many positive n, then there is a largest positive integer $m = m(x)$ such that $\tau^m x \in A$. Clearly then $\tau^m x \in W$, so that $x \in \tau^{-m} W \subseteq N$ which is a contradiction. So $\tau^n x$ returns to A for infinitely many positive n. Similarly $\tau^n x$ returns to A for infinitely many negative n. This proves the lemma.

Remark 1. It is to be noted that if $x \in A - N$ then $\tau^k x$ in fact returns to $A - N$ for infinitely many positive and infinitely many negative n because N is invariant under τ and $x \notin N$.