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TEXTS AND READINGS IN MATHEMATICS 35

Mathematical Foundations of Quantum Mechanics

K. R. Parthasarathy

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Preface

This is a brief introduction to the mathematical foundations of quantum mechanics based on lectures given by the author to Ph.D.students at the Delhi Centre of the Indian Statistical Institute during the years 1980-1985 in order to initiate active research in the emerging field of quantum probability. The material in the first chapter is included in the author's book "An Introduction to Quantum Stochastic Calculus" published by Birkhauser Verlag in 1992 and the permission of the publishers to reprint it here is acknowledged. Apart from quantum probability, an understanding of the role of group representations in the development of quantum mechanics is always a fascinating theme for mathematicians. In this context the books by G.W.Mackey [**8**] and V.S.Varadarajan [**11**] have exerted a considerable influence in my pedagogical approach to the subject. Graduate students to whom I had recommended these books felt somewhat uncomfortable with them and I hope these notes would prove to be more encouraging in getting a quicker introduction to this theme.

The first chapter deals with the definitions of states, observables and automorphisms of a quantum system through Gleason's theorem, Hahn-Hellinger theorem and Wigner's theorem. Mackey's imprimitivity theorem and the theorem of inducing representations of groups in stages are proved directly for projective unitary antiunitary representations in the second chapter. Based on a discussion of multipliers on locally compact groups in the third chapter all the well-known observables of classical quantum theory like linear momenta, orbital and spin angular momenta, kinetic and potential energies, gauge operators etc., are derived solely from Galilean covariance in the last chapter. A very short account of observables concerning a relativistic free particle is included. In conclusion, the spectral theory of Schrödinger operators of one and two electron atoms is discussed in some detail.

I have benefited greatly from discussions with R.Bhatia, M.Krishna, P.L.Muthuramalingam and K.B.Sinha in writing these notes. The first three participated actively in the preparation of an earlier version and correcting innumerable mistakes. To all of them I express my sincere thanks and I take the responsibility for all the surviving errors. Finally, I thank M.Krishna for revising the notes, organising the TEXing and making it possible to bring out this edition in the form of a lecture note volume.

I am grateful to Mr.V.P.Sharma for typing the original lecture notes and Mrs.T.S.Bagya Lakshmi for typing a part of the current version. Generous support from the Institute of Mathematical Sciences, Chennai in the completion of the revised version during the period 19 June - 31 July 2005 is gratefully acknowledged.

K.R.Parthasarathy
August 2005, New Delhi

CHAPTER 1

PROBABILITY THEORY ON THE LATTICE OF PROJECTIONS IN A HILBERT SPACE

1.1. Gleason's theorem

In classical probability theory one assumes that all the events concerning a statistical experiment constitute a Boolean σ-algebra and defines a probability measure as a completely additive non-negative function which assigns the value unity for the identity element of the σ-algebra. Invariably, the σ-algebra is the Borel σ-algebra $\mathcal{B}_X$ (i.e., the smallest σ-algebra generated by the open subsets) of a nice topological space X. Under very general conditions it turns out that all probability measures on $\mathcal{B}_X$ constitute a convex set whose extreme points are degenerate probability measures.

We observe that $\mathcal{B}_X$ admits a null element, namely $\emptyset$, a unit element, namely X, a partial order $\subseteq$ and operations union ($\cup$), intersection ($\cap$) and complementation ($'$). We shall, in this chapter, develop an analogous probability theory by replacing the σ-algebra $\mathcal{B}_X$ of events by a lattice of projections on a Hilbert space $\mathcal{H}$. Most of the computations of quantum mechanics is done in such a lattice.

Let $\mathcal{H}$ be a real or complex separable Hilbert Space and let $\mathcal{P}(H)$ denote the set of all orthogonal projection operators on $\mathcal{H}$, where 0 denotes the zero projection and I denotes the identity operator. If $P_1, P_2 \in \mathcal{P}(\mathcal{H})$ we say that $P_1 \leq P_2$ if the range of P_1 is contained in the range of P_2. Then $\leq$ makes $\mathcal{P}(H)$ a partially ordered set. For any operator A on $\mathcal{H}$ let $R(A)$ denote its range. For $\{P_\alpha, \alpha \in T\} \subset \mathcal{P}(H)$, let $\bigvee_{\alpha \in T} P_\alpha$ be the orthogonal projection on the smallest closed linear span of all the subspaces $R(P_\alpha), \alpha \in T$. Let $\bigwedge_{\alpha \in T} P_\alpha$ be the orthogonal projection on $\bigcap_{\alpha \in T} R(P_\alpha)$. For any $P \in \mathcal{P}(H), I - P$ is the orthogonal projection on the orthogonal complement $R(P)^\perp$ of the range of P. We may compare $0, I, \leq \vee, \wedge$ and the map $P \to I - P$ in $\mathcal{P}(H)$ with $\emptyset, X, \subset, \cup, \cap$ and complementation $'\prime'$ of standard set theory in the space X. The chief distinction lies in the fact that $\cup$ distributes with $\cap$ but $\vee$ need not distribute with $\wedge$. For example in the real Hilbert space $\mathbb{R}^2$, let P_1, P_2, P_3 be the projections on the one dimensional subspaces S_1, S_2, S_3 respectively. See Figure 1. We have $S_1 \cap S_2 = S_1 \cap S_3 = 0, S_2 + S_3 = \mathbb{R}^2$. Hence $(P_1 \wedge P_2) \vee (P_1 \wedge P_3) = 0, P_1 \wedge (P_2 \vee P_3) = P_1$. This failure of distributivity in $\mathcal{P}(H)$ has very important consequences as we shall observe in the book.

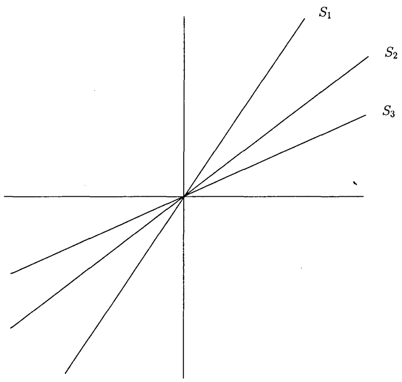

FIGURE 1

Definition 1.1.1. *A* **state** *is a map $\mu : \mathcal{P}(H) \to [0,1]$ satisfying the following properties:*

(i) $\mu(0) = 0, \mu(I) = 1$;

(ii) $\mu\left(\bigvee_{i=1}^{\infty} P_i\right) = \sum_{i=1}^{\infty} \mu(P_i)$ *whenever $P_iP_j = 0$ for every $i \neq j$.*

Example 1.1.2. *Let T be any non-negative compact self-adjoint operator on $\mathcal{H}$ of trace unity. Define*

$$\mu(P) = tr\ TP.$$

Then μ is a state on $\mathcal{P}(H)$.

Example 1.1.3. *Let $\mathcal{H} = \mathbb{R}^2$ and let $f(\theta), 0 \leq \theta \leq \frac{\pi}{2}$ be a function satisfying $0 \leq f(\theta) \leq 1$ for all θ. Define a state μ_f as follows: $\mu_f(0) = 0$, $\mu_f(I) = 1$; if l_θ is a line through the origin making the angle θ with the x axis and P_θ is the orthogonal projection on l_θ then*

$$\mu_f(P_\theta) = \begin{cases} f(\theta) & \text{if } 0 \leq \theta < \frac{\pi}{2}, \\ 1 - f(\theta) & \text{if } \frac{\pi}{2} \leq \theta < \pi. \end{cases}$$

Since every orthogonal projection is either 0,1 or 2 dimensional and the only 2 dimensional projection is I it is clear that μ *is a state.*

When $\dim \mathcal{H} \geq 3$ *the situation changes drastically and every state is determined by a non-negative self-adjoint operator of trace unity as in Example 1.1.2. This is precisely Gleason's theorem.*

Remark 1.1.4. *It may be noted that if* μ *and* ν *are two states and* $0 \leq p, q \leq 1, p+q=1$ *we can define a new state* $p\mu + q\nu$ *by the equation* $(p\mu+q\nu)(P) = p\mu(P) + q\nu(P)$ *for all* $P \in \mathcal{P}(\mathcal{H})$. *Then* $p\mu + q\nu$ *is called a* **mixture** *of* μ *and* ν. *Thus the set of all states on* $\mathcal{P}(H)$ *is a convex set. We may compare a state on* $\mathcal{P}(\mathcal{H})$ *with a probability measure on a* σ*-algebra.*

Definition 1.1.5. *A* **frame function** f *of* **weight** W *for a separable Hilbert space* $\mathcal{H}$ *is a complex-valued function defined on the unit sphere of* $\mathcal{H}$ *satisfying (i)* $f(x) = f(\lambda x)$ *for all scalars* λ *of modulus unity; (ii) for every complete orthonormal basis* $\{x_j, j = 1, 2, \cdots\}$ *the infinite series* $\sum_{j=1}^{\infty} f(x_j)$ *converges absolutely and* $\sum_{j=1}^{\infty} f(x_j) = W$. *A frame function* f *is said to be* **regular** *if there exists a bounded operator* T *such that* $f(x) = \langle Tx, x \rangle$ *where* $\langle ., . \rangle$ *denotes inner product.*

It may be noted that for every state μ *on* $\mathcal{P}(\mathcal{H})$ *we can define a frame function* f_μ *by putting* $f_\mu(x) = \mu(P_x)$ *where* P_x *is the orthogonal projection on the one dimensional subspace generated by the unit vector* x. *Conversely if* f *is any non-negative frame function of weight unity then we can define a state* μ_f *by the identity* $\mu_f(P) = \sum_{j=1}^{\infty} f(x_j)$ *where* $\{x_j, j = 1, 2, \cdots\}$ *is any complete orthonormal basis for the range of* P. *Thus, in order to prove Gleason's theorem we have to only show that every non-negative frame function of unit weight is regular.*

Lemma 1.1.6. *Every infinitely differentiable frame function in* $\mathbb{R}^3$ *is regular.*

PROOF. Let $\mathcal{F}$ denote the space of all frame functions in $\mathbb{R}^3$. We shall represent any point p on the unit sphere S^2 by its spherical polar coordinates $(\theta, \phi), 0 \leq \theta < \pi, 0 \leq \phi < 2\pi$. See Figure 2. Thus any function f on S^2 can be viewed upon as a function of θ and ϕ which is periodic in θ of period π and periodic in ϕ of period 2π. Let O_3 denote the group of all rotations in $\mathbb{R}^3$ about the origin 0. For any $g \in O_3$ let $g \circ (\theta, \phi)$ be the point obtained by applying g to the point (θ, ϕ).

Let now f be any infinitely differentiable frame function. If g_α^x and g_α^z denote rotations through an angle α about the x and z axis respectively, then

$$\begin{aligned} g_\alpha^z \circ (\theta, \phi) &= (\theta, \phi + \alpha) \mod (\pi, 2\pi) \\ g_\alpha^x \circ (\theta, \phi) &= (\theta', \phi') \end{aligned}$$

where (θ, ϕ) and (θ', ϕ') are related by the equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \sin\theta'\cos\phi' \\ \sin\theta'\sin\phi' \\ \cos\theta' \end{pmatrix}$$

For any rotation g it is clear that $f(g \circ (\theta, \phi))$ is also a frame function.

Further $\mathcal{F}$ is a linear space. Thus

$$f(g_\alpha^z \circ (\theta, \phi)) = f(\theta, \phi + \alpha) \in \mathcal{F} \text{ for all } \alpha, \tag{1.1.1}$$

$$\frac{d}{d\alpha} f(g_\alpha^z \circ (\theta, \phi))|_{\alpha=0} = \sin\phi \frac{\partial f}{\partial \theta} + \cot\theta \cos\phi \frac{\partial f}{\partial \phi} \in \mathcal{F}. \tag{1.1.2}$$

Hence for every integer n,

$$\int_0^{2\pi} f(\theta, \phi+\alpha) e^{-in\alpha} d\alpha = e^{in\phi} \int_0^{2\pi} f(\theta, \alpha) e^{-in\alpha} d\alpha = e^{in\phi} f_n(\theta), \tag{1.1.3}$$

say, is an element of $\mathcal{F}$ which is differentiable.

We shall now investigate frame functions of the form $F(\theta)e^{in\phi}$ where F is differentiable. Applying (1.1.2) we obtain

$$(F'(\theta)\sin\phi + in\ F(\theta)\cot\theta\cos\phi)e^{in\phi} \in \mathcal{F}. \tag{1.1.4}$$

Changing ϕ to $\phi + \frac{\pi}{2}$ in this relation we get

$$(F'(\theta)\cos\phi - in\ F(\theta)\cot\theta\sin\phi)e^{in\phi} \in \mathcal{F}. \tag{1.1.5}$$

Multiplying (1.1.4) by i and adding and subtracting from (1.1.5) we conclude

$$\begin{aligned} &F(\theta)e^{\text{in}\ \phi} \in \mathcal{F}, \\ &(F'(\theta) - nF(\theta)\cot\theta)e^{i(n+1)\phi} \in \mathcal{F}, \\ &(F'(\theta) + nF(\theta)\cot\theta)e^{i(n-1)\phi} \in \mathcal{F}. \end{aligned} \tag{1.1.6}$$

In the plane $\phi = 0$ we find that $F(\theta)$ and $nF(\theta)\cot\theta$ are frame functions. Hence there exist constants a and b such that

$$\begin{aligned} &F(\theta) + F\left(\theta + \frac{\pi}{2}\right) = a, \\ &F(\theta)\cot\theta + F\left(\theta + \frac{\pi}{2}\right)\cot\left(\theta + \frac{\pi}{2}\right) = b, \text{ if } n \neq 0. \end{aligned}$$

Solving for $F(\theta)$ we get

$$\left.\begin{aligned} F(\theta) &= a\sin^2\theta + b\sin\theta\cos\theta \\ F'(\theta) &= a\sin 2\theta + b\cos 2\theta \end{aligned}\right\} \text{if } n \neq 0 \tag{1.1.7}$$

Now consider a frame function of the form $F(\theta)e^{in\phi}, n \neq 0$ where F is given by (1.1.7) and the orthonormal basis $(\theta, \phi), (\theta+\frac{\pi}{2}, \phi), (\frac{\pi}{2}, \phi + \frac{\pi}{2})$. Then there exists a constant c such that

$$F(\theta)e^{in\phi} + F\left(\theta + \frac{\pi}{2}\right)e^{in\phi} + F\left(\frac{\pi}{2}\right)e^{in\pi/2}e^{in\phi} = c$$

for all θ, ϕ. Hence

$$c = 0, \;\; a(1 + i^n) = 0.$$

Applying the middle relation in (1.1.6) with F, F' from (1.1.7) and using the same orthonormal basis we obtain $b(n + i^{n+1}) = 0$.

In other words $b \neq 0$ implies that $n = \pm 1$ and $a = 0$. If $b = 0$, then $a\sin^2\theta e^{\text{i}n\phi} \in \mathcal{F}$ and the middle relation in (1.1.6) implies that $ae^{i(n+1)\phi}(2-n)\sin\theta\cos\theta \in \mathcal{F}$. Applying (1.1.6) again we conclude

$$\begin{array}{llll} a = 0 & \textit{if} & n = \pm 1 & \text{or} \quad |n| > 2, \\ b = 0 & \textit{if} & |n| \geq 2. & \end{array}$$

Now consider a frame function of the form $F(\theta)$. Then (1.1.6) implies that $F'(\theta)e^{i\phi}$ is a frame function. Hence

$$F'(\theta) = a\sin^2\theta + b\sin\theta\cos\theta \text{ for some } a, b.$$

Thus

$$F(\theta) = c_1 + c_2\left(\theta - \frac{1}{2}\sin 2\theta\right) + c_3\sin^2\theta$$

for some constants c_1, c_2, c_3. Since $c_1 + c_3\sin^2\theta$ is a frame function but $\theta - \frac{1}{2}\sin 2\theta$ is not, if follows that $c_2 = 0$. Thus the most general differentiable frame function is of the form

$$\begin{aligned} f(\theta, \phi) = a_1 + a_2\sin^2\theta \;&+\; a_3e^{i\phi}\sin\theta\cos\theta + a_4e^{-i\phi}\sin\theta\cos\theta \\ &+\; a_5e^{2i\phi}\sin^2\theta + a_6e^{-2i\phi}\sin^2\theta. \end{aligned}$$

Since the cartesian coordinates (x, y, z) and polar coordinates (θ, ϕ) are related by $x = \sin\theta\cos\phi, y = \sin\theta\sin\phi, z = \cos\theta$ it follows that $f(\theta, \phi)$ is a quadratic form in x, y, z. In other words f is regular and the proof is complete. □

Lemma 1.1.7. *Every continuous frame function in* $\mathbb{R}^3$ *is regular.*

PROOF. Let S^2 denote the unit sphere in $\mathbb{R}^3$ and let $f : S^2 \to \mathbb{C}$ be a continuous frame function. On the group O_3 of rotations define the lifted function:

$$\widetilde{f}(g) = f(g(0,0))$$

where $(0,0)$ denotes the polar coordinates of the north pole. For any smooth function ϕ on O_3 consider

$$(\widetilde{f} * \phi)(g) = \int \widetilde{f}(h^{-1}g)\phi(h)dh.$$

where dh denotes integration with respect to the normalised Haar measure of O_3. Then $(\widetilde{f} * \phi)(gk) = (\widetilde{f} * \phi)(g)$ for all rotations k about the z axis. Further $\widetilde{f} * \phi$ is smooth. Thus there exists a smooth function ψ on S^2 such that

$$(\widetilde{f} * \phi)(g) = \psi(g(0,0)) \text{ for all } g.$$

Since frame functions constitute a vector space it also follows that ψ is a smooth frame function. Thus ψ is regular. choose a sequence of non-negative smooth functions ϕ_n on G such that

$$\begin{aligned} \int \phi_n(g)dg &= 1 \\ \int_{N'} \phi_n(g)dg &\to 0 \text{ as } n \to \infty, \end{aligned}$$

for every neighbourhood N of the identity in O_3, N' denoting the complement of N. Then $\widetilde{\psi}_n = \widetilde{f} * \phi_n$ converges uniformly to $\widetilde{f}$ as $n \to \infty$. In other words the corresponding ψ_n converges uniformly to f as $n \to \infty$. Since each ψ_n is regular it follows that f is regular. This completes the proof. □

We shall now establish a few lemmas leading to the result that every non-negative frame function in $\mathbb{R}^3$ is, indeed, continuous. We write for any map $f : S^2 \to \mathbb{C}$ and any neighbourhood $V \subset S^2$,

$$Osc(f, V) = \sup_{x,y \in V} |f(x) - f(y)|.$$

Lemma 1.1.8. *Let f be a frame function defined on S^2. If for some point p there exists a neighbourhood V_p such that $Osc(f, V_p) < \eta$ then for every point q there exists a neighbourhood V_q of q such that $Osc(f, V_q) < 4\eta$.*

PROOF. Without loss of generality we may take p to be the north pole and V_p to be the spherical cap with centre p and arc length θ. See

Figure 3. Let q_o be any point on the equator. Draw the great circle pq_o and consider the point r at distance $\frac{\theta}{2}$ from the equator along this great circle. Construct a neighbourhood V_{q_0} of q_0 so that for any $q \in V_{q_o}$ the points at right angles to r and q along the great circle rq fall in V_p. Denote the points by (r', q'). Note that the map $q \rightarrow (r', q')$ is continuous in V_{q_0}. If q_1, q_2 belong to V_{q_0} and the images are (r_1', q_1') and (r_2', q_2') respectively then

$$f(r) + f(r_i') = f(q_i) + f(q_i'), i = 1, 2.$$

Taking difference we have

$$f(q_1) - f(q_2) = [f(r_1') - f(r_2')] - [f(q_1') - f(q_2')].$$

This implies $\mathrm{Osc}(f, V_{q_o}) < 2\eta$. If s is any point on S^2 then the equatorial circles for p and s meet at a point q_o. Then s lies on the equatorial circle of q_0. Hence we can find a neighbourhood V_s of s such that $\mathrm{Osc}(f, V_s) < 4\eta$. This completes the proof. □

Definition 1.1.9. *Let p denote the north pole and N the northern open hemisphere in a unit sphere with centre O. For any $p' \in N - \{p\}$ the great circles on the equatorial planes corresponding to p and p' intersect at two points q, q'. The great circle $qp'q'$ is called the* **east west great circle** *through p' and denoted by $EW_{p'}$. The angles $p'Oq$ and $p'Oq'$ are $\frac{\pi}{2}$ each (See Figure 4).*

Lemma 1.1.10. *Let $z \in N - \{p\}$. Then the set*

$$U = \{x : y \in EW_x \cap (N - \{p\}), z \in EW_y\}$$

contains a nonempty open set.

PROOF. Without loss of generality we may assume that z has cartesian coordinates $(\cos\theta, 0, \sin\theta)$. Let $y_o = (\xi_0, \eta_o, \zeta_0)$ be a point in $N - \{p\}$ such that the great circle zy_0 is also EW_{y_0} which meets the equator at t. (See Figure 5).

Let $t = (\cos\alpha, \sin\alpha, 0)$ and let (l, m, n) be the direction cosines of the normal to the plane zOt. Then

$$\begin{aligned} l\cos\theta + m0 + n\sin\theta &= 0, \\ l\xi_0 + m\eta_0 + n\zeta_0 &= 0, \\ l\cos\alpha + m\sin\alpha + n0 &= 0. \end{aligned}$$

Eliminating l, m, n we have

$$(\xi_0 \sin\theta - \zeta_0 \cos\theta)\sin\alpha - \eta_0 \sin\theta\cos\alpha = 0. \quad (1.1.8)$$

Since $t \perp y_0$ we have

$$\eta_0 \sin\alpha + \xi_0 \cos\alpha = 0. \tag{1.1.9}$$

Eliminating α from (1.1.8) and (1.1.9) we get

$$(\xi_0^2 + \eta_0^2)\sin\theta - \xi_0\zeta_0\cos\theta = 0. \tag{1.1.10}$$

Let $\psi(\xi,\eta,\zeta) = (\xi^2+\eta^2)\sin\theta - \xi\zeta\cos\theta$. Then $\psi(\xi,\eta,0) > 0$ and $\psi(\xi,\eta,\zeta) < 0$ when $(1-\zeta^2)^{1/2} > \xi > \zeta^{-1}(1-\zeta^2) tan\theta$ and $\zeta > \sin\theta$. This implies that if we take a point $x = (\xi,\eta,\zeta)$ such that $\psi(\xi,\eta,\zeta) < 0$ then EW_x meets the equator at a point $t = (\xi',\eta',0)$ where $\psi(\xi',\eta',0) > 0$. Hence there exists an intermediate point $y_0 = (\xi_0,\eta_0,\zeta_0)$ on EW_x where (1.1.10) is fulfilled. Then $z \in EW_{y_0}$. Thus

$$U \supset \{x = (\xi,\eta,\zeta) \in N - \{p\}, \quad \psi(\xi,\eta,\zeta) < 0\}.$$

Since ψ is continuous U contains an open set. The proof is complete. □

Lemma 1.1.11. *Every non-negative frame function in $\mathbb{R}^3$ is regular.*

PROOF. Let $f \geq 0$ be a frame function of weight W. Without loss of generality we may assume that $\inf_{x\in S^2} f(x) = 0$. Let $\eta > 0$ be arbitrary. Choose p such that $f(p) \leq \eta$. Let σ be the rotation through $\frac{\pi}{2}$ about the axis through p. Put $g(x) = f(x) + f(\sigma x)$. Then g is a frame function of weight 2W. For any point q on the equator of $p, g(q) = W - f(p)$. Let $r \in N - \{p\}$ and let s, t be two points on $EW_r \cap N - \{p\}$ such that $s \perp t$. If q is a point of intersection of EW_r with the equator of p then

$$2W \geq g(s) + g(t) = g(r) + g(q) \geq g(r) + W - f(p) \geq g(r) + W - \eta.$$

Hence

$$g(r) \leq W + \eta \text{ for all } r \in N - \{p\}.$$

In particular

$$g(r) + W - \eta \leq g(s) + g(t) \leq g(s) + W + \eta.$$

Thus for any $r \in N - \{p\}$ and $s \in EW_r$

$$g(r) \leq g(s) + 2\eta. \tag{1.1.11}$$

Let $\beta = \inf\{g(x) | x \in N - \{p\}\}$. Let $z \in N - \{p\}$ be such that $g(z) \leq \beta + \eta$. Let now $x \in N - \{p\}$ be such that there is a $y \in (N - \{p\}) \cap EW_x$ and $z \in EW_y$. Then by (1.1.11)

$$g(x) \leq g(y) + 2\eta, g(y) \leq g(z) + 2\eta.$$

Thus

$$\beta \leq g(x) \leq g(z) + 4\eta \leq \beta + 5\eta.$$

By Lemma 1.1.10 there exists a neighbourhood V such that

$$Osc(g, V) \leq 5\eta.$$

By Lemma 1.1.8 there exists a neighbourhood V_p of p such that

$$Osc(g, V_p) \leq 20\eta.$$

Hence for any $x \in V_p$

$$0 \leq g(x) \leq g(x) - g(p) + g(p) \leq 20\eta + 2g(p) \leq 22\eta.$$

Since $f(x) \leq g(x)$ for all x we have $0 \leq f(x) \leq 22\eta$ for all $x \in V_p$. Thus $\mathrm{Osc}(f, V_p) \leq 22\eta$. By Lemma 1.1.8 every point q has a neighbourhood V_q such that $\mathrm{Osc}(f, V_p) \leq 88\eta$. Since η is arbitrary it follows that f is continuous. The regularity of f is now immediate from Lemma 1.1.7. This completes the proof. □

Lemma 1.1.12. *If f is a non-negative regular frame function of weight W in a real Hilbert space, then for any two unit vectors x and y,*

$$|f(x) - f(y)| \leq 2W\|x - y\|.$$

PROOF. Since f is regular there is a positive symmetric operator T such that $f(x) = \langle Tx, x\rangle$. Then

$$\begin{aligned} |f(x) - f(y)| &= |\langle T(x + y), x - y\rangle| \\ &\leq \|T\|\|x + y\|\|x - y\| \\ &\leq 2W\|x - y\|. \end{aligned}$$

□

Definition 1.1.13. *Let $\mathcal{H}$ be a complex Hilbert space. A closed set S is said to be a* **completely real subspace** *if for all $x, y \in S$, $\langle x, y\rangle$ is real and $ax + by \in S$ for all real a, b.*

Lemma 1.1.14. *Let $\mathcal{H}$ be a complex Hilbert space of dimension 2 and let f be a non-negative frame function in $\mathcal{H}$. If f is regular in every completely real subspace then it is regular in $\mathcal{H}$.*

PROOF. Let $\sup f(x) = M$. For any $x \in \mathcal{H}$, let

$$F(x) = \begin{cases} \|x\|^2 f\left(\frac{x}{\|x\|}\right) & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Choose unit vectors x_n such that

$$\lim_{n\to\infty} f(x_n) = M, \lim_{n\to\infty} x_n = x_0$$