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Diophantine Approximation and Dirichlet Series

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Preface

The birth act of the analytic theory of Dirichlet series

$$A(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{0.0.1}$$

can be rightly claimed to be the Dirichlet Arithmetic Progression Theorem. In that case, the arithmetical function is $n \mapsto a_n$, the indicator function of the integers n congruent to q mod b for some given pair (q, b) of coprime integers, and its properties are reflected in a subtle way in the "analytic" properties of the function A, although for Dirichlet the variable s remains real. Later on, in the case of the zeta function, Riemann in his celebrated Memoir allowed complex values for s and opened the way to the proof by Hadamard and de la Vallée-Poussin of the Prime Number Theorem.

The utility of those Dirichlet series for the study of arithmetical functions and of their summatory function

$$A^*(x) = \sum_{n \leq x} a_n$$

was widely confirmed during the first half of the twentieth century, with the expansion of tauberian theorems, including those related to Fourier and harmonic analysis, in the style of Wiener, Ikehara, Delange, etc...The hope was of course that progress on those series would imply progress on the distribution of primes, and perhaps a solution to the Riemann hypothesis, the last big question left open in Riemann's Memoir.

A parallel aspect also appeared in the work of H.Bohr, where the series (0.0.1) and their generalization

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \tag{0.0.2}$$

began to be studied for themselves. In particular, Bohr proved a fundamental theorem relating the uniform convergence of a Dirichlet series (and therefore almost-periodicity properties) and the boundedness of its sum A in some half-plane. This naturally led him to his famous question on the maximal gap between abscissas of uniform and absolute convergence. Surprisingly, this question turned out to be very deep, and led him to develop fairly sophisticated tools of other branches, either of complex or harmonic analysis or of diophantine approximation, through the Kronecker approximation theorem (what is called nowadays the Bohr point of view : the unique factorization in primes is seen as the linear independence of the logarithms of those primes). The central importance of this theorem in the theory of Dirichlet series was quickly recognized by him. A solution to his question, found by Bohnenblust and Hille in a famous paper of the Annals, was obtained along the lines suggested by Bohr. Many notions of harmonic analysis (Littlewood's multilinear inequality, p-Sidon sets, Rudin-Shapiro polynomials, etc) were underlying in that work.

The Kronecker theorem (simultaneous, non-homogeneous, approximation) points at two other aspects : on the one hand, at ergodic theory through its formulation and proof, which will be used again in the final chapter on universality. On the other hand at diophantine approximation, which as a consequence is very present in the book. In particular, a thorough treatment of the continued fraction expansion of a real number is presented, as well as its ergodic aspects through the Gauss map (ergodic theory again). This in turn allows a sharp study of the abscissas of convergence of classes of Dirichlet series, which extends a previous study by Hardy-Littlewood for the (easier) case of Taylor series. The simultaneous approximation is still not well understood, except in some cases as the sequence of powers of some given real number, like the Euler basis e, through the use of Padé approximants. A detailed presentation of those approximants, and their applications to a streamlined proof of the transcendency of e, is given in Chapter 3.

Needless to say, the hope of solving Riemann's hypothesis through the study of series (0.0.1) has not been completely met, in spite of many efforts. But along the lines of Bohr, Landau (also S.Mandelbrojt as concerns the series (0.0.2)) and others, those series *continued to be studied for themselves.* Then came a period of relative lack of interest for that point of view, about from 1960 to 1995, with several noticeable exceptions, among which the Voronin theorem (1975) which emphasized the universal role of the zeta function, even if it made no specific progress on the Riemann hypothesis. It seems that the subject was rather suddenly revived by an important paper of Hedenmalm, Lindqvist, and Seip (1997), where several of the forgotten properties of Dirichlet series were successfully revisited for the solution of a hilbertian problem dating back to Beurling (Riesz character of a system of dilates of a given function), and new Hilbert and Banach spaces of Dirichlet series defined and studied. That paper stimulated a series of other, related, works, and this is part of those works, dating back to the last 30 years, which is exposed in those pages.

The aim of this introductory book, which has the ambition of being essentially self-contained, is therefore two-fold:

(1) On the one-hand, the basic tools of diophantine approximation, of ergodic theory, harmonic analysis, probability, necessary to understand the fundamentals of the analytic theory of Dirichlet series, are displayed in detail in the first chapters, as well as general facts about those series, and their products.
(2) On the other-hand, in the last two chapters, especially in Chapter 6, more recent and striking aspects of the analytic theory of Dirichlet are presented, as an application of the techniques coined before.

One fascinating aspect of that theory is that it touches many other aspects of number theory (obviously!) but also of functional, harmonic or complex analysis, so that its detailed comprehension requires a certain familiarity with several other subjects. Accordingly, this book has been divided in seven chapters, which we now present one by one.

1. Chapter one is a review of harmonic analysis on locally compact abelian groups, with its most salient features, including Haar measure, dual group, Plancherel and Pontryagin's theorems. It also insists on some more recent aspects, like the uncertainty principle for the line or a finite group (Tao's version), and on the connection with Dirichlet series (embedding theorem of Montgomery and Vaughan).

2. Chapter two presents the basics of ergodic theory (von Neumann, Oxtoby and Birkhoff theorems) with special emphasis on the applications to the Kronecker theorem (whose precised forms will be of essential use in Chapter 7), to one or multi-dimensional equidistribution problems, and also to some classes of algebraic numbers (Salem numbers).

3. Chapter three deals more specifically with diophantine approximation (continued fractions) in relationship with ergodic theory (Gauss transformation, which is proved to be strong mixing) and aims at giving a classification of real numbers according to their rate of approximation by rationals with controlled denominator. This classification is given by a theorem of Khintchine, fully proved here. As a corollary, the transcendency of the Euler basis e is completely proved.

4. Chapter four presents the basics of general Dirichlet series of the form (0.0.1), with the Perron formulas and the way to compute the three abscissas of simple, uniform, absolute, convergence, and with some comments and examples on a fourth abscissa (the holomorphy abscissa).

Several classes of examples are examined in detail, including the series

$$\sum_{n=1}^{\infty} \frac{n^{-s}}{\|n\theta\|} \tag{0.0.3}$$

according to the diophantine properties of the real number θ. An exact formula for the abscissa of convergence of this series is given in terms of the continued fraction expansion of θ. The problem of products of Dirichlet series and some of its specific aspects, is examined in depth, with emphasis on the role of the translation 1/2. And the Bohr point of view, which allows to look at a Dirichlet series as at a holomorphic function in several complex variables, is revisited, with some applications like the form of Wiener's lemma for Dirichlet series (Hewitt-Williamson's theorem). The chapter ends with a striking application of this point of view to a density result of Jessen and Bohr.

5. Chapter five is a short intermediate chapter establishing the basics of random Dirichlet polynomials through a multidimensional Bernstein inequality and an approach due to Kahane. It will play, technically speaking, an important role in the rest of the book. The tools introduced here remain quite elementary, but will turn out to be sufficient for our purposes.

6. Chapter six is the longest of the book. It is devoted to the detailed study of new Banach spaces of Dirichlet series (the $\mathcal{H}^p$-spaces), which extend the initial work of Bohr and turn out to be of basic importance in completeness problems for systems of dilates in the Hilbert space $L^2(0,1)$, and seem to open the way to new directions of study, like those of Hankel operators (Helson operators) in infinite-dimension. A complete presentation of a recent, very sharp, version of the Bohnenblust-Hille theorem, is also given, using the tools of the previous chapters as well as tools borrowed from number theory, in particular the properties of the function $\psi(x,y)$, the number of integers $\leq x$ which are free of prime divisors $> y$.

7. Chapter seven gives a complete proof of the universality theorems of Voronin (zeta function) and Bagchi (L-functions), and needs in passing a reminder of some properties of those functions in the critical strip. This complete proof is long and involved, but some essential tools (like the Birkhoff-Oxtoby ergodic theorem) have already been introduced in the previous chapters. New, important, tools are an extended version of Carlson's identity seen in Chapter 6, and hilbertian (Bergman) spaces of analytic functions. Those two results have the advantage of showing the pivotal role of zeta and L-functions in analysis and function theory, in the wide sense, and more or less independently of the Riemann hypothesis.

Each of the seven chapters is continued by quite a few exercises, of reasonable difficulty for whoever has read the corresponding chapter. We hope that they can bring additional information, and be useful to the reader.

Acknowledgments The idea of writing this book on diophantine approximation and Dirichlet series had been for some time in our minds, under a rather vague form. But the "passage to the act" followed rather quickly a long stay and a complete course on those topics in the Harish-Chandra Institute of Allahabad in January and February 2011. This course was at the invitation of Surya Ramana, Reader in this Institute and a specialist in Number Theory. We take here the opportunity of deeply thanking him for the numerous mathematical discussions we had, as well as for his kindness and efficiency during our stay, and for his patience before the successive delays in the polishing of the final aspects of the book.

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Contents

1 A review of commutative harmonic analysis

1.1. The Haar measure

1.1.1. Locally compact abelian groups. This chapter might be skipped at first reading. But we have the feeling that a minimal knowledge of basic facts in harmonic analysis is necessary to understand certain aspects of the analytic theory of Dirichlet series, especially those connected with almost-periodicity, ergodic theory, the Bohr point of view to be developed later, and also universality problems. Therefore, in this introductory chapter, we begin with reminding several basic results of commutative harmonic analysis. Those results, although standard by now, are not so easy to prove, and deserve a careful treatment.

Let G be an additive abelian group equipped with a Hausdorff topology τ, which is compatible with the group structure. This means that the operations of the group (addition and inverse) are continuous for that topology. We then say that G is a topological group. Throughout that book, the topology τ will be locally compact, and G will be called a locally compact, abelian group (in short, a LCA group). In most cases, G will indeed be compact. A basic example is that of the compact multiplicative group $\mathbb{T}$ of unimodular complex numbers, that is the unit circle of the complex plane $\mathbb{C}$. *This particular group plays a fundamental role in the theory.*

For $a \in G$, we will denote by T_a the operator of translation by a (a homeomorphism of G, also acting on functions), namely

$$T_a x = x + a, \quad T_a f(x) = f(x+a). \tag{1.1.1}$$

A simple and useful result is the following :

Proposition 1.1.1. *Let G be a topological group and H a subgroup of G. If H has non-empty interior, H is open and closed in G.*

Proof : Let a be interior to H and V a neighbourhood of 0 with $a + V \subset H$. For any $b \in H$, we have $b + V = (b - a) + (a + V) \subset H$, showing that H is open, as well as its cosets $x + H$. Now, we can write G as a disjoint union $G = H \sqcup E$, where E is a union of cosets mod H and is open. So that $H = G \backslash E$ is closed. □

1.1.2. Existence and properties of the Haar measure. A basic fact, expressing the strong ties between the topology and the group structure, is the following theorem :

Theorem 1.1.2. *A locally compact abelian group G always possesses a non-zero, positive and regular Borel measure m which is translation invariant, i.e.*

$$(1.1.2) \quad \int_G f(x)dm(x) = \int_G f(T_a x)dm(x) \quad \forall f \in L^1(G,m), \quad \forall a \in G.$$

This measure (also written dx) is unique up to multiplication by a positive scalar and we write $L^1(G)$ instead of $L^1(G,m)$.

A very simple proof can be found in **[88]** (page 570-571) in the case of a compact, metrizable, group, *abelian or not.* A clear and modern proof can be found for the general (abelian, but not necessarily metrizable) case in **[34]**, Chapter 9. This generality will sometimes be needed, as shown by the forthcoming examples. The measure m is called the Haar measure of G. Three important properties of m are the following :

Proposition 1.1.3. *The Haar measure verifies :*

$$(1.1.3) \qquad m(V) > 0 \text{ for each open non-void set } V \subset G.$$

$$(1.1.4) \qquad m(-B) = m(B) \text{ for all Borel subsets of } G.$$

$$(1.1.5) \qquad m(G) < \infty \Longleftrightarrow G \text{ is compact .}$$

Indeed, suppose that $m(V) = 0$. Let $K \subset G$ be a compact set. This set can be covered by finitely many translates of V, and therefore, $m(K) = 0$. But since m is regular, we have $m(G) = \sup_{K \subset G} m(K)$ so that $m(G) = 0$, which is absurd. Now, the measure $\tilde{m}$ defined by $\tilde{m}(B) = m(-B)$ is translation-invariant, therefore $\tilde{m} = cm$ where c is a scalar. Let then V be a compact and symmetric neighbourhood of 0, so that by (1.1.3) we have $0 < m(V) < \infty$. The equation $cm(V) = m(V)$ therefore implies $c = 1$ and $\tilde{m} = m$. Suppose that G is not compact, and observe that (just take x outside the compact set $K - L$) :

(1.1.6) If $K, L \subset G$ are copac t , there exists $x \in G$; $(x + L) \cap K = \emptyset$.

Now, let V be a compact neighbourhood of 0, so that $m(V) > 0$ by (1.1.3). Using (1.1.6), we can inductively find a sequence $(x_n) \subset G$ such that the translated sets $x_j + V$ are disjoint. Therefore, for any $n \geq 1$:

$$m(G) \geq m\Big[\bigcup_{j=1}^{n}(x_j + V)\Big] = \sum_{j=1}^{n} m(x_j + V) = n \times m(V),$$

and this shows that $m(G) = \infty$. If G is compact, m is clearly finite and we always normalize it to have $m(G) = 1$, i.e. m is a probability measure. □

In the general case, let $M(G)$ be the set of regular, complex Borel measures on G, normed with the total variation of measures. By the Riesz representation theorem, $M(G)$ can be isometrically identified with the dual of the Banach space $C_0(G)$ of continuous functions $f : G \to \mathbb{C}$ which tend to zero at infinity, namely :

$$\forall \varepsilon > 0, \exists K \subset G, K \text{ compact} ; \; x \notin K \Longrightarrow |f(x)| \leq \varepsilon.$$

The convolution $\lambda * \mu$ of λ and μ in $M(G)$ is the element σ of $M(G)$ defined on Borel sets E by :

$$\sigma(E) = \int_G \lambda(E - x)d\mu(x) = \int_G \mu(E - x)d\lambda(x),$$

$$\text{equivalently} \quad \int f d\sigma = \iint f(x + y)d\lambda(x)d\mu(y).$$

One can define an involution $\mu \mapsto \tilde{\mu}$ on $M(G)$ by the formula :

$$\tilde{\mu}(E) = \overline{\mu(-E)}.$$

Once equipped with the variation-norm, convolution, and involution, $M(G)$ is a commutative, unital (the unit being the Dirac measure δ_0 at the origin), stellar (meaning that $\|\tilde{\mu}\| = \|\mu\|$) Banach algebra. But this is not a C^*-algebra : the equation $\|\mu * \tilde{\mu}\| = \|\mu\|^2$ does not hold in general. The Banach space $L^1(G) = L^1(G, m)$ is a closed ideal of $M(G)$, the ideal of measures which are absolutely continuous with respect to m. It is itself a commutative Banach algebra once equipped with the convolution $f * g$ as multiplication :

$$(f * g)(x) = \int_G f(x - y)g(y)dm(y) = (g * f)(x),$$

for almost every $x \in G$. We have $\|f * g\|_1 \leq \|f\|_1\|g\|_1$ and the algebra $L^1(G)$ is unital if and only if G is compact. This is an involutive algebra wit the induced involution defined by $\tilde{f}(x) = \overline{f(-x)}$, i.e. we have $\|\tilde{f}\|_1 = \|f\|_1$. But this is not a C^*-algebra either : the equation $\|f * \tilde{f}\|_1 = \|f\|_1^2$ does not hold in general (see the exercices). Another important property of $L^1(G)$ is the following general fact :

Theorem 1.1.4. *Let $f \in L^p(G), 1 \leq p < \infty$. Then, the mapping $a \mapsto T_a f : G \to L^p(G)$ is (uniformly) continuous.*

Proof : The result holds, by uniform continuity, for $h \in C_{00}(G)$, the space of functions $: G \to \mathbb{C}$ which are continuous and compactly supported. This space is dense in $L^p(G)$ since $p < \infty$. And by translation invariance of m, we clearly have :

$$\|T_a f - f\|_p \leq 2\|h - f\|_p + \|T_a h - h\|_p,$$

which gives the general result, since $\|T_a f - T_b f\|_p = \|T_{a-b} f - f\|_p$. □

We will see that the spectrum of $L^1(G)$ can be identified, whereas a complete description of the spectrum of $M(G)$ is difficult to obtain, and to work with ([**74**]). To that effect, we first have to define the dual of a LCA group.

1.2. The dual group and the Fourier transform

1.2.1. Characters and the algebra $L^1(G)$. The dual group $\widehat{G}$ or Γ of the LCA group G is the group of all *continuous* morphisms $\gamma : G \to \mathbb{T}$, i.e.

$$|\gamma(x)| = 1; \quad \gamma(x+y) = \gamma(x)\gamma(y) \quad \forall x, y \in G.$$

The elements of Γ are called the (*continuous*, or *strong*) characters of G. Sometimes, we will consider all the characters, continuous or not, on G. They are called the *weak characters.* The set Γ, equipped with the natural multiplication of characters, is itself an abelian group (*for multiplication*) whose zero element is the character identical to one. And $\gamma^{-1} = \overline{\gamma}$ for each $\gamma \in \Gamma$. This group appears naturally for the following reason :

Theorem 1.2.1. *The spectrum $\mathcal{L}$ of the Banach algebra $L^1(G)$ can be naturally identified with Γ, in the following sense :*

(1) *Each $\gamma \in \Gamma$ defines $h_\gamma \in \mathcal{L}$ by the formula*

$$h_\gamma(f) = \int_G \gamma(-x) f(x) dm(x).$$

(2) *Each element $h \in \mathcal{L}$ is of the form $h = h_\gamma$.*

Generally, $\int_G \gamma(-x) f(x) dm(x)$ is denoted by $\widehat{f}(\gamma)$ and is called *the Fourier transform* of f at γ. If G is compact and moreover $f \in L^2(G)$, we see that $\widehat{f}(\gamma) = \langle f, \gamma \rangle$, the scalar product of f and γ. In view of Theorem 1.2.1 (see [**114**], page 7 for a detailed proof), we will naturally equip Γ with the Gelfand topology of $\mathcal{L}$, which is the weak-star topology inherited from the dual space Y of $L^1(G)$. This makes Γ a compact Hausdorff space if $\mathcal{L}$ is unital, which happens if and only if G is discrete, and a locally compact Hausdorff space in the general case (since $\mathcal{L} \cup \{0\}$ is weak-star-closed and therefore weak-star-compact in the unit ball of Y). But this topology is fairly abstract and difficult to describe, and we will see later a more concrete and tractable definition. It is first useful to study in detail this Fourier transform, whose main properties are listed in the simple, following theorem, and with obvious notations :

Theorem 1.2.2. *The Fourier transform on $L^1(G)$ verifies :*

(1) If $f \in L^1(G)$, $\widehat{f} \in C_0(\Gamma)$ and $\|\widehat{f}\|_\infty \leq \|f\|_1$
(2) If $\gamma_1 \neq \gamma_2$, there exists $f \in L^1(G) \cap L^2(G)$; $\widehat{f}(\gamma_1) \neq \widehat{f}(\gamma_2)$
(3) For any $\gamma \in \Gamma$, there exists $f \in L^1(G) \cap L^2(G)$; $\widehat{f}(\gamma) \neq 0$
(4) If $f, g \in L^1(G)$, $\widehat{f * g} = \widehat{f}\widehat{g}$
(5) $f * \gamma = \widehat{f}(\gamma)\gamma$; $\widehat{T_a f} = \gamma(a)\widehat{f}$ and $\widehat{\gamma_0 f}(\gamma) = \widehat{f}(\gamma\overline{\gamma_0})$.

Let us denote by $A(\Gamma)$ the subspace of $C_0(\Gamma)$ formed by functions of the form $g(\gamma) = \widehat{f}(\gamma)$ for some $f \in L^1(G)$. This set is called the *Wiener algebra* of Γ. We have the following corollary of Theorem 1.2.2 :

Corollary 1.2.3. *The space $A(\Gamma)$ is a dense, self-adjoint, subalgebra of $C_0(\Gamma)$, stable under translation and multiplication by a character.*

Proof : Using the items of Theorem 1.2.2, we see that $A(\Gamma)$ is a subalgebra, which separates points of Γ and has no common zeros. If $g = \widehat{f} \in A(\Gamma)$, so does $\overline{g} = \widehat{\tilde{f}}$ as we easily see. Therefore, the complex Stone-Weierstrass theorem for locally compact spaces applies and $A(\Gamma)$ is uniformly dense in $C_0(\Gamma)$. □

1.2.2. Topology on the dual group. Here is now an alternative description of the topology on Γ ([**114**], pages 10-11). One interest of this description is that it shows the following : the set Γ, which is so far an abelian group and a locally compact Hausdorff space, is indeed a locally compact abelian group.

Theorem 1.2.4. *The natural topology on Γ is that of uniform convergence on compact subsets of G. More precisely, K, C being compact subsets of G and Γ respectively, and r a positive number, we have :*

(1) *The function $(x, \gamma) \mapsto \gamma(x)$ is continuous on $G \times \Gamma$.*

(2) *Let* $N(K,r) = \{\gamma \in \Gamma \ ; \ |1-\gamma(x)| < r \text{ for all } x \in K\}$. *Then,* $N(K,r)$ *is an open subset of* Γ.
(3) *The family of all sets* $N(K,r)$ *and their translates is a base for the topology of* Γ.
(4) *Let* $M(C,r) = \{x \in G \ ; \ |1-\gamma(x)| < r \text{ for all } \gamma \in C\}$. *Then,* $M(C,r)$ *is an open subset of* G.
(5) Γ *itself is a locally compact abelian group.*

Proof : (1) Let $(x_0, \gamma_0) \in G \times \Gamma$. By Theorem 1.2.2, there is $f \in L^1(G)$ such that $\widehat{f}(\gamma_0) \neq 0$, and we can write, near (x_0, γ_0) :

$$\gamma(x) = \frac{\widehat{T_x f}(\gamma)}{\widehat{f}(\gamma)}.$$

The denominator is continuous at (x_0, γ_0) by Theorem 1.2.2. The numerator as well, since setting $g = T_{x_0} f$, we see that :

$$\begin{aligned} |\widehat{T_x f}(\gamma) - \widehat{T_{x_0} f}(\gamma_0)| &\leq |\widehat{T_x f}(\gamma) - \widehat{T_{x_0} f}(\gamma)| + |\widehat{T_{x_0} f}(\gamma) - \widehat{T_{x_0} f}(\gamma_0)| \\ &\leq \|T_x f - T_{x_0} f\|_1 + |\widehat{g}(\gamma) - \widehat{g}(\gamma_0)|, \end{aligned}$$

and the right-hand-side tends to 0 as $(x, \gamma) \to (x_0, \gamma_0)$, by Theorems 1.1.4 and 1.2.2.

(2) Now, fix $\gamma_0 \in N(K,r)$. For each $x \in K$, there are open neighbourhoods V_x and W_x of x and γ_0 respectively such that :

$$y \in V_x \text{ and } \gamma \in W_x \Longrightarrow |\gamma(y) - 1| < r.$$

Let $V_{x_1}, \ldots, V_{x_p}$ be a finite covering of K and $W = \cap_{j=1}^p W_{x_j}$. The set W is a neighbourhood of γ_0 and $W \subset N(K,r)$, so that $N(K,r)$ is open in Γ.

(3) Conversely, let V be a neighbourhood of γ_0. We may assume that $\gamma_0 = 1$. By definition of the Gelfand topology on Γ, there are functions $f_1, \ldots, f_n \in L^1(G)$ and $\varepsilon > 0$ such that

$$W = \bigcap_{j=1}^{n} \{\gamma \ ; \ |\widehat{f_j}(\gamma) - \widehat{f_j}(1)| < \varepsilon\} \subset V. \tag{1.2.1}$$

By density, we may assume that $f_1, \ldots, f_n \in C_{00}(G)$, so that they vanish outside a compact set $K \subset G$. If $r < \varepsilon / \max_j \|f_j\|_1$, one easily checks that $N(K,r) \subset W \subset V$, since

$$|\widehat{f_j}(\gamma) - \widehat{f_j}(1)| \leq \int_K |1-\gamma(-x)||f_j(x)|dx = \int_K |1-\gamma(x)||f_j(x)|dx < \varepsilon.$$

(4) The same proof applies to $M(C,r)$, with a significant difference : the sets $M(C,r)$ and their translates will turn out to be a base for the

topology of G. *But so far we are unable to establish that fact, which will be proved and used later, and have to content ourselves with the sets* $N(K,r)$.

(5) The obvious inequality

$$|1-\delta_1(x)\overline{\delta_2(x)}| \leq |1-\delta_1(x)| + |1-\delta_2(x)|, \ \forall \delta_1, \delta_2 \in \Gamma, \forall x \in G$$

shows that $[\gamma_1 \times N(K,r/2)]\overline{[\gamma_2 \times N(K,r/2)]} \subset \gamma_1\overline{\gamma_2} \times N(K,r)$. This and the previous description of the topology of Γ shows that the map $(\gamma_1,\gamma_2) \mapsto \gamma_1\overline{\gamma_2}$ is continuous, so that Γ is a LCA group. □

1.2.3. Examples and basic facts. Let us now list, sometimes without proof, some basic *examples and facts* about Haar measures and dual groups.

1.2.3.1. The dual of a compact group is a discrete one, and the dual of a discrete group is a compact one.

1.2.3.2. $\widehat{\mathbb{T}^d} = \mathbb{Z}^d$ and if $\gamma = (n_1, \ldots, n_d) \in \mathbb{Z}^d, z = (z_1, \ldots, z_d) \in \mathbb{T}^d$ we have $\gamma(z) = \prod_{j=1}^d z_j^{n_j}$. The Haar measure m of $\mathbb{T}^d$ acts on continuous functions by the formula

$$\int_{\mathbb{T}^d} f dm = \int_0^1 \int_0^1 \cdots \int_0^1 f(e^{2i\pi t_1}, \ldots, e^{2i\pi t_d}) dt_1 \ldots dt_d.$$

Similarly, $\widehat{\mathbb{Z}^d} = \mathbb{T}^d$. This last fact will later appear as a consequence of the Pontryagin duality theorem. More generally, if $G_1, \ldots, G_d$ are locally compact abelian groups with Haar measures $m_1, \ldots, m_d$ and dual groups $\Gamma_1, \ldots, \Gamma_d$, the product group $G = G_1 \times \cdots \times G_d$ has the Haar measure $m = m_1 \otimes \cdots \otimes m_d$ and its dual group is $\Gamma = \Gamma_1 \times \cdots \times \Gamma_d$.

1.2.3.3. $\widehat{\mathbb{T}^\infty} = \mathbb{Z}^{(\infty)}$ where the LHS is the product of countably many copies of $\mathbb{T}$ and the RHS is the set of all sequences $\nu = (n_1, \ldots, n_d, \ldots)$ of integers which vanish for d large enough, with $\gamma(z) = \prod_{j=1}^\infty z_j^{n_j}$, all but a finite number of the factors being equal to 1. The Haar measure of $\mathbb{T}^\infty$ is the tensor product of countably many copies of the Haar measure of $\mathbb{T}$. This fact has an obvious generalization to the countable product of compact abelian groups, as in example 2.

1.2.3.4. $\widehat{\mathbb{R}^d} = \mathbb{R}^d$ and if $\gamma = (t_1, \ldots, t_d) \in \mathbb{R}^d, \ x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have $\gamma(x) = e^{i\sum_{j=1}^d t_j x_j}$. The Haar measure of $\mathbb{R}^d$ is simply the Lebesgue measure on $\mathbb{R}^d$. Those facts follow from the general remark of example 2.

1.2.3.5. Let G be a compact abelian group with dual Γ. Then, we have the equivalence :

$$G \text{ metrizable} \Longleftrightarrow \Gamma \text{ countable.} \tag{1.2.2}$$

We use the following fact : if X is a topological compact space and $C(X)$ the space of continuous functions $f : X \to \mathbb{C}$ equipped with the norm $\|f\|_\infty = \sup_{t\in X} |f(t)|$, we have the equivalence :

$$X \text{ metrizable} \Longleftrightarrow C(X) \text{ separable.} \tag{1.2.3}$$

Indeed, if the topology of X is defined by a metric d, let (x_n) be a dense sequence of X, and $\varphi_n(x) = d(x, x_n)$. The algebra generated by the φ_n is separable, and dense in $C(X)$ by the Stone-Weierstrass theorem. Conversely, if (f_n) is a dense subset of $C(X)$, the distance d defined by :

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|f_n(x) - f_n(y)|}{1 + |f_n(x) - f_n(y)|}$$

is easily seen to define the topology of X.

To prove (1.2.2), we observe that $\Gamma \subset C(G)$ and that, if $\gamma, \gamma' \in \Gamma$ are distinct, we have by orthogonality :

$$\|\gamma - \gamma'\|_\infty \geq \|\gamma - \gamma'\|_2 = \sqrt{2}. \tag{1.2.4}$$

Now, if Γ is countable, the set $\mathcal{P}$ of trigonometric polynomials is separable, and dense in $C(G)$ by Theorem 1.3.4 to come. Therefore, G is metrizable by (1.2.3). Conversely, if G is metrizable, $C(G)$ is separable, and then Γ has to be countable in view of (1.2.4). This ends the proof of (1.2.2).

1.2.3.6. Let $\mathbb{R}$ be equipped with the Haar measure dx, the usual Lebesgue measure. Its dual Γ can be identified to $\mathbb{R}$, but then the Haar measure corresponding to the forthcoming inversion Theorem 1.4.1 is $dx/2\pi$. Indeed, if $f(t) = e^{-|t|}$, one easily computes $\widehat{f}(x) = 2/(1 + x^2)$ and the change of variable $x = \tan t$ shows that

$$\frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f}(x)|^2 dx = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos^2 t dt = 1 = \int_{\mathbb{R}} |f(t)|^2 dt.$$

1.2.3.7. If $G = \{x_1, \ldots, x_N\}$ is a finite abelian group with dual $\Gamma = \{\gamma_1, \ldots, \gamma_N\}$ (isomorphic to G), and if we equip G with the normalized Haar measure $m = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$, the Haar measure on Γ corresponding to the inversion theorem is the non-normalized measure $\mu = \sum_{j=1}^N \delta_{\gamma_j}$ as is easily checked. This corresponds to the fact that the matrix $(\frac{1}{\sqrt{N}}\gamma_j(x_i))_{(i,j)}$ is unitary. This example is very important for Dirichlet characters.

1.2.3.8. As an important specialization of example 3, we have the following : let G be the Cantor group, i.e. the compact abelian and metrizable group $\{-1,1\}^{\mathbb{N}}$ of all choices of signs $\omega = (\varepsilon_n)_{n\geq 1}$ with $\varepsilon_n = \pm 1$ and coordinatewise multiplication, equipped with its normalized Haar measure m. Its dual group (discrete and countable) is called the Walsh group and can be described as the group of words w_A indexed by the finite subsets of $\mathbb{N}^* := \{1,2,\ldots\}$ defined by

$$w_A(\omega) = \prod_{n\in A} \varepsilon_n(\omega), \ \ w_\emptyset(\omega) = 1.$$

The coordinate functions ε_n are independent random variables on the probability space (G,m) and are sometimes called the Rademacher, or centered Bernoulli, variables. *They will play a very important role in the study of random polynomials and random Dirichlet series.*

1.3. The Bochner-Weil-Raikov and Peter-Weyl theorems

1.3.1. An abstract theorem. The structure of stellar, Banach algebra of $M(G)$ is interesting for us with a view to the following fundamental theorem. Let A denote a commutative, stellar, unital Banach algebra with unit e, with dual space A^* (in the sense of Banach spaces), involution $x \mapsto \tilde{x}$ and spectrum $\mathcal{M}$. We recall that $\mathcal{M}$ is the set of non-zero homomorphisms $\varphi : A \to \mathbb{C}$, which are automatically continuous with norm 1. This is a compact Hausdorff space with the usual Gelfand topology, namely the weak-star topology induced by A^* on $\mathcal{M}$. We denote by $\widehat{x}(\gamma) = \gamma(x)$ the Gelfand transform of $x \in A$ at $\gamma \in \mathcal{M}$, and by $r(x) := \|\widehat{x}\|_\infty$ the spectral radius of $x \in A$. We then have the :

Theorem 1.3.1 (Bochner-Weil-Raikov). *Let L be a positive linear form on A, namely $L(x\tilde{x}) \geq 0$ for all $x \in A$. Then, we have :*

(1) *L is continuous.*
(2) *$|L(x)| \leq L(e)r(x)$ and $|L(x\tilde{x})| \leq L(e)r(x)^2$ for all $x \in A$.*
(3) *There is a positive measure μ on $\mathcal{M}$ such that*

$$L(x) = \int_{\mathcal{M}} \widehat{x}(\gamma)d\mu(\gamma), \ \forall x \in A.$$

(4) *If $L(x\tilde{x}) \neq 0$, there exists $\chi \in \mathcal{M}$ such that $\chi(x) \neq 0$.*

Proof : (1) First note that $\tilde{e} = e$ since $\tilde{e}$ is also a unit for A. Now recall that, for t real and $|t| \leq 1$, we have :

$$\sqrt{1-t} = \sum_{n=0}^{\infty} a_n t^n \text{ with } a_n \text{ real and } \sum_{n=0}^{\infty} |a_n| < \infty.$$

So that, if $x \in A$ and $\|x\| \leq 1$, we can write

$$e - x\tilde{x} = y^2 \text{ with } y = \tilde{y} = \sum_{n=0}^{\infty} a_n (x\tilde{x})^n.$$

This proves that $L(e - x\tilde{x}) = L(y\tilde{y}) \geq 0$ and that $L(x\tilde{x}) \leq L(e)$. Moreover, the assumptions imply that the map $(x, y) \mapsto L(x\tilde{y})$ is a positive, hermitian, form on A, therefore we have the Cauchy-Schwarz inequality :

$$|L(x\tilde{y})|^2 \leq L(x\tilde{x})L(y\tilde{y}).$$

Taking $y = e$, we get, for $\|x\| \leq 1$, the following :

(1.3.1) $$|L(x)|^2 \leq L(e)L(x\tilde{x}) \leq [L(e)]^2.$$

(2) Using (1.3.1) and then iterating, we get :

$$\begin{aligned} |L(x)| &\leq L(e)^{1/2+1/4+\cdots+1/2^n} \left[L(x\tilde{x})^{2^{n-1}}\right]^{1/2^n} \\ &\leq L(e)^{1/2+1/4+\cdots+1/2^n} \|L\|^{1/2^n} \left\|(x\tilde{x})^{2^{n-1}}\right\|^{1/2^n}. \end{aligned}$$

Recall that, according to the spectral radius theorem, $r(x)$ is given by :

(1.3.2) $$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n}.$$

So that, letting n tend to infinity in the above, we get the first claimed inequality

$$|L(x)| \leq L(e)r(x\tilde{x})^{1/2} \leq L(e)r(x).$$

Indeed, if $\chi \in \mathcal{M}$, so does ψ defined by $\psi(x) = \overline{\chi(\tilde{x})}$, and $\chi(x\tilde{x}) = \chi(x)\overline{\psi(x)}$, so that $r(x\tilde{x}) \leq r(x)^2$. The second inequality follows by changing x into $x\tilde{x}$.

(3) Let $\widehat{A}$ the subspace of $C(\mathcal{M})$ formed by Gelfand transforms of elements of A. Define a linear form S on $\widehat{A}$ by the formula $S(\widehat{x}) = L(x)$. The preceding shows that S is well-defined and that

$$|S(\widehat{x})| \leq L(e)\|\widehat{x}\|_\infty.$$

Therefore, S is continuous on $\widehat{A}$ and $\|S\| \leq L(e)$. The Hahn-Banach extension theorem and the Riesz representation theorem now show that there exists a regular, complex measure μ on $\mathcal{M}$, with $\|\mu\| \leq L(e)$, such that :

$$L(x) = S(\widehat{x}) = \int_{\mathcal{M}} \widehat{x}(\gamma)d\mu(\gamma).$$

In particular, $L(e) = S(1) = \int_{\mathcal{M}} d\mu(\gamma) \geq \|\mu\|$, so that μ is positive with norm $L(e)$.

(4) If $L(x\tilde{x}) \neq 0$, item (1) shows that $r(x) = \|\widehat{x}\|_\infty \neq 0$, which ends the proof. □

1.3.2. Applications to harmonic analysis. An important consequence of Theorem 1.3.1 is :

Theorem 1.3.2. *The Fourier transform* $f \mapsto \widehat{f} : L^1(G) \to A(\Gamma)$ *is injective.*

Proof : Consider the unital subalgebra $\mathcal{A} = L^1(G) + \mathbb{C}\delta_0$ of $M(G)$. Fix a function φ in $C_{00}(G)$, the set of continuous, compactly supported functions $G \to \mathbb{C}$. Then, define a linear form $L = L_\varphi$ on that algebra by the formula : $L(\sigma) = \big(\tilde{\varphi} * \varphi * \sigma\big)(0)$, that is, if $\sigma = fdm + c\delta_0 \in \mathcal{A}$:

$$L(\sigma) = \big(\tilde{\varphi} * \varphi * f\big)(0) + c\big(\tilde{\varphi} * \varphi\big)(0).$$

This linear form is positive, since one easily sees that : $L(\sigma * \tilde{\sigma}) = \|\sigma * \varphi\|_2^2$. (Observe in passing that, by Cauchy-Schwarz, Fubini and the translation-invariance of m, one has for $\sigma \in M(G)$: $\varphi * \sigma \in L^2(G)$, with moreover $\|\varphi * \sigma\|_2 \leq \|\varphi\|_2\|\sigma\|$). Now, let $f \in L^1(G), f \neq 0$. Choose $\varphi \in C_{00}(G)$ such that

$$(f * \varphi)(0) = \int_G \varphi(-x)f(x)dx \neq 0. \tag{1.3.3}$$

This implies that $L(f * \tilde{f}) = \|f * \varphi\|_2^2 \neq 0$ since $f * \varphi$ is continuous, does not vanish at 0 by (1.3.3), and since the Haar measure charges all non-void open sets by (1.1.3). Therefore, by Theorem 1.3.1, there is a character h of A such that $h(f) \neq 0$. But the characters of $\mathcal{A}$ are of the form :

$$h(fdm + c\delta_0) = \widehat{f}(\gamma) + c, \text{ for soe } \quad \gamma \in \Gamma.$$

Taking $c = 0$ here, we obtain $h(f) = \widehat{f}(\gamma) \neq 0$, which gives the result. □

In functional analysis, the dual of a normed space has many elements thanks to the Hahn-Banach theorem. It turns out that the dual of a locally compact abelian group G has many elements as well. Namely, as a consequence of Theorem 1.3.2, we have the *Peter-Weyl theorem* in the abelian case :

Theorem 1.3.3 (Peter-Weyl theorem). *The dual* Γ *of any LCA group* G *separates the points of* G*, namely :*

(1.3.4) *If* $x \neq y$*, there exists* $\gamma \in \Gamma$; $\gamma(x) \neq \gamma(y)$.

Proof : Let $x, y \in G$ with $x \neq y$. By the Tietze-Urysohn theorem, there exists $\varphi \in C_{00}(G)$ such that $\varphi(x) \neq \varphi(y)$, that is $T_x\varphi(0) \neq T_y\varphi(0)$. Applying Theorem 1.3.2, we can find $\gamma \in \Gamma$ such that $\widehat{T_x\varphi}(\gamma) \neq \widehat{T_y\varphi}(\gamma)$, equivalently :

$$\widehat{\varphi}(\gamma)\gamma(x) \neq \widehat{\varphi}(\gamma)\gamma(y), \text{ so that } \gamma(x) \neq \gamma(y).$$

□

Let us indicate some important consequences of the Peter-Weyl theorem. We will denote by $\mathcal{P}$ be the algebra of *trigonometric polynomials on* G, i.e. the vector space of functions generated by Γ.

Theorem 1.3.4. *If G is a compact abelian group, the set $\mathcal{P}$ of trigonometrical polynomials is uniformly dense in the space $C(G)$ of complex, continuous functions on G. Conversely, if Δ is a subgroup of Γ separating the points of G, we have $\Delta = \Gamma$.*

Proof : The set $\mathcal{P}$ is a self-adjoint algebra, since the conjugate of a character, and the products of two of them, is still a character. It separates points of G by the Peter-Weyl theorem 1.3.3, and contains the constant 1, the zero-character. Therefore, it is dense in $C(G)$ by the Stone-Weierstrass theorem. Now, let $\mathcal{Q}$ be the set of trigonometric polynomials generated by Δ, i.e. the vector space generated by Δ. This is a self-adjoint algebra since Δ is a subgroup, and it separates points of G, therefore is uniformly dense in $C(G)$ by the Stone-Weierstrass theorem again. Now, suppose that $\gamma \in \Gamma \backslash \Delta$, and let $Q \in \mathcal{Q}$. By orthogonality, we have :

$$\|\gamma - Q\|_\infty \geq \|\gamma - Q\|_2 = \left(1 + \|Q\|_2^2\right)^{1/2} \geq 1,$$

which contradicts the uniform density of $\mathcal{Q}$ in $C(G)$. □

A nice partial consequence of Theorem 1.3.4 is a kind of Hahn-Banach extension theorem for certain subgroups. A more complete description will be given once we have the Pontryagin theorem at our disposal.

Corollary 1.3.5. *Let H be a subgroup of the LCA group G. Then :*
a) Any weak character of H extends to a weak character of G.
b) If H is compact or open, any continuous character on H extends to a continuous character on G.

Proof : a) We use a transfinite induction (or Zorn's lemma) as follows : Let (K, δ) be a maximal pair formed by a subgroup K with $H \subset K \subset G$ and a weak character δ on K extending γ. If $K \neq G$, let $x \notin K$ and L be the group generated by K and x. We separate two cases :