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TEXTS AND READINGS IN MATHEMATICS

50

Notes on Functional Analysis

Rajendra Bhatia

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Preface

These notes are a record of a one semester course on Functional Analysis that I have given a few times to the second year students in the Master of Statistics program at the Indian Statistical Institute, Delhi.

I first taught this course in 1987 to an exceptionally well prepared batch of five students, three of whom have gone on to become very successful mathematicians. Ten years after the course one of them suggested that my lecture notes could be useful for others. I had just finished writing a book in 1996 and was loathe to begin another soon afterwards. I decided instead to prepare an almost verbatim record of what I said in the class the next time I taught the course. This was easier thought than done. The notes written in parts over three different years of teaching were finally ready in 2004.

This background should explain the somewhat unusual format of the book. Unlike the typical text it is not divided into chapters and sections, and it is neither self-contained nor comprehensive. The division is into lectures each corresponding to a 90 minutes class room session. Each is broken into small units that are numbered.

Prerequisites for this course are a good knowledge of Linear Algebra, Real Analysis, Lebesgue Integrals, Metric Spaces, and the rudiments of Set Topology. Traditionally, all these topics are taught before Functional Analysis, and they are used here without much ado. While all major ideas are explained in full, several smaller details are left as exercises. In addition there are other exercises of varying difficulty, and all students are encouraged to do as many of them as they can.

The book can be used by hard working students to learn the basics of Functional Analysis, and by teachers who may find the division into lectures helpful in planning their courses. It could also be used for training and refresher courses for Ph.D. students and college teachers.

The contents of the course are fairly standard; the novelties, if any, lurk in the details. The course begins with the definition and examples of a Banach space and ends with the spectral theorem for bounded self-adjoint operators in a Hilbert space. Concrete examples and connections with classical analysis are emphasized where possible. Of necessity many interesting topics are left out.

There are two persons to whom I owe special thanks. The course follows, in spirit but not in detail, the one I took as a student from K. R. Parthasarathy. In addition I have tried to follow his injunction that each lecture should contain (at least) one major idea. Ajit Iqbal Singh read the notes with her usual diligence and pointed out many errors, inconsistencies, gaps and loose statements in the draft version. I am much obliged for her help. Takashi Sano read parts of the notes and made useful suggestions. I will be most obliged to alert readers for bringing the remaining errors to my notice so that a revised edition could be better.

The notes have been set into type by Anil Shukla with competence and care and I thank him for the effort.

A word about notation

To begin with I talk of real or complex vector spaces. Very soon, no mention is made of the field. When this happens, assume that the space is complex. Likewise I start with normed linear spaces and then come to Banach spaces. If no mention is made of this, assume that X stands for a complete normed linear space.

I do not explicitly mention that a set has to be nonempty or a vector space nonzero for certain statements to be meaningful. Bounded linear functionals, after some time are called linear functionals, and then just functionals. The same happens to bounded linear operators.

A sequence is written as $\{x_n\}$ or simply as "the sequence x_n".

Whenever a general measure space is mentioned, it is assumed to be σ-finite.

The symbol $\overline{E}$ is used for two different purposes. It could mean the closure of the subset E of a topological space, or the complex conjugate of a subset E of the complex plane. This is always clear from the context, and there does not seem any need to discard either of the two common usages.

There are twenty six Lectures in this book. Each of these has small parts with numbers. These are called Sections. A reference such as "Section m" means the section numbered m in the same Lecture. Sections in other lectures are referred to as "Section m in Lecture n". An equation number $(m.n)$ means the equation numbered n in Lecture m.

Do I contradict myself?

Very well then I contradict myself

(I am large, I contain multitudes)

...

—Walt Whitman

Contents

Banach Spaces

The subject Functional Analysis was created at the beginning of the twentieth century to provide a unified framework for the study of problems that involve *continuity* and *linearity*. The basic objects of study in this subject are Banach spaces and linear operators on these spaces.

1. Let X be a vector space over the field $\mathbb{F}$, where $\mathbb{F}$ is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. A *norm* $\|\cdot\|$ on X is a function that assigns to each element of X a nonnegative real value, and has the following properties:

$$(i) \qquad \|x\| = 0 \text{ if, and only if, } x = 0.$$

$$(ii) \qquad \|\alpha x\| = |\alpha|\ \|x\|, \quad \text{for all } \alpha \in \mathbb{F}, x \in X.$$

$$(iii) \qquad \|x+y\| \le \|x\| + \|y\|, \quad \text{for all } x, y \in X.$$

Property (iii) is called the *triangle inequality.*

A vector space equipped with a norm is called a *normed vector space* (or a *normed linear space).*

From the norm arises a *metric* on X given by $d(x,y) = \|x-y\|$. If the metric space (X,d) is complete, we say that X is a *Banach space.* (Stefan Banach was a Polish mathematician, who in 1932 wrote the book *Théorie des Opérations Linéaires,* the first book on Functional Analysis.)

It follows from the triangle inequality that

$$\big|\, \|x\| - \|y\| \,\big| \le \|x-y\|.$$

This shows that the norm is a continuous function on X.

Examples Aplenty

2. The absolute value $|\cdot|$ is a norm on the space $\mathbb{F}$, and with this $\mathbb{F}$ is a Banach space.

3. The *Euclidean space* $\mathbb{F}^n$ is the space of n-vectors $x = (x_1, \ldots, x_n)$ with the norm

$$\|x\|_2 := \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}.$$

4. For each real number $p, 1 \leq p < \infty$ the space ℓ_p^n is the space $\mathbb{F}^n$ with the *p-norm* of a vector $x = (x_1, \ldots, x_n)$ defined as

$$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}.$$

The ∞-*norm* of x is defined as

$$\|x\|_\infty = \max_{1 \leq j \leq n} |x_j|.$$

It is easy to see that $\|x\|_p$ is a norm in the special cases $p = 1, \infty$. For other values of p, the proof goes as follows.

(i) For each $1 \leq p \leq \infty$, its *conjugate index* (the *Hölder conjugate*) is the index q that satisfies the equation

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $1 < p < \infty$, and $a, b \geq 0$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.1}$$

This is called the *generalised arithmetic-geometric mean inequality* or *Young's inequality.* (When $p = 2$, this is the arithmetic-geometric mean inequality.)

(ii) Given two vectors x and y, let xy be the vector with coordinates $(x_1y_1, \ldots, x_ny_n)$. Use (1.1) to prove the *Hölder inequality*

$$\|xy\|_1 \leq \|x\|_p \|y\|_q, \tag{1.2}$$

for all $1 \le p \le \infty$. When $p = 2$, this is the more familiar *Cauchy-Schwarz inequality.*
(iii) Use (1.2) to prove the *Minkowski inequality*

$$\|x + y\|_p \le \|x\|_p + \|y\|_p. \tag{1.3}$$

5. The justification for the symbol $\|\cdot\|_\infty$ is the fact

$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty.$$

6. Why did we restrict ourselves to $p \ge 1$? Let $0 < p < 1$ and take the same definition of $\|\cdot\|_p$ as above. Find two vectors x and y in $\mathbb{F}^2$ for which the triangle inequality is violated.

7. A slight modification of Example 4 is the following. Let $\alpha_j, 1 \le j \le n$ be given positive numbers. Then, for each $1 \le p < \infty$,

$$\|x\| := (\sum \alpha_j |x_j|^p)^{1/p},$$

is a norm.

All the spaces in the examples above are finite-dimensional and are Banach spaces when equipped with the norms we have defined.

8. Let $C[0,1]$ be the space of (real or complex valued) continuous functions on the interval $[0,1]$. Let

$$\|f\| = \sup_{0\le t\le 1} |f(t)|.$$

Then $C[0,1]$ is a Banach space.

The space consisting of all polynomial functions (of all degrees) is a subspace of $C[0,1]$. This subspace is not complete. Its completion is the space $C[0,1]$.

9. More generally, let X be any compact metric space, and let $C(X)$ be the space of (real or complex valued) continuous functions on X. Let

$$\|f\| := \sup_{x\in X} |f(x)|.$$

It is clear that this defines a norm. The completeness of $C(X)$ is proved by a typical use of epsilonics. This argument is called the $\varepsilon/3$ *argument.*

Let f_n be a Cauchy sequence in $C(X)$. Then for every $\varepsilon > 0$ there exists an integer N such that for $m, n \geq N$ and for all x

$$|f_n(x) - f_m(x)| \leq \varepsilon.$$

So, for every x, the sequence $f_n(x)$ converges to a limit (in $\mathbb{F}$) which we may call $f(x)$. In the inequality above let $m \to \infty$. This gives

$$|f_n(x) - f(x)| \leq \varepsilon$$

for $n \geq N$ and for all x. In other words, the sequence f_n converges uniformly to f. We now show that f is continuous. Let x be any point in X and let ε be any positive number. Choose N such that $|f_N(z) - f(z)| \leq \varepsilon/3$ for all $z \in X$. Since f_N is continuous at x, there exists δ such that $|f_N(x) - f_N(y)| \leq \varepsilon/3$ whenever $d(x, y) \leq \delta$. Hence, if $d(x, y) \leq \delta$, then

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|.$$

Each of the three terms on the right hand side of this inequality is bounded by $\varepsilon/3$. Thus $|f(x) - f(y)| \leq \varepsilon$, and f is continuous at x.

10. For each natural number r, let $C^r[0,1]$ be the space of all functions that have continuous derivatives $f^{(1)}, f^{(2)}, \ldots, f^{(r)}$ of order upto r. (As usual, the derivatives are one-sided limits at the endpoints 0 and 1.) Let

$$\|f\| := \sum_{j=0}^{r} \sup_{0 \leq t \leq 1} |f^{(j)}(t)|.$$

The space $C^r[0,1]$ is a Banach space with this norm. (Recall that if the sequences f_n and f'_n converge uniformly on $[0,1]$ to f, g respectively, then f is differentiable and $f' = g$.)

11. Now let X be any metric space, not necessarily compact, and let $C(X)$ be the

space of *bounded* continuous functions on X. Let

$$\|f\| := \sup_{x \in X} |f(x)|.$$

Then $C(X)$ is a Banach space.

Sequence Spaces

12. An interesting special case of Example 11 is obtained by choosing $X = \mathbb{N}$, the set of natural numbers. The resulting space is then the space of *bounded sequences.* This is the space ℓ_∞; if $x = (x_1, x_2, \ldots)$ is an element of this space then its norm is

$$\|x\|_\infty := \sup_{1 \le j < \infty} |x_j|.$$

13. Let c be the subspace of ℓ_∞ that consists of all convergent sequences. Use an $\varepsilon/3$ argument to show that it is a closed subspace of ℓ_∞.

Let c_0 be the collection of all sequences converging to 0. This is also a closed linear subspace of ℓ_∞.

We use the symbol c_{00} to denote the collection of all sequences whose terms are zero after some stage. This is a linear subspace of ℓ_∞, but is not closed. The space c_0 is the *completion* of c_{00} (the smallest closed space in ℓ_∞ that contains c_{00}).

14. For each real number $1 \le p < \infty$, let ℓ_p be the collection of all sequences $x = (x_1, x_2, \ldots)$ such that $\sum_{j=1}^{\infty} |x_j|^p < \infty$.

(i) Use the convexity of the function $f(t) = t^p$ on $[0, \infty)$ to show that ℓ_p is a vector space.

(ii) Note that $\ell_p \subset c_0 \subset \ell_\infty$.

(iii) The inclusions in (ii) are proper. (Consider the sequence with terms $x_n = \frac{1}{\log n}$.)

(iv) The space ℓ_p for any $1 \le p < \infty$ is not closed in ℓ_∞.

(v) For $x \in \ell_p$, define

$$\|x\|_p := (\sum_{j=1}^{\infty} |x_j|^p)^{1/p}.$$

Show that this is a norm. Imitate the steps in Example 4. Some modifications are necessary. The Hölder inequality (1.2) is now the statement: *if* $x \in \ell_p$ *and* $y \in \ell_q$, *then their termwise product* xy *is in* ℓ_1 *and the inequality* (1.2) *holds.* With this norm ℓ_p is a Banach space.

(vi) Let $1 \leq p < p' < \infty$. If the series $\Sigma\,|x_j|^p$ converges, then so does $\Sigma\,|x_j|^{p'}$. Thus the vector space ℓ_p is contained in $\ell_{p'}$. Further, for every $x \in \ell_p$ we have

$$\|x\|_{p'} \leq \|x\|_p. \tag{1.4}$$

This inequality can be proved as follows. Assume first that $\|x\|_p = 1$. Then $|x_j| \leq 1$ for all j, and hence, $|x_j|^{p'} \leq |x_j|^p$. This shows that

$$\Sigma\,|x_j|^{p'} \leq \Sigma\,|x_j|^p = 1,$$

and the inequality (1.4) follows. If x is an arbitrary element of ℓ_p, then let $y = x/\|x\|_p$. Then $\|y\|_p = 1$, and hence, $\|y\|_{p'} \leq \|y\|_p$. This shows (1.4) is true for all $x \in \ell_p$.

Lebesgue Spaces

15. Let I be the interval $[0,1]$ with the Lebesgue measure μ. Let X be the collection of all bounded measurable functions on I, and for $f \in X$ let

$$\|f\| := \sup_{t \in I} |f(t)|.$$

Then X is a Banach space. (To prove completeness, recall that uniform convergence of a sequence f_n is enough to ensure that the limit f is measurable.)

16. Since sets of measure zero are of no consequence, it is more natural to consider *essentially bounded* functions rather than bounded ones. Let f be a measurable function on I. If there exists an $M > 0$ such that

$$\mu(\{t \in I : |f(t)| > M\}) = 0,$$

we say f is essentially bounded. The infimum of all such M is called the *essential* *supremum* of $|f|$, and is written as

$$\|f\|_\infty = \text{ess sup}\,|f|.$$

The collection of all (equivalence classes of) such functions is the space $L_\infty[0,1]$. It is a Banach space with this norm.

17. For $1 \le p < \infty$, let $L_p[0,1]$ be the collection of all measurable functions on $[0,1]$ for which $\int_0^1 |f(t)|^p dt$ is finite. Then $L_p[0,1]$ is a vector space and

$$\|f\|_p := \left(\int_0^1 |f(t)|^p dt\right)^{1/p}$$

is a norm on it. To prove this, one uses versions of Hölder and Minkowski inequalities (1.2) and (1.3) in which sums are replaced by integrals.

The completeness of $L_p[0,1]$ is standard measure theory. The assertion that $L_p[0,1]$ is complete is called the *Riesz–Fischer Theorem.* (Warning: There are other theorems going by the same name.)

18. The interval I can be replaced by a general measure space $(X, \mathcal{S}, \mu)$ in which X is a set, $\mathcal{S}$ a σ-algebra of subsets of X, and μ any measure. The spaces $L_p(X, \mathcal{S}, \mu)$, $1 \le p \le \infty$, can then be defined in the same way as above. (It is often necessary to put some restrictions like σ-finiteness to prevent unruly behaviour of different sorts.) When $X = \mathbb{N}$, and μ is the counting measure, we get sequence spaces.

If $\mu(X)$ is finite, and $1 \le p < p' \le \infty$, then the space $L_{p'}$ is a linear subspace of L_p. In this case we have

$$\|f\|_p \le \mu(X)^{1/p-1/p'} \|f\|_{p'} \tag{1.5}$$

for all $f \in L_{p'}$. (This can be seen using the Hölder inequality, choosing one of the functions to be identically 1.) This is just the opposite of the behaviour of sequence spaces in Example 14.

If $\mu(X) = \infty$, no inclusion relations of this kind can be asserted in general.

Separable Spaces

A metric space is called *separable* if it has a subset that is countable and dense. Separable Banach spaces are easier to handle than nonseparable ones. So, it is of interest to know which spaces are separable.

19. The space $C[0,1]$ is separable. Polynomials with rational coefficients are dense in this space.

20. For $1 \leq p < \infty$, the space c_{00} is dense in ℓ_p. Within this space those that have rational entries are dense. So the spaces ℓ_p, $1 \leq p < \infty$ are separable.

21. The space ℓ_∞ is not separable. Consider the set S of sequences whose terms are 0 or 1. Then S is an uncountable subset of ℓ_∞. (It is uncountable because every point in the unit interval has a binary decimal expansion and thus corresponds to a unique element of S.) If x, y are any two distinct elements of S, then $\|x - y\|_\infty = 1$. So the open balls $B(x, 1/2)$, with radii $1/2$ and centred at points $x \in S$, form an uncountable disjoint collection. Any dense set in ℓ_∞ must have at least one point in each of these balls, and hence can not be countable.

The subspace c_0 of ℓ_∞ is separable (c_{00} is dense in it) as is the subspace c (consider sequences whose terms are constant after some stage).

22. For $1 \leq p < \infty$, the spaces $L_p[0,1]$ are separable. Continuous functions are dense in each of them. The space $L_\infty[0,1]$ is not. (Consider the characteristic functions of the intervals $[0,t]$, $0 \leq t \leq 1$).

23. What about the spaces $L_p(X, \mathcal{S}, \mu)$? These can not be "smaller" than the spaces $(X, \mathcal{S}, \mu)$. If we put $d(E,F) = \mu(E \Delta F)$, where $E \Delta F$ is the symmetric difference of the sets E and F, then $d(E,F)$ is a metric on $\mathcal{S}$. It can be proved (with standard but elaborate measure theory) that for $1 \leq p < \infty$, the space $L_p(X, \mathcal{S}, \mu)$ is separable if and only if the metric space $(\mathcal{S}, d)$ is separable. Further, this condition is satisfied if and only if the σ-algebra $\mathcal{S}$ is countably generated. (The statements about ℓ_p and

$L_p[0,1]$, $1 \le p < \infty$ are included in this more general set up.)

More examples

24. A function f on $[0,1]$ is said to be *absolutely continuous* if, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(t_i') - f(t_i)| < \varepsilon$$

for every finite disjoint collection of intervals $\{(t_i,\ t_i')\}$ in $[0,1]$ with $\sum_{i=1}^{n} |t_i' - t_i| < \delta$. The *Fundamental Theorem of Calculus* says that if f is absolutely continuous, then it is differentiable almost everywhere, its derivative f' is in $L_1[0,1]$, and $f(t) = \int_0^t f'(s)ds + f(0)$ for all $0 \le t \le 1$. Conversely, if g is any element of $L_1[0,1]$, then the function G defined as $G(t) = \int_0^t g(s)ds$ is absolutely continuous, and then G' is equal to g almost everywhere.

For each natural number r, let $L_p^{(r)}[0,1]$ be the collection of all $(r-1)$ times continuously differentiable functions f on $[0,1]$ with the properties that $f^{(r-1)}$ is absolutely continuous and $f^{(r)}$ belongs to $L_p[0,1]$. For f in this space define

$$\|f\| := \|f\|_p + \|f^{(1)}\|_p + \cdots + \|f^{(r)}\|_p.$$

Then $L_p^{(r)}[0,1]$, $1 \le p < \infty$ is a Banach space. (The proof is standard measure theory.) These are called *Sobolev spaces* and are used often in the study of differential equations.

25. Let D be the unit disk in the complex plane and let X be the collection of all functions analytic on D and continuous on its closure $\bar{D}$. For f in X, let

$$\|f\| := \sup_{z \in \bar{D}} |f(z)|.$$

Then X is a Banach space with this norm. (The uniform limit of analytic functions is analytic. Use the theorems of Cauchy and Morera.)

Caveat

We have now many examples of Banach spaces. We will see some more in the course. Two remarks must be made here.

There are important and useful spaces in analysis that are vector spaces and have a natural topology on them that does not arise from any norm. These are *topological vector spaces* that are not normed spaces. The spaces of *distributions* used in the study of differential equations are examples of such spaces.

All the examples that we gave are not hard to describe and come from familiar contexts. There are Banach spaces with norms that are defined inductively and are not easy to describe. These Banach spaces are sources of counterexamples to many assertions that seem plausible and reasonable. There has been a lot of research on these exotic Banach spaces in recent decades.

Dimensionality

Algebraic (Hamel)Basis

1. Let X be a vector space and let S be a subset of it. We say S is *linearly independent* if for every finite subset $\{x_1, \ldots, x_n\}$ of S, the equation

$$a_1 x_1 + \cdots + a_n x_n = 0 \tag{2.1}$$

holds if and only if $a_1 = a_2 = \cdots = a_n = 0$. A (finite) sum like the one in (2.1) is called a *linear combination* of $x_1, \ldots, x_n$.

Infinite sums have a meaning only if we have a notion of convergence in X.

2. A linearly independent subset B of a vector space X is called a *basis* for X if every element of X is a linear combination of (a finite number of) elements of B. To distinguish it from another concept introduced later we call this a *Hamel basis* or an *algebraic basis.*

Every (nonzero) vector space has an algebraic basis. This is proved using Zorn's Lemma. We will use this Lemma often.

Zorn's Lemma

3. Let X be any set. A binary relation $\leq$ on X is called a *partial order* if it satisfies three conditions

(i) $x \leq x$ for all $x \in X$, (*reflexivity*)

(ii) if $x \leq y$ and $y \leq x$, then $x = y$, (*antisymmetry*)

(iii) if $x \leq y$ and $y \leq z$, then $x \leq z$. (*transitivity*)

A set X with a partial order is called a *partially ordered set.*

The sets $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ (natural numbers, rational numbers, and real numbers) are partially ordered if $x \leq y$ means "x is less than or equal to y". Another partial order on $\mathbb{N}$ can be defined by ordaining that $x \leq y$ means "x divides y". The class of all subsets of a given set is partially ordered if we say $E \leq F$ whenever $E \subseteq F$.

An element x_0 of a partially ordered set X is called a *maximal element* if there is no element bigger than it; i.e., $x_0 \leq x$ if and only if $x = x_0$. Such an element need not exist; and if it does it need not be unique.

Let E be a subset of a partially ordered set X. An element x_0 of X is an *upper bound* for E if $x \leq x_0$ for all $x \in E$. We say E is *bounded above* if an upper bound for E exists.

A partially ordered set X is *totally ordered* if in addition to the conditions (i) - (iii), the binary relation $\leq$ satisfies a fourth condition:
(iv) if $x, y \in X$, then either $x \leq y$ or $y \leq x$.

Zorn's Lemma says:

If X is a partially ordered set in which every totally ordered subset is bounded above, then X contains a maximal element.

This Lemma is logically *equivalent* to the *Axiom of Choice* (in the sense that one can be derived from the other). This axiom says that if $\{X_\alpha\}$ is any family of sets, then there exists a set Y that contains exactly one element from each X_α.

See J.L. Kelley, *General Topology* for a discussion.

4. **Exercises.** (i) Use Zorn's Lemma to show that every vector space X has an algebraic basis. (This is a maximal linearly independent subset of X.)

(ii) Show that any two algebraic bases of X have the same cardinality. This is called the *dimension* of X, written as $\dim X$.

(iii) If B is an algebraic basis for X then every element of X can be written uniquely as a linear combination of elements of B.

(iv) Two vector spaces X and Y are isomorphic if and only if $\dim X = \dim Y$.

5. The notion of an algebraic basis is not of much use in studying Banach spaces since it is not related to any topological property. We will see if X is a Banach space, then either $\dim X < \infty$ or $\dim X \geq c$, the cardinality of the continuum. Thus there is no Banach space whose algebraic dimension is countably infinite.

Topological (Schauder) Basis

6. Let $\{x_n\}$ be a sequence of elements of a Banach space X. We say that the series $\sum_{n=1}^{\infty} x_n$ *converges* if the sequence $s_N = \sum_{n=1}^{N} x_n$ of its partial sums has a limit in X.

7. A sequence $\{x_n\}$ in a Banach space X is a *topological basis* (*Schauder basis*) for X if every element x of X has a unique representation $x = \sum_{n=1}^{\infty} a_n x_n$. Note that the order in which the elements x_n are enumerated is important in this definition.

A Schauder basis is necessarily a linearly independent set.

8. If $\{x_n\}$ is a Schauder basis for a Banach space X, then the collection of all finite sums $\sum_{n=1}^{N} a_n x_n$, in which a_n are scalars with rational real and imaginary parts, is dense in X. So, X is separable. Thus a nonseparable Banach space can not have a Schauder basis.

For $n = 1, 2, \ldots$, let e_n be the vector with all entries zero except an entry 1 in the nth place. Then $\{e_n\}$ is a Schauder basis for each of the spaces $\ell_p, 1 \leq p < \infty$, and for the space c_0.

9. Is there any obvious Schauder basis for the space $C[0,1]$ of real functions? The one constructed by Schauder is described below.

Exercise. Let $\{r_i : i \geq 1\}$ be an enumeration of dyadic rationals in $[0,1]$: $0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \frac{3}{16}, \cdots$ Let $f_1(t) \equiv 1, f_2(t) = t$; and for $n > 2$ define f_n as follows. Let $f_n(r_j) = 0$ if $j < n$, $f_n(r_n) = 1$, and let f_n be linear between any two neighbours among the first n dyadic rationals. Draw the graphs of f_3, f_4 and f_5. Show that every element g of $C[0,1]$ has a unique representation $g = \sum a_i f_i$:

(i) Note a_1 must be $g(0)$;

(ii) a_2 must be $g(1) - a_1$;

(iii) proceed inductively to see that
$$a_n = g(r_n) - \sum_{i=1}^{n-1} a_i f_i(r_n);$$

(iv) draw the graph of $\sum_{i=1}^{n} a_i f_i$;

(v) since the sequence r_i is dense in $[0,1]$, these sums converge uniformly to g, as $n \to \infty$.

Note that $\|f_n\| = 1$ for all n. Thus we have a *normalised* basis for $C[0,1]$.

10. Does every separable Banach space have a Schauder basis?

This question turns out to be a difficult one. In 1973, P. Enflo published an example to show that the answer is in the negative. (This kind of problem has turned out to be slippery ground. For example, it is now known that every l_p space with $p \neq 2$ has a *subspace* without a Schauder basis.)

Equivalence of Norms

11. Let $\|\cdot\|$ and $\|\cdot\|'$ be two norms on a vector space X. We say these norms are *equivalent* if there exist positive real numbers C and C' such that

$$\|x\| \leq C\|x\|', \ \|x\|' \leq C'\|x\|$$

for all x. Clearly this is an equivalence relation between norms.

The metrics arising from equivalent norms are equivalent. Any sequence that converges in the metric induced by a norm also converges in the one induced by an equivalent norm.

We will see that if X is finite dimensional, then all norms on X are equivalent to one another.

12. Let $x_1, \ldots, x_n$ be orthonormal vectors in the Euclidean space $\mathbb{C}^n$. Then for all scalars $a_1, \ldots, a_n$,

$$\|a_1x_1 + \cdots + a_nx_n\|_2^2 = |a_1|^2 + \cdots + |a_n|^2. \tag{2.2}$$

The next lemma provides a good working substitute for this. It says that if $x_1, \ldots, x_n$ are linearly independent vectors in any Banach space, then the norm of any linear combination $a_1x_1 + \cdots + a_nx_n$ can not be too small.

Lemma. Let $\{x_1, \ldots, x_n\}$ be linearly independent vectors in any normed linear space X. Then there exists a constant $C > 0$, such that for all scalars $a_1, \ldots, a_n$

$$\|a_1x_1 + \cdots + a_nx_n\| \geq C(|a_1| + \cdots + |a_n|). \tag{2.3}$$

Proof. Divide both sides of the inequality (2.3) by $|a_1| + \cdots + |a_n|$. The problem reduces to showing that there exists C, such that if $\sum |a_j| = 1$, then

$$\|a_1x_1 + \cdots + a_nx_n\| \geq C.$$

If this were not the case, for each positive integer m there would exist $a_1^{(m)}, \ldots, a_n^{(m)}$ with $\sum |a_j^{(m)}| = 1$ such that

$$\|a_1^{(m)} x_1 + \cdots + a_n^{(m)} x_n\| < \frac{1}{m}. \tag{2.4}$$

The sequence $(a_1^{(m)}, \ldots, a_n^{(m)})$ indexed by m is a bounded sequence in $\mathbb{C}^n$. So, by the Bolzano-Weierstrass Theorem it has a convergent subsequence. The limit of this subsequence is an n-tuple $(a_1, \ldots, a_n)$ with $\sum |a_j| = 1$. Since x_j are linearly independent, this means

$$a_1 x_1 + \ldots + a_n x_n \neq 0.$$

This contradicts (2.4) which says that $a_1^{(m)} x_1 + \cdots + a_n^{(m)} x_n$ converges to zero as $m \to \infty$. ■

13. **Theorem.** Any two norms on a finite dimensional vector space are equivalent.

Proof. Let $\{x_1, \ldots, x_n\}$ be a basis for X. If $x = a_1 x_1 + \cdots + a_n x_n$, set

$$\|x\|_1 = |a_1| + \cdots + |a_n|.$$

This is a norm on X. Let $\|\cdot\|$ be any other norm. By the Lemma in 12, there exists a constant C such that

$$\|x\| \geq C \|x\|_1.$$

On the other hand if $C' = \max \|x_j\|$, then

$$\|x\| \leq \sum_j |a_j| \; \|x_j\| \leq C' \sum |a_j| = C' \|x\|_1.$$

Thus $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. ■

14. **Exercises.** (i) Consider the space $\mathbb{C}^n$ with the p-norms $1 \leq p \leq \infty$. Given two indices p and p', find the smallest numbers $C_{p,p'}$ such that

$$\|x\|_p \leq C_{p,p'} \|x\|_{p'} \text{ for all } x.$$