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TEXTS AND READINGS IN MATHEMATICS 42

Surprises and Counterexamples in

Real Function Theory

A. R. Rajwade

A. K. Bhandari

HINDUSTAN BOOK AGENCY

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Preface

Our aim in this book is to consider a variety of intriguing, surprising and appealing topics, and nonroutine proofs of the usual results of real function theory. The reader is expected to have done a first course in real analysis (or advanced calculus), since the book assumes a knowledge of continuity and differentiability of functions, Rolle's theorem, the mean value theorem, Taylor expansion and Riemann integration. However, no sophisticated knowledge of analysis is required and a student at the masters or advanced undergraduate level should have no difficulty in going through the book.

Though this book has some part of the title common with the book "Counterexamples in Analysis" by Gelbaum and Olmstead, it is totally different in nature and contents. Some examples and counterexamples (fifteen or twenty) in our book are essentially the same as given in the book by Gelbaum and Olmstead, but otherwise the intersection is small.

This book contains a number of surprising and unexpected results. It is meant to be a reference book and is expected to be a book to which one turns for finding answers to curiosities which one comes across while studying or teaching elementary analysis. *For example: We know that continuous functions defined on an interval satisfy the intermediate value property. Are there functions which are not continuous but have this property in every interval?(Example 4.1.2) If a one to one onto function is continuous at a point, is its inverse also continuous at the image of that point?(Example 2.4.17) Most would believe that " if* $y = f(x)$ *is a function defined on* $[a, b]$ *and* $c \in (a, b)$ *is any point, then the tangent to its graph* Γ_f *exists at the point* $(c, f(c))$ *if and only if* f *is differentiable at* c*"– see §5.2 for a negative answer. Where does one find easily accessible details of everywhere continuous, nowhere differentiable functions?(Chapter 3)*

Chapter 1 of the book gives an introduction to algebraic, irrational and transcendental numbers. It has several results about the numbers e and π and contains a detailed account of the construction of the curious Cantor ternary set.

In Chapter 2, we consider functions with extraordinary properties. For example, an increasing function $f : [0, 1] \to [0, 1]$, $f(0) = 0$, $f(1) = 1$, the length of whose graph is equal to 2. Another example studied is a function defined on the entire real line that is differentiable at each point but is monotone in no interval.

Chapter 3 discusses, in detail, functions that are continuous at each point but differentiable at no point.

Chapters 4 and 5 include the intermediate value property, periodic functions, properties of derivatives, Rolle's theorem, Taylor's theorem, L'Hospital's rule, points of inflexion, tangents to curves etc. The geometric interpretation of the second and the third derivatives and some intricate aspects of Riemann integration are also included in this chapter.

Chapter 6 discusses sequences, series and Euler's constant γ. The restricted harmonic series is a beautiful topic included here. The surprising rearrangements of alternating harmonic series, leading to Riemann's theorem are also discussed in this Chapter. Some number theoretic aspects are also treated.

In Chapter 7, the infinite exponential $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ with its peculiar range of convergence is studied in detail. We have included an analytic proof of its convergence as well as a very revealing graphical proof.

Appendix I deals with Stirling's formula, a specialized topic of Schwartz's differentiability and some curious properties of Cauchy's functional equation.

Some exercises, conforming to the spirit and style of the book, are included at the end of each chapter. Hints and/or full solutions for the exercises are provided in Appendix II.

References for most of the material in the book are given at the end. Certainly, many other interesting topics could have easily found a place in the book, but there are limitations of time and space.

We hope that the book will be useful to students and teachers alike.

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Chapter 1

Introduction to the real line $\mathbb{R}$ and some of its subsets

§1.1. The real number system

The system of real numbers has evolved as a result of a process of successive extensions of the system of natural numbers (i.e., the positive whole numbers). We shall denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively, the sets of natural numbers, integers, rational numbers, real numbers and complex numbers. The sets $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ form a field with respect to the usual operations of addition and multiplication. The fields $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields:

Definition 1.1.1 An ordered field is a field $\mathbf{F}$ that contains a subset $\mathcal{P}$ such that
(i) $\mathcal{P}$ is closed with respect to addition; that is,

$$x \in \mathcal{P}\,,\ y \in \mathcal{P} \implies x + y \in \mathcal{P}.$$

(ii) $\mathcal{P}$ is closed with respect to multiplication, that is,

$$x \in \mathcal{P}\,,\ y \in \mathcal{P} \implies x.y \in \mathcal{P}.$$

(iii) For all $x \in \mathbf{F}$, exactly one of the following three statements is true:

$$x \in \mathcal{P}\ ;\ x = 0\ ;\ -x \in \mathcal{P},$$

where 0 is the additive identity of the field $\mathbf{F}$.

A member x of $\mathbf{F}$ is called positive if and only if $x \in \mathcal{P}$ and is called negative if and only if $-x \in \mathcal{P}$. Inequalities in an ordered field are defined by: $x < y$ if and only if $y - x \in \mathcal{P}$ and $x \leq y$ if and only if $y - x \in \mathcal{P}$ or $x = y$. If $\mathbf{F}$ is an ordered field and if $x \in \mathbf{F}$, then $|\ x\ |$, called the absolute value of x, is defined to be x in case $x \geq 0$ and to be $-x$ in case $x < 0$.

Suppose that $\mathbf{F}$ is an ordered field. Let $u \in \mathbf{F}$ and $A \subseteq \mathbf{F}$. If $x \leq u$ for every $x \in A$, then u is called an upper bound of A. A non empty subset A of $\mathbf{F}$ is called bounded above in $\mathbf{F}$ if and only if there exits an element of $\mathbf{F}$ which is an upper bound of A. If s is an upper bound of A and if s is less than or equal to every other upper bound of A, then s is called the least upper bound or supremum of A, denoted by $\sup A$. Similarly one defines the notion of being bounded below, lower bound and the greatest lower bound or infimum of a non-empty set A. The infimum of A is denoted by $\inf A$.

Definition 1.1.2. A complete ordered field is an ordered field $\mathbf{F}$ in which a least upper bound exists for every non-empty subset of $\mathbf{F}$ which is bounded above in $\mathbf{F}$.

From any of the well-known constructions of the real number system, it follows that *the set* $\mathbb{R}$ *of real numbers is a complete ordered field.*

Let $\mathbf{F}$ be an ordered field. A sequence in $\mathbf{F}$ is a function with values in $\mathbf{F}$ whose domain is the set of natural numbers $\mathbb{N}$. Its values are denoted by a_n and the sequence itself by $\{a_n\}$. A sequence $\{a_n\}$ is said to converge and to have a limit a if and only if for every ϵ in the set $\mathcal{P}$ of positive numbers, there exists $N \in \mathbb{N}$ such that $| a_n - a | < \epsilon$ for all $n > N$. A sequence that is not convergent is said to be a divergent sequence. A sequence $\{a_n\}$, where the terms a_n are members of an ordered field $\mathbf{F}$, is called a Cauchy sequence if for every $\epsilon \in \mathcal{P}$, there exists $N \in \mathbb{N}$ such that $| a_m - a_n | < \epsilon$ for all $m, n > N$. It follows that every convergent sequence is a Cauchy sequence. The completeness property of $\mathbb{R}$ is equivalent to the fact that every Cauchy sequence in $\mathbb{R}$ is convergent. Not all fields can be ordered in the sense of Definition 1.1.1. We have:

Example 1.1.3 The field $\mathbb{C}$ can not be ordered, i.e., it possesses no subset $\mathcal{P}$ satisfying the properties of Definition 1.1.1. Indeed, assume that there does exist such a subset $\mathcal{P}$ of $\mathbb{C}$. Consider the complex number ι. Since $\iota \neq 0$, there are two possibilities. The first is that $\iota \in \mathcal{P}$, in which case $\iota^2 = -1 \in \mathcal{P}$, whence $\iota^4 = 1 \in \mathcal{P}$. Since ι^2 and ι^4 are additive inverses of each other, it is impossible for both of them to be in $\mathcal{P}$. We thus obtain a contradiction. The other alternative is that $-\iota \in \mathcal{P}$, in which case $(-\iota)^2 = -1 \in \mathcal{P}$, whence $(-\iota)^4 = 1 \in \mathcal{P}$, and we arrive at the same contradiction.

It can be shown that the fields $\mathbb{Q}$ as well as $\mathbb{R}$ admit only one ordering (see, for example [61]). The set $\mathcal{P}$ of positive elements of $\mathbb{R}$ is the set of squares of the elements of $\mathbb{R}$. The set of positive elements of $\mathbb{Q}$ is $\mathbb{Q} \cap \mathcal{P}$.

However, there are fields which can be ordered in more than one ways.

Example 1.1.4. Let m be any positive integer which is not a perfect square. Let $\mathbf{F} = \{a + b\sqrt{m} \mid a, b \in \mathbb{Q}\}$. It is easy to see that $\mathbf{F}$ is a field under the usual operations of addition and multiplication of real numbers. Let $\mathcal{P}$ be the set of usual positive elements of $\mathbb{R}$ (i.e., squares) and let $\mathcal{P}' = \mathcal{P} \cap \mathbf{F}$. Then $\mathcal{P}'$ serves as a subset of positive elements of $\mathbf{F}$ according to Definition 1.1.1. A second way in which $\mathbf{F}$ is an ordered field is provided by the subset $\mathcal{P}''$ defined by

$$a + b\sqrt{m} \in \mathcal{P}'' \iff a - b\sqrt{m} \in \mathcal{P} \; ;$$

that $\mathcal{P}''$ satisfies the three requirements of Definition 1.1.1 can be easily verified.

Example 1.1.5. The ordered field $\mathbb{Q}$ of rational numbers is not complete (i.e., does not satisfy the requirements of Definition 1.1.2). Let

$$A = \{r \in \mathbb{Q} \mid r > 0 \; , \; r^2 < 2\}.$$

The set A is non empty ($1 \in A$) and is bounded above by 2. Let us assume that $\mathbb{Q}$ is complete. Then there must be a positive rational number c that is the supremum of A. Since there is no rational number whose square is equal to 2, either $c^2 < 2$ or $c^2 > 2$. Assume first that $c^2 < 2$ and let d be the positive number $d = \frac{1}{2}\min\{\frac{2-c^2}{(c+1)^2}\ ,\ 1\}$. Then $c+d$ is a positive rational number greater than c whose square is less than 2, i.e., $(c+d)^2 < c^2 + d(c+1)^2 < 2$; but then $c+d \in A$, whereas c is an upper bound of A. Assuming $c^2 > 2$, let d be the positive number $d = \frac{c^2-2}{2(c+1)^2}$. Then $c-d$ is a positive rational number less than c whose square is greater than 2, i.e., $(c-d)^2 > c^2 - d(c+1)^2 > 2$. Since $c-d$ is an upper bound of A which is less than the least upper bound c, we arrive at a contradiction.

Recall that the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$ (in the usual distance topology). We close this section by recording a somewhat surprising result, which will be needed later.

Theorem 1.1.6. *Let θ be an irrational number. Then the set of numbers of the form $m + n\theta$, $m, n \in \mathbb{Z}$, is dense in $\mathbb{R}$.*

Proof. Let $\epsilon > 0$ be an arbitrary real number. We shall first find m, n such that $0 < m + n\theta < \epsilon$. Choose N such that $\frac{1}{N} < \epsilon$ and then for each $n = 0, 1, 2, \ldots, N$, choose m $(= m(n)$, i.e., m depends on $n)$ such that $0 < m + n\theta < 1$ (for example, $m = -[n\,\theta]$ will do, where for a real number x, $[x]$ denotes the largest integer less than or equal to x). Then we have $N+1$ distinct numbers $m(n) + n\,\theta$, $0 \le n \le N$, in $[0,1]$ and therefore there exist two such numbers which are at a distance less than $\frac{1}{N}$ apart (box principle), say, $|\,(m_1 + n_1\,\theta) - (m_2 + n_2\,\theta)\,| < \frac{1}{N} < \epsilon$, i.e., $|\,(m_1 - m_2) - (n_1 - n_2)\,\theta\,| < \epsilon$. Thus, we find a number of the form $m + n\,\theta$ such that $|\,m + n\,\theta\,| < \epsilon$. By changing signs we get a number $0 < m + n\,\theta < \epsilon$. Now, given any real number r, find an integer k such that $k(m + n\,\theta) < r < (k+1)(m + n\,\theta)$. We thus get a number of the required form which is as close to r as we want.

§1.2. Irrational and transcendental numbers

A real number α is called an *algebraic number*, if α is a root of a nonconstant polynomial with rational coefficients. Let $\mathbb{A}$ denote the set of all such real numbers. In the usual notations, the inclusions

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{A} \subseteq \mathbb{R}$$

give rise to the following disjoint unions:

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \smallsetminus \mathbb{Q}) = \mathbb{Q} \cup \mathbb{I} \ \ ; \ \ \mathbb{R} = \mathbb{A} \cup (\mathbb{R} \smallsetminus \mathbb{A}) = \mathbb{A} \cup \mathbb{T}\,,$$

where, $\mathbb{I} = \mathbb{R} \smallsetminus \mathbb{Q}$ and $\mathbb{T} = \mathbb{R} \smallsetminus \mathbb{A}$. As $\mathbb{Q} \subseteq \mathbb{A}$, it follows that $\mathbb{T} \subseteq \mathbb{I}$.

Definition 1.2.1. Elements of $\mathbb{I}$ are called irrational numbers while those of $\mathbb{T}$ are called transcendental numbers.

Perhaps it is very surprising that while on one hand it is not immediately clear that $\mathbb{T}$ is non empty, on the other hand, it turns out that (in a well defined sense) almost all real numbers are transcendental. Not only that, it is hard to identify individual numbers as being transcendental and such identifications, as have been made are mathematical epics. According to a Cambridge story, G.H. Hardy was prepared to resign his chair in favour of anyone who proved that the Euler's constant

$$\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n)$$

was irrational, let alone transcendental ! It was in 1873 that Hermite proved that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

is transcendental and in 1882, Lindemann did the same for π; the status of γ is still undecided.

It will give some idea of the difficulty of transcendence questions if we note that the transcendence of $2^{\sqrt{2}}$ was the seventh in the list of the famous 23 unsolved problems that David Hilbert presented, at the International Congress of 1900 in Paris, as signposts for twentieth century Mathematics. Not only that but speaking informally at a seminar in Göttingen, twenty years later, Hilbert declared that none of the audience would live to see a solution of this problem. As it happened, Hilbert was wrong : the problem was settled some twelve years later by Gelfond and Schneider, while several questions that Hilbert believed to be easier are still unanswered.

A beautiful result that settles the transcendence of $2^{\sqrt{2}}$ is the following:

Theorem 1.2.2.(Gelfond-Schneider, 1934) *If* $\alpha \neq 0$ *or* 1, $\alpha \in \mathbb{A}$, $\beta \in \mathbb{I} \cap \mathbb{A}$, *then* $\alpha^\beta \in \mathbb{T}$.

For a self-contained proof of this result the reader is referred to [44]. Taking $\alpha = 2$, $\beta = \sqrt{2}$ shows that $2^{\sqrt{2}}$ is transcendental.

In 1851, Liouville was the first to exhibit a class of transcendental numbers, viz. numbers of the form $a_1/10 + a_2/10^{2!} + a_3/10^{3!} + \cdots$, $0 \leq a_i \leq 9$, with infinitely many a_i's nonzero. This class is uncountable whereas the set $\mathbb{A}$ can be shown to be countable. This already shows that *most numbers are transcendental.*

Theorem 1.2.3. (Liouville, 1851) *The real number*

$$L = \sum_{i=1}^{\infty} \frac{1}{10^{i!}} = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^6} + \frac{1}{10^{24}} + \cdots \quad = 0.1100010\ldots$$

is transcendental.

Proof. Let α be an algebraic number. Then it satisfies a polynomial $P(x)$ with integral coeffcients. Suppose that the degree of $P(x)$ is n. We first claim that then α can not be the limit of a sequence of distinct rational numbers p_i/q_i satisfying

$$|\, p_i/q_i - \alpha \,| < k/q_i^{n+1} \;, \qquad (*)$$

where k is a fixed number. For otherwise, the mean value theorem gives $P(p_i/q_i) - P(\alpha) = (p_i/q_i - \alpha)P'(\xi_i)$ for some ξ_i between p_i/q_i and α, i.e.,

$$\begin{aligned} |\, P(p_i/q_i) \,| &= |\, p_i/q_i - \alpha \,||\, P'(\xi_i) \,| && (\text{since } P(\alpha) = 0) \\ &< (k/q_i^{n+1}) \,|\, P'(\xi_i) \,| && (\text{by}(*)) \\ &< (k/q_i^{n+1})(|\, P'(\alpha) \,| + 1) \;, \end{aligned}$$

if i is large enough, since $\xi_i \to \alpha$, (as $i \to \infty, p_i/q_i \to \alpha$) and P' is a polynomial, so continuous. This gives $|\, q_i^n P(p_i/q_i) \,| < (k/q_i)(|\, P'(\alpha) + 1 \,|)$, if i is large. Here the left hand side $q_i^n P(p_i/q_i) \in \mathbb{Z}$, since $P(p_i/q_i)$ is a polynomial in p_i/q_i of degree n with integer coefficients, while the right hand side tends to 0 as i tends to infinity, i.e., the absolute value of the right hand side is less than 1 (note that p_i/q_i being distinct implies that $q_i \to \infty$ as $i \to \infty$). It follows that the left hand side is equal to 0 for all $i \geq J$, say, i.e., $P(p_i/q_i) = 0$ for $i \geq J$, which is a contradiction since P is a polynomial of degree n and so has no more than n zeros. This proves the claim.

Now going back to the number L, we have $L = 1/10 + 1/10^2 + 0/10^3 + 0/10^4 + 0/10^5 + 1/10^6 + 0/10^7 + \cdots$, so that its decimal expansion equals 0.1100010.... as stated in the theorem, with the i^{th} nonzero decimal digit equal to 1 in the $i!^{\text{th}}$ place. If we truncate this decimal expansion after the i^{th} nonzero digit, we obtain a rational approximation $p_i/10^{i!}$, which differs from L by less than $2/10^{i!+1}$. Indeed, we have

$$L = \underbrace{(1/10 + 1/10^2 + 0/10^3 + 0/10^4 + 0/10^5 + 1/10^6 + \cdots + 1/10^{i!})}_{\textit{truncated at this digit}} + 0/10^{i!+1} + \cdots$$

and we call the quantity in brackets p_i/q_i, which is equal to $p_i/10^{i!}$ (common denominator equal to $10^{i!}$), and then

$$\begin{aligned} L - p_i/q_i &= 0/10^{i!+1} + \cdots + 1/10^{(i+1)!} + \cdots \\ &= 0 + \cdots + 1/10^{(i+1)!} + \cdots \\ &= (1/10^{(i+1)!})(1 + 1/10^{i+2} + 1/10^{(i+3)(i+2)} + \cdots) \\ &< (1/10^{(i+1)!}) \cdot 2 \quad (\text{the sum being a sub-geometric progression}) \\ &= 2/(10^{i!})^{i+1} \\ &\leq 2/(10^{i!})^{n+1} \;, \end{aligned}$$

for any $n \geq i$; and so, since the q_i are all distinct ($q_i = 10^{i!}$), the claim above yields that L can not satisfy a polynomial equation of degree n with integer coefficients and hence it follows that L is not algebraic.

For the general Liouville number $L = \sum a_i/10^{i!}$, $0 \le a_i \le 9$, a slight modification of the above argument shows L to be transcendental. Alternatively, one could use binary expansion and write L in the form $L = \sum a_i/2^{i!}$ $(a_i = 0, 1)$ and then, exactly as in Theorem 1.2.3, L may be shown to be transcendental. This was the first encounter of a class of transcendental numbers (in 1851). The Gelfond-Schneider theorem mentioned above guarantees another such class of transcendental numbers. To check specific numbers for transcendence is an extremely difficult job. Results that help in this direction are the following:

Theorem 1.2.4. *If β is a positive number such that 2^β, 3^β, 5^β, 7^β, 11^β, ... are all integers, i.e., if p^β is an integer for every prime p then β itself is an integer.*

For a proof see [41].

The hypothesis of the above theorem can be weakened and indeed a deeper argument given in [41] to prove Theorem 1.2.4 yields the following result.

Theorem 1.2.5.(Siegel) *If β is a positive real number such that 2^β, 3^β, 5^β are integers, then β is an integer.*

It is an open question whether the hypothesis of Theorem 1.2.5 can be reduced to requiring only that 2^β and 3^β be integers. If that were possible, as is conjectured, the result would be the best possible, since $2^{\log 3/\log 2} = 3$ is an integer but $\log 3/\log 2$ is irrational (for, if it is equal to p/q, then $q \log 3 = p \log 2$, or $3^q = 2^p$, which is impossible). Theorem 1.2.5 tells us that if β is irrational, at least one of 2^β, 3^β, 5^β is not an integer.

The following far-reaching extension of Theorem 1.2.2 was proved by Baker (see [6]) in 1966.

Theorem 1.2.6. *If $\alpha_1, \dots, \alpha_n$ are algebraic numbers, $\alpha_i \neq 0, 1$, and if $\beta_1, \dots, \beta_n$ are algebraic numbers which are different from 0 and 1 and are linearly independent over $\mathbb{Q}$, then $\alpha_1^{\beta_1}\alpha_2^{\beta_2}\dots\alpha_n^{\beta_n}$ is transcendental.*

The case $n = 1$ is Theorem 1.2.2. As an example, it follows that the number $2^{\sqrt{2}}.3^{\sqrt{3}}.5^{\sqrt{5}}$ is transcendental.

Example 1.2.7. Can a rational or an irrational number raised to a rational or an irrational power be rational or irrational? All the eight possibilities are as follows: (i) $(irrational)^{irrational} = rational$: Observe that $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2}.\sqrt{2}} = 2$ and that $\sqrt{2}^{\sqrt{2}}$ is irrational, being the square root of Hilbert's number $2^{\sqrt{2}}$ (Theorem 1.2.2). It was actually proved to be transcendental by Kuzmin in 1930.
(ii) $(irrational)^{irrational} = irrational$: Yes; $(\sqrt{2})^{\sqrt{2}}$, as above.
(iii) $(rational)^{irrational} = irrational$: Yes; $2^{\sqrt{2}}$ is irrational, by Theorem 1.2.2.
(iv) $(rational)^{irrational} = rational$: Yes; $2^{\log_2 3} = 3$. However, a rational number raised to an algebraic irrational power is irrational, by Theorem 1.2.2.

(v) $(irrational)^{rational} = rational$: Yes; $(\sqrt{2})^2 = 2$.
(vi) $(irrational)^{rational} = irrational$:Yes; trivially, $(\sqrt{2})^1 = \sqrt{2}$.
(vii) $(rational)^{rational} = rational$: Yes; $2^2 = 4$.
(viii) $(rational)^{rational} = irrational$:Yes; $2^{\frac{1}{2}} = \sqrt{2}$ is irrational.

The reader is referred to [52] for more such results.

Example 1.2.8. The number $\xi = \sum_{i=1}^{\infty} \frac{1}{c_i}$ is transcendental, where $c_0 = 0$, $c_{i+1} = 2^{c_i}$.

To see this, observe that c_i are rapidly increasing powers of 2. Write out a few and then check (by induction) that if $k \geq 1$, then $k \leq c_{k-1}$ and that $c_{k-1}^2 \leq 2^{c_{k-1}}$. It follows that $kc_{k-1} \leq c_{k-1}^2 \leq 2^{c_{k-1}} = c_k$, i.e., $kc_{k-1} \leq c_k$ and so $2^{kc_{k-1}} \leq 2^{c_k} = c_{k+1}$, or $(2^{c_{k-1}})^k \leq c_{k+1}$, or $c_k^k \leq c_{k+1}$ (note that $x^y - y^x \leq 0$ for (x, y) in regions IV and II of the plane (see Chapter 7); hence $c_k^2 - 2^{c_k} \leq 0$, since $(c_k, 2)$ is in the region IV).

Now, let $\xi_k = \sum_{i=1}^{k} \frac{1}{c_i}$. Then $\xi = p_k/c_k$, where $(p_k, c_k) = 1$ (for $\xi_k =$ (a sum of even integers+1)$/c_k$). Now suppose ξ is algebraic of degree n ($n > 1$), then

$$\xi - \frac{p_k}{c_k} = \sum_{i \geq k+1} c_i = \frac{1}{c_{k+1}} + \left\{ \frac{1}{c_{k+2}} + \cdots \right\} < \frac{2}{c_{k+1}} \leq \frac{2}{c_k^k} \, ,$$

because, $\{\frac{1}{c_{k+2}} + \cdots\} \leq \frac{1}{c_{k+2}}$ + terms of a geometric progression, all of the form $1/2^r$, from c_{k+2} onward $\leq 2/c_{k+2} < 1/c_{k+1}$ and $c_k^k \leq c_{k+1}$, by above.

It follows that

$$0 \leq \lim_{k \to \infty} c_k^{n+1}(\xi - p_k/c_k) \leq \lim_{k \to \infty} c_k^{n+1}.(2/c_k^k) = 0 \, ,$$

and so $| \, c_k^{n+1}(\xi - p_k/c_k) \, | < 1$, if k is large, which gives that $| \, p_k/c_k - \xi \, | < 1/c_k^{n+1}$ and since p_k/c_k are distinct, this contradicts the claim in the beginning of the proof of Liouville's theorem.

Finally, there remains the case $n = 1$, i.e., ξ is rational, say, $\xi = p/q$ (p, q integers). Choose k such that $c_k > q$. Then $0 < (\xi - \xi_k)qc_k = (p/q - p_k/c_k)qc_k = pc_k - p_kq$, which is an integer.

However, $(\xi - \xi_k)qc_k = qc_k(\sum_{i \geq k+1} 1/c_i) < qc_k.(2/c_{k+1}) \leq 2q/c_k^{k-1} < 2/c_k^{k-2}$, since $c_k^k \leq c_{k+1}$ and k is so chosen that $c_k > q$; and it follows that for $k \geq 3$, $(\xi - \xi_k)qc_k < 1$, contradicting the above observation that it is an integer.

There are many outstanding problems regarding the irrationality of numbers. In particular, many irrational numbers can be explicitly exhibited, e.g., $\sqrt{n}$ is irrational for all positive integers n which are not perfect squares. Also $\sqrt{2} + \sqrt{3}$, $\sqrt{2} + \sqrt{3} + \sqrt{6}$ etc. are irrational. In what follows, we give a couple of such interesting results, which yield irrationality of some classes of numbers. The proofs of these results require familiarity with the basic concepts of field extensions.

Theorem 1.2.9. *Let $a_1, \ldots, a_n$ be positive integers, none perfect squares and* *coprime in pairs. Then the 2^n algebraic numbers $(a_1^{\varepsilon_1}a_2^{\varepsilon_2}\dots a_n^{\varepsilon_n})^{\frac{1}{2}}$, $\varepsilon_i = 0, 1$, are linearly independent over $\mathbb{Q}$.*

Proof. We use induction on n. For $n = 1$, the result is trivial. So suppose the result is true for $n-1$ and we prove it for n.. Define a tower of fields: $K_0 = \mathbb{Q}$, $K_1 = \mathbb{Q}(\sqrt{a_1})$, $K_2 = K_1(\sqrt{a_2})$, ... , $K_n = K_{n-1}(\sqrt{a_n})$. We first prove that each K_i is of degree 2 over K_{i-1}. To see this we again use induction. For $n = 1$, clearly $[K_1 : K_0] = 2$. So, suppose the result is true for $n-1$ and we are to show that $[K_n : K_{n-1}] = 2$. If not, then $K_n = K_{n-1}$, i.e., $\sqrt{a_n} \in K_{n-1} = K_{n-2}(\sqrt{a_{n-1}})$ and so $\sqrt{a_n} = \beta + \gamma\sqrt{a_{n-1}}$, $\beta, \gamma \in K_{n-2}$, or $a_n = \beta^2 + \gamma^2 a_{n-1} + 2\beta\gamma\sqrt{a_{n-1}}$. Here $\sqrt{a_{n-1}} \notin K_{n-2}$, by induction hypothesis, so $\beta\gamma = 0$. If $\gamma = 0$, then $\sqrt{a_n} = \beta \in K_{n-2}$, which contradicts the induction hypothesis for $n-1$ numbers $a_1, a_2, \dots, a_{n-2}, a_n$. If $\beta = 0$, then $\sqrt{a_n} = \gamma\sqrt{a_{n-1}}$ and so $\sqrt{(a_{n-1}a_n)} = a_{n-1}\gamma \in K_{n-2}$, which again contradicts the induction hypothesis for $n-1$ numbers $a_1, \dots, a_{n-2}, a_{n-1}a_n$. This completes the proof of the observation and hence $[K_n : \mathbb{Q}] = 2^n$.

By induction hypothesis, the set $\{(a_1^{\varepsilon_1}\dots a_{n-1}^{\varepsilon_{n-1}})^{1/2} \mid \varepsilon_i = 0, 1\}$ is linearly independent over $\mathbb{Q}$ and therefore constitutes a basis of K_{n-1} over $\mathbb{Q}$, as $[K_{n-1} : \mathbb{Q}] = 2^{n-1}$, by above.

Let $E \subseteq K \subseteq L$ be a tower of fields. Let $\{\alpha_1, \dots, \alpha_s\}$ be a basis of K over E and $\{\beta_1, \dots, \beta_t\}$ be a basis of L over K. Then it is easy to see that $\{\alpha_i\beta_j \mid 1 \le i \le s, 1 \le j \le t\}$ is a basis of L over E.

Now, $\{(a_1^{\varepsilon_1}\dots a_{n-1}^{\varepsilon_{n-1}})^{1/2} \mid \varepsilon_i = 0, 1\}$ is a basis of K_{n-1} over $\mathbb{Q}$ and $\{1, \sqrt{a_n}\}$ is a basis of K_n over K_{n-1}, the result follows.

Example 1.2.10. As an application of the above result it follows that an expression like $7\sqrt{19}/4 - 3\sqrt{7} + 8\sqrt{6}/5$ is irrational.

While we are at it, we prove the following very useful:

Theorem 1.2.11. *Let m be a square-free integer greater than 1 and let $\alpha, \beta, \gamma \in \mathbb{Q}(\sqrt{m}) = \{a + b\sqrt{m} \mid a, b \in \mathbb{Q}\}$. A necessary and sufficient condition that there exist $u, v, w \in \mathbb{Q}$ such that*

$$u\sqrt{\alpha} + v\sqrt{\beta} + w\sqrt{\gamma} = 0 \qquad (1)$$

is that there exist $\lambda, \mu, \nu \in \mathbb{Q}(\sqrt{m})$ such that

$$\alpha : \beta : \gamma = \lambda^2 : \mu^2 : \nu^2. \qquad (2)$$

Proof. Write $\alpha = a_1 + a_2\sqrt{m}$, $\beta = b_1 + b_2\sqrt{m}$, $\gamma = c_1 + c_2\sqrt{m}$, $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{Q}$. Then (1) $\Rightarrow (u\sqrt{\alpha} + v\sqrt{\beta})^2 = (-w\sqrt{\gamma})^2 \Rightarrow u^2\alpha + v^2\beta + 2uv\sqrt{\alpha\beta} = w^2\gamma \Rightarrow \sqrt{\alpha\beta} \in \mathbb{Q}(\sqrt{m})$, i.e., that $\alpha\beta$ is a square in $\mathbb{Q}(\sqrt{m})$, say, $\alpha\beta = 1/\nu^2$ and similarly $\beta\gamma = 1/\lambda^2$, $\gamma\alpha = 1/\mu^2$, with $\lambda, \mu, \nu \in \mathbb{Q}(\sqrt{m})$. On dividing, it follows that $\alpha/\beta = \lambda^2/\mu^2$ and $\alpha/\gamma = \lambda^2/\nu^2$, which together give (2).

Conversely, let (2) hold. Then $\sqrt{\alpha} : \sqrt{\beta} : \sqrt{\gamma} = \lambda : \mu : \nu$, i.e., say, $\sqrt{\alpha} = c\lambda, \sqrt{\beta} = c\mu, \sqrt{\gamma} = c\nu$, where $c \in \mathbb{Q}(\sqrt{m})$. Then $\sqrt{\alpha} = c(a_1 + a_2\sqrt{m})$, $\sqrt{\beta} = c(b_1 + b_2\sqrt{m})$, $\sqrt{\gamma} = c(c_1 + c_2\sqrt{m})$ and we want to solve for u, v, w (not all zero) in $\mathbb{Q}$, with $u\sqrt{\alpha} + v\sqrt{\beta} + w\sqrt{\gamma} = 0$, i.e., $u(a_1 + a_2\sqrt{m}) + v(b_1 + b_2\sqrt{m}) +$

$w(c_1 + c_2\sqrt{m}) = 0$, i.e., we want to solve the equations $ua_1 + vb_1 + wc_1 = 0$, $ua_2 + vb_2 + wc_2 = 0$, simultaneously for u, v, w, nontrivially, in $\mathbb{Q}$. As these are two equations in three variables, there exist a nontrivial solution u, v, w in $\mathbb{Q}$, as required.

Observe that when (2) is satisfied, (1) becomes $u\lambda + v\mu + w\nu = 0$.

As an application, we have the following:

Example 1.2.12. $59\sqrt{(90 - 14\sqrt{7})}+4\sqrt{(4555 + 1721\sqrt{7})} = 145\sqrt{(26 + 2\sqrt{7})}$.

Indeed, let $m = 7$ and let $\alpha = 90-14\sqrt{7}$, $\beta = 4555+1721\sqrt{7}$, $\gamma = 26+2\sqrt{7}$. We need to verify that there exist u, v, w (given respectively as 59, 4, -145) such that $u\sqrt{\alpha} + v\sqrt{\beta} + w\sqrt{\gamma} = 0$ (i.e., (1)). It is enough to verify (2), i.e., that $\alpha : \beta : \gamma = \lambda^2 : \mu^2 : \nu^2$. We have $\alpha/\gamma = (90 - 14\sqrt{7})(26 + 2\sqrt{7}) = (317-68\sqrt{7})/81 = ((17-2\sqrt{7})/9)^2$, and $\beta/\gamma = (4555+1721\sqrt{7})/(26+2\sqrt{7}) = (47168+17818\sqrt{7})/324 = ((151+59\sqrt{7})/18)^2$, so $\alpha = ((17-2\sqrt{7})/9)^2 \cdot \gamma$, $\beta = ((151 + 59\sqrt{7})/18)^2 \cdot \gamma$ and of course $\gamma = 1^2 \cdot \gamma$. These give us the ratios $\lambda^2 : \mu^2 : \nu^2$ and proceeding as in the proof of Theorem 1.2.11, one checks easily that $u = 59, v = 4, w = -145$ is a solution of the system of equations

$$\begin{aligned} \frac{17}{9}u + \frac{151}{18}v + w &= 0 \\ \frac{-2}{9}u + \frac{59}{18}v &= 0 \end{aligned}$$

Remark 1.2.13. From elementary field theory it follows that if K/F and L/K are algebraic extensions, then so is L/F. From this, it is easy to deduce that the set of all algebraic real numbers is a subfield of $\mathbb{R}$, containing $\mathbb{Q}$, i.e., the sum, difference, product and quotient of two algebraic numbers is algebraic.

If θ is a rational number, it follows that $\cos\pi\theta$ and $\sin\pi\theta$ and hence $\cot\pi\theta = \frac{\cos\pi\theta}{\sin\pi\theta}$ are algebraic numbers (indeed, for $\theta = p/q$, $(\cos\frac{\pi p}{q} + \iota\sin\frac{\pi p}{q})^q = (e^{\iota\pi p/q})^q = e^{\iota\pi p} = \pm 1$, yields polynomial equations with rational coefficients satisfied by $\cos\pi\theta$ and $\sin\pi\theta$). As an application, we have:

Example 1.2.14. The number $g(\theta) = \sum_{n=1}^{\infty} \theta/(n(n+\theta))$ is transcendental (and hence irrational) for infinitely many rational numbers $\theta \in (0, 1)$.

Proof (sketch). Assuming the partial fraction expansion of $\pi\cot\pi\theta$, i.e.,

$$\pi\cot\pi\theta = \sum_{n=2}^{\infty} \frac{2\theta}{\theta^2 - n^2} \quad ,$$

the proof proceeds as follows:

$$g(\theta) = 1 - 1/(1+\theta) + \sum_{n=2}^{\infty}(1/n - 1/(n+\theta)) \;, \tag{1}$$

$$\begin{aligned} g(1-\theta) &= \sum_{n=1}^{\infty}(1/n - 1/(n+1-\theta)) \\ &= (1 - 1/(2-\theta)) + (1/2 - 1/(3-\theta)) + \cdots \\ &= 1 + \sum_{n=2}^{\infty}(1/n - 1/(n-\theta)) \,. \end{aligned} \tag{2}$$

(1) and (2) imply that

$$\begin{aligned} g(1-\theta) - g(\theta) &= \frac{1}{1+\theta} + \sum_{n=2}^{\infty}(2\,\theta/(\theta^2 - n^2)) \\ &= \frac{1}{1-\theta} + \sum_{n=1}^{\infty}(2\,\theta/(\theta^2 - n^2)) \\ &= \frac{1}{1-\theta} + \pi\cot\pi\theta \quad . \end{aligned}$$

Since for θ rational, $\cot\pi\theta$ is algebraic, whence $\pi\cot\pi\theta$ is transcendental (as π is transcendental, see next section), therefore, at least one of $g(1-\theta)$ and $g(\theta)$ is transcendental.

§1.3. The numbers e and π

In this section we investigate two special real numbers e and π, their irrationality and transcendence. We have tried to collect interesting facts about them at one place. We begin with the following:

Theorem 1.3.1. *(i) The series* $1 + 1/1! + 1/2! + \cdots$ *is convergent. Its limit is called* e*. Moreover* $2 < e < 3$.
(ii) The number e *is irrational.*
(iii) Let $a_n = (1+1/n)^{n+1}$, $b_n = (1-1/n)^n$, $c_n = (1+1/n)^n$*. Then* c_n *increases,* $a_n > c_n$ *for all* n *and* $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} c_n$ *both exist and are equal to* e*. Further,* $\lim_{n\to\infty} b_n$ *also exists and equals* $1/e$.

Proof. (i) The partial sums $S_n = 1 + 1/1! + 1/2! + \cdots + 1/n! \; < 1 + 1 + 1/2 + 1/2^2 + \cdots + 1/2^{n-1} = 3$, are bounded above by 3 and below by 2 and are clearly increasing as n increases and so tend to a limit e say, as required.

(ii) We have $0 < e - S_n$ and that

$$\begin{aligned} e - S_n &= 1/(n+1)! + 1/(n+2)! + \cdots \\ &= (1/(n+1)!)(1 + 1/(n+2) + 1/(n+3)(n+2) + \cdots) \\ &< (1/(n+1)!)(1 + 1/(n+1) + 1/(n+1)^2 + \cdots) \\ &= (1/(n+1)!)(1/(1 - 1/(n+1))) \\ &= 1/n! \cdot n\,. \end{aligned}$$

Now if $e = p/q$, say, with $1 \le p, q$, $(p, q) = 1$, then $0 < p/q - S_n < 1/n! \cdot n$. In this taking $n = q$ we get $0 < p/q - S_q < 1/q!\,.\,q$, and multiplying by $q\,!$, this becomes $0 < (q-1)! \cdot p - q! \cdot S_q < 1/q \le 1$. Here $(q-1)! \cdot p$ and $q\,! \cdot S_q$ are both integers and so their difference ρ say, is an integer, i.e., $0 < \rho < 1$, which is not possible.
(iii) We first prove that $c_n = (1 + 1/n)^n \to e$ as $n \to \infty$. Using the binomial theorem

$$\begin{aligned} (1+1/n)^n &= 1 + n(1/n) + (n(n-1)/2!)(1/n^2) + (n(n-1)(n-2)/3!)(1/n^3) + \cdots \\ &= 1 + 1 + (1/2!)(1 - 1/n) + (1/3!)(1 - 1/n)(1 - 2/n) + \cdots + \\ &\qquad + (1/n!)(1 - 1/n)(1 - 2/n) \cdots (1 - (n-1)/n) \qquad (*) \\ &< 1 + 1 + 1/2! + 1/3! + \cdots + 1/n! \\ &< e \end{aligned}$$

Further, each term on the right hand side of (*) increases as n increases and the number of terms increases too as n increases, i.e., $(1 + 1/n)^n$ increases as n increases, and it is bounded above by e and so tends to a limit $\eta \le e$. However, $(1+1/n)^n \ge 1+1+(1/2!)(1-1/n)+(1/m!)(1-1/n)(1-2/n)\ldots(1-(m-1)/n)$, (by (*) again), if $n \ge m$. Now keeping m fixed and letting $n \to \infty$, we get that $\eta \ge 1 + 1 + 1/2! + \cdots + 1/m!$ and this is true for all m. Letting $m \to \infty$, we get $\eta \ge e$.

Short proofs of the other parts are as follows. The arithmetic mean of the $n + 2$ numbers $1,\ n/(n+1),\ n/(n+1),\ \ldots\ ,\ n/(n+1)$ is greater than their geometric mean. This gives $(1 + n)/(n + 2) > ((n/(n + 1))^{n+1})^{1/(n+2)}$, i.e., on taking reciprocals $1+1/(n+1) < (1+1/n)^{(n+1)/(n+2)}$, i.e., $(1+1/(n+1))^{(n+2)} < (1 + 1/n)^{(n+1)}$, showing that a_n decreases as n increases.

Next, $b_{n+1} = (1 - 1/(n+1))^{(n+1)} = (n/(n+1))^{(n+1)} = 1/a_n$; hence b_n increases, as required.

Finally, the arithmetic mean of the $n+1$ numbers $1,\ 1+1/n,\ 1+1/n,\ \ldots\ ,\ 1+1/n$ is greater than their geometric mean. Hence $(1 + n(1 + 1/n))/(n + 1) > ((1+1/n)^n)^{1/(n+1)}$, i.e., $((1+1/(n+1))^{(n+1)} > (1+1/n)^n$, showing that c_n increases as n increases. Since $c_n \to e$ (already proved), so $a_n = c_n(1+1/n) \to e$ too. Finally, $1/b_n = 1/a_{n-1} \to 1/e$ as required.

Remark 1.3.2. The value of e has been calculated with great accuracy (see [63], page 101):

$$2.71828182\ldots < e < 2.71828184\ldots$$

We next consider the number π. A reference to an excellent account of π, its computation and all other developments through the ages, is the exhaustive article " The Ubiquitous π ", by Dario Castellanos [22].

With the help of computers, π has now been calculated to thousands of decimal places:

$$\pi = \mathbf{3.14159265358979323846}\ldots .$$

Many *mnemonics* have been composed to remember and write down the value of π, the number of letters in each word representing successive digits of π. A well known mnemonic is: *How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics.*(15 digits).

One of the most basic result in which π appears is the sum of the following infinite series:

Theorem 1.3.3. $1 + 1/2^2 + 1/3^2 + 1/4^2 + \cdots \;=\; \pi^2/6.$

We shall present here a simple proof given in [78]. First we prove the following simple

Lemma 1.3.4. $\sum_{k=1}^{m} \cot^2 (k\pi/(2m+1)) \;=\; m(2m-1)/3$.

Proof. By equating the imaginary parts in the formula

$$\begin{aligned}\cos n\,\theta + \iota \sin n\,\theta = (\cos\theta + \iota\sin\theta)^n &= \sin^n\theta\,(\cot\theta + \iota)^n \\ &= \sin^n\theta \sum_{k=0}^{n} \binom{n}{k} \cdot \iota^k \cot^{n-k}\theta ,\end{aligned}$$

we obtain the identity

$$\sin n\theta = \sin^n\theta \left[\binom{n}{1}\cot^{n-1}\theta - \binom{n}{3}\cot^{n-3}\theta + \binom{n}{5}\cot^{n-5}\theta - \cdots\right],$$

where $0<\theta<\pi/2$. Take $n = 2m+1$ and write this in the form $\sin(2m+1)\theta = \sin^{2m+1}\theta \cdot P_m(\cot^2\theta)$, where $P_m(x) = \binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \binom{2m+1}{5}x^{m-2} - \cdots + \binom{2m+1}{2m+1}$ is a polynomial of degree m. Since $\sin\theta \neq 0$, if $0 < \theta < \pi/2$, it follows from above that $P_m(\cot^2\theta) = 0$ if and only if $(2m+1)\theta = k\pi$, $k \in \mathbb{Z}$. Therefore, $P_m(x)$ vanishes at the m distinct points $x_k = \cot^2(\pi k/(2m+1))$, for $k = 1, 2, \ldots, m$. Since the degree of $P_m(x)$ is m, these are all the zeros of $P_m(x)$ and their sum is $-(-\binom{2m+1}{3}/\binom{2m+1}{1})$, which proves the lemma.

Proof of Theorem 1.3.3. Start with the inequality $\sin x < x < \tan x$ for $0< x <\pi/2$. Take reciprocals and square to obtain $\cot^2 x < 1/x^2 < 1 + \cot^2 x$. Now put $x = k\pi/(2m+1)$, where k, m are integers, $1 \le k \le m$, and sum from $k = 1$ to $k = m$ to get

$$\sum_{k=1}^{m} \cot^2(k\pi/(2m+1)) < ((2m+1)^2/\pi^2) \sum_{k=1}^{m} 1/k^2 < m + \sum_{k=1}^{m} \cot^2(k\pi/(2m+1)).$$

Putting in the value of the sums, using the lemma, we get

$$m(2m-1)/3 < ((2m+1)^2/\pi^2)\sum_{k=1}^{m} 1/k^2 < m + m(2m-1)/3\,,$$

which, on dividing throughout by $(2m+1)^2/\pi^2$ and then letting $m \to \infty$, proves the theorem.

One of the most beautiful expressions for π, given by John Wallis in 1655, called Wallis' product, is the following:

Theorem 1.3.5.

$$\frac{\pi}{2} = \lim_{m\to\infty}\left(\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdots(2m)\cdot(2m)}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdots(2m-1)\cdot(2m+1)}\right).$$

Proof. Let $I_n = \int_0^{\pi/2}\sin^n\theta\,dx$. Integration by parts easily gives $I_n = ((n-1)/n)I_{n-2}$ $(n>1)$. Since $I_0 = \pi/2$, $I_1 = 1$, it follows that

$$I_{2m} = ((2m-1)/2m)\cdot((2m-3)/(2m-2))\cdots(1/2).(\pi/2) \qquad (1)$$

$$I_{2m+1} = (2m/(2m+1))\cdot((2m-2)/(2m-1))\cdots(2/3).1 \qquad (2)$$

Now in the range $0 \le \theta \le \pi/2$, we have $0 \le \sin\theta \le 1$ (in fact < 1 if $\theta < \pi/2$) and so $\sin^n\theta > \sin\theta.\sin^n\theta = \sin^{n+1}\theta$ for all $\theta \in (0,\pi/2)$. It follows that (see Figure 1.1) $0 < I_{2m+1} < I_{2m} < I_{2m-1}$.

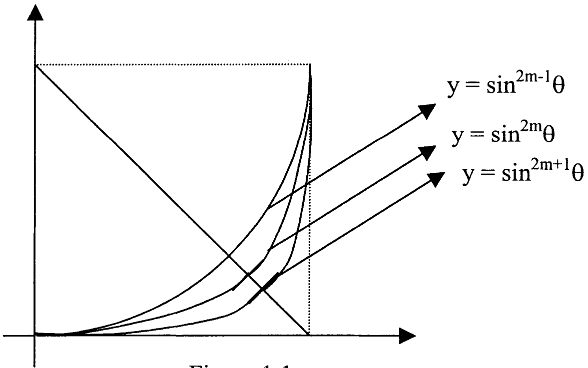

Figure 1.1

Dividing by I_{2m+1} gives

$$1 < I_{2m}/I_{2m+1} < I_{2m-1}/I_{2m+1} \qquad (*)$$

Here the extreme right hand term is equal to $(2m+1)/2m$ (see (1) and (2)), which tends to 1 as $m \to \infty$. So letting $m \to \infty$ in (*), we get $1 \le \lim_{m\to\infty} I_{2m}/I_{2m+1} \le 1$. However using (1) and (2), we get

$$\frac{I_{2m}}{I_{2m+1}} = \left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{2m-1}{2m-2}\right)\left(\frac{2m-1}{2m}\right)\left(\frac{2m+1}{2m}\right),$$

and letting $m \to \infty$, we get the result.

Corollary 1.3.6.

$$\lim_{m\to\infty} \frac{(2^m\, m!)^4}{((2m)!)^2 \cdot (2m+1)} = \frac{\pi}{2}.$$

Proof. The right side of Wallis' product is

$$\begin{aligned}\frac{2\cdot 2\cdot 4\cdot 4\cdots 2m\cdot 2m}{1\cdot 3\cdot 5\cdots(2m-1)(2m+1)} &= \frac{(2\cdot 4\cdots 2m)^2}{(1\cdot 3\cdot 5\cdots(2m-1))^2(2m+1)}\\ &= \frac{(2^m\cdot m!)^2}{(1\cdot 3\cdot 5\cdots(2m-1))^2(2m+1)}\cdot\frac{(2\cdot 4\cdots 2m)^2}{(2\cdot 4\cdots 2m)^2}\\ &= \frac{(2^m\cdot m!)^2\cdot(2^m\cdot m!)^2}{(1\cdot 2\cdot 3\cdot 4\cdots(2m-1)\cdot 2m)^2(2m+1)}\\ &= \frac{(2^m\cdot m!)^4}{((2m)!)^2(2m+1)}.\end{aligned}$$

We shall now describe some methods to compute the value of π.

1. By inscribed and circumscribed polygons. Euclid, in the fourth century B.C., had proved that $3 < \pi < 4$, but it was not until the third century B.C. that Archimedes attacked the problem of the determination of π systematically. Using polygons inscribed in and circumscribed to the circle, whose number of sides are successively doubled, he obtained for π the bounds $3\frac{10}{71} < \pi < 3\frac{1}{7}$. The bound $3\frac{1}{7} = \frac{22}{7}$ is often referred to, erroneously, as the Archimedian value. Archimedes meant this as an upper bound on the value of π.

Archimedes' method remained essentially unchanged except for better approximations to π obtained by taking larger and larger doublings, until the advent of Calculus.

2. An analytic expression for π . Consider the identity

$$\begin{aligned}(\sin\theta)/\theta &= \cos(\theta/2)\cdot\sin(\theta/2)/(\theta/2)\\ &= \cos(\theta/2)\cdot\cos(\theta/4)\cdot\sin(\theta/4)/(\theta/4)\\ &\cdots\quad\cdots\quad\cdots\\ &= \cos(\theta/2)\cdot\cos(\theta/4)\cdot\ \ldots\ \cdot\cos(\theta/2^n)\cdot\sin(\theta/2^n)/(\theta/2^n).\end{aligned}$$

As $n\to\infty$, $\sin\theta/2^n/(\theta/2^n)\to 1$ and we obtain Euler's formula

$$(\sin\theta)/\theta = \lim_{n\to\infty}(\cos(\theta/2)\cdots\cos(\theta/2^n)) = \cos(\theta/2)\cdot\cos(\theta/4)\cdot\cos(\theta/8)\ldots$$

Putting $\theta = \pi/2$, this gives, on use of the formula $\cos(\theta/2) = \sqrt{(1+\cos\theta)/2}$, the following result, giving $2/\pi$ as a limit of a sequence, which was first given by Fancois Vieta in 1593:

$$2/\pi = \sqrt{(1/2 + 1/2\sqrt{(1/2)})}\cdot\sqrt{(1/2+1/2\sqrt{(1/2+1/2\sqrt{(1/2)})})}\cdot\cdots\ .$$

The convergence of this expression was proved by F.Rudio in 1891 (see [98]). Vieta's formula is the first analytical expression ever obtained for π.

Now, taking logarithms in the Euler's formula above (noting that the logarithm is a continuous function):

$$\log\sin\theta - \log\theta = \log\cos\theta/2 + \log\cos\theta/4 + \log\cos\theta/8 + \cdots$$

On differentiating with respect to θ, this gives

$$1/\theta = \cot\theta + (1/2)\tan(\theta/2) + (1/4)\tan(\theta/4) + (1/8)\tan(\theta/8) + \cdots$$

Putting $\theta = \pi/4$, we obtain

$$4/\pi = 1 + (1/2)\tan(\pi/8) + (1/4)\tan(\pi/16) + (1/8)\tan(\pi/32) + \cdots .$$

The calculation of the number π by this formula is equivalent to a geometrical calculation. It was undertaken by Rene Descartes in the seventeenth century .

There have since been many formulae giving π as a rapidly converging series. These make use of various methods, as for example Euler's summation formula, Γ-function and the Taylor's series for $\tan^{-1} x$. We give here an arctan series formula for the calculation of π.

3. The arctan method. The formula

$$\pi/4 = 4\tan^{-1}(1/5) - \tan^{-1}(1/239) \tag{i}$$

can be derived as follows:

Let α be the angle given by $\tan\alpha = 1/5$; then $\tan 2\alpha = 2\tan\alpha/(1 - \tan^2\alpha) = 5/12$ and $\tan 4\alpha = 2\tan 2\alpha/(1-\tan^2 2\alpha) = 120/119 = \tan(\pi/4) + 1/119$. Thus $4\alpha > \pi/4$, say $4\alpha - \pi/4 = \beta$, so that $\tan\beta = (\tan 4\alpha - \tan\pi/4)/(1+\tan 4\alpha\tan\pi/4) = 1/239$, which finally gives $4\alpha - \pi/4 = \beta = \tan^{-1}(1/239)$, i.e., $\pi/4 = 4\tan^{-1}(1/5) - \tan^{-1}(1/239)$, as required. This formula was discovered by John Machin in 1706.

We also have the expression $1/(1+x^2) = 1 - x^2 + x^4 - x^6 + \cdots$, $\mid x \mid < 1$. As the above series is uniformly convergent, we may integrate term by term to get

$$\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1) \qquad (\mid x \mid < 1) \tag{ii}$$

This is the series discovered by James Gregory in 1671, which we shall obtain in Chapter 7. For $x = 1$, it gives the Leibnitz's celebrated series $\pi/4 = 1 - 1/3 + 1/5 - \cdots$ which requires 2000 terms to give three decimal figures of π. Using (i) and (ii), we get

$$\begin{aligned}\pi = {} & (16/5)(1 - 1/(3\cdot 25) + 1/(5\cdot 25^2) - 1/(7\cdot 25^3) + \cdots) \\ & -(4/(239))(1 - 1/(3\cdot 57121) + 1/(5\cdot 57121^2) - 1/(7\cdot 57121^3) + \cdots)\end{aligned}$$

which is well suited for the calculation of π. In 1706, using this series, Machin did the computation to 100 decimal digits. Similar series for arctan have been used by many to compute π accurately. The reader is referred to pages 90–96 of [22].

In 1882, Lindemann proved that π is transcendental (see [74]). Here we give Niven's ingenious proof (see [75]) that π is irrational.

Theorem 1.3.7. *The real number π is irrational.*

Proof. Suppose $\pi = a/b$ $(a, b \in \mathbb{N})$. Define polynomials $f(x) = x^n(a-bx)^n/n!$ and $g(x) = f(x) - f''(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)$, where the positive integer n will be chosen later. Now,

$$\begin{aligned} n!\,f(x) &= x^n\left(a^n + \binom{n}{1}a^{n-1}(-bx) + \binom{n}{2}a^{n-2}(-bx)^2 + \cdots + (-bx)^n\right) \\ &= a^n x^n - \binom{n}{1}a^{n-1}bx^{n+1} + \binom{n}{2}a^{n-2}b^2x^{n+2} - \cdots + (-1)^n b^n x^{2n}\,. \end{aligned}$$

Differentiating successively we get

$$\begin{aligned} n!f'(x) =& na^nx^{n-1} - \binom{n}{1}(n+1)a^{n-1}bx^n + \binom{n}{2}(n+2)a^{n-2}b^2x^{n+1} - \cdots + \\ & (-1)^n b^n 2nx^{2n-1}\,, \\ n!f''(x) =& n(n-1)a^nx^{n-2} - \binom{n}{1}(n+1)na^{n-1}bx^{n-1} + \\ & + \binom{n}{2}(n+2)(n+1)a^{n-2}b^2x^n - \cdots + (-1)^n b^n 2n(2n-1)x^{2n-2}\,, \end{aligned}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$n!f^{(2n)}(x) = (-1)^n b^n . 2n(2n-1). \ldots .2.1\,.$$

The last equation is the only one which is free of x. It follows that $f(0) = f'(0) = \ldots f^{(2n-1)}(0) = 0$, while $n!f^{(2n)}(0) = (2n)!\,(-1)^n\,b^n$, so that $f^{(2n)}(0)$ is an integer. In any case:

$$f(0)\,,\ f'(0)\,,\ \ldots\,,\ f^{(2n)}(0) \text{ are all integers.} \tag{1}$$

Further, it can be easily checked that $f(x) = f(a/b - x)$ for all x and so differentiating successively, we get $f'(x) = -f'(a/b - x)$, $f''(x) = +f''(a/b - x)$, ... Hence $f(\pi) = f(a/b - \pi) = f(0) = 0$, $f'(\pi) = f'(0) = 0, \ldots,$ $f^{(2n-1)}(\pi) = f^{(2n)}(0) = ((2n)!/n!)(-1)^n b^n \in \mathbb{Z}$. In any case

$$f(\pi),\ f'(\pi),\ f''(\pi), \ldots,\ f^{(2n-1)}(\pi),\ f^{(2n)}(\pi) \text{ are all integers.} \tag{2}$$

Note that in (1) and (2), only $f^{(2n)}(0)$ and $f^{(2n)}(\pi)$ are nonzero, the rest are in fact zero. Now,

$$\begin{aligned}\frac{d(g'(x)\sin x - g(x)\cos x)}{dx} &= g'(x)\cos x + \sin x g''(x) + g(x)\sin x - \cos x g'(x) \\ &= (g(x) + g''(x))\sin x\,.\end{aligned}$$

But $g = f - f^{(2)} + f^{(4)} - \cdots + (-1)^n f^{(2n)}$, and so $g'' = f^{(2)} - f^{(4)} + f^{(6)} - \cdots + (-1)^n f^{(2n+2)}$. Adding, we get $g + g'' = f + (-1)^n f^{(2n+2)} = f$, as $f^{(2n)} = (-1)^n b^n (2n)!/n!$, and so $f^{(2n+1)} = f^{(2n+2)} = 0$. Thus $\frac{d(g'(x)\sin x - g(x)\cos x)}{dx} = f(x)\sin x$. Hence

$$\begin{aligned}\int_0^\pi f(x)\sin x\,dx &= \Big[g'(x)\sin x - g(x)\cos x\Big]_0^\pi \\ &= g(\pi) + g(0) \\ &= f(\pi) - f^{(2)}(\pi) + \cdots + (-1)^n f^{(2n)}(\pi) + f(0) - f^{(2)}(0) + \cdots \\ &\quad + (-1)^n f^{(2n)}(0) \\ &= (-1)^n (f^{(2n)}(\pi) + f^{(2n)}(0)) \quad \text{(the rest being all zero)}\,, \\ &= 2(-1)^n (2n)!\, b^n\, /n\,!\,,\end{aligned}$$

which is an integer, since $(2n)!\,/n!$ is an integer. Thus

$$\int_0^\pi f(x)\sin x\,dx \quad \text{is an integer.} \tag{3}$$

However, for $0 < x < \pi$, we have $0 < f(x)\sin x < \pi^n a^n / n! 2^{2n}$, because, $f(x) = x^n (a - bx)^n / n!$; hence $f'(x) = x^n . n(a - bx)^{n-1}(-b) + (a - bx)^n . n x^{n-1}/n! = 0, i.e., x^{n-1}(a - bx)^{n-1}(-bx + (a - bx)) = 0$, giving $x = 0$, or $a/b = \pi$, or $a/2b = \pi/2$. But as $0 < x < \pi$, the only relevant solution is $x = a/2b = \pi/2$. The value at 0 and π is 0, so that $\pi/2$ is a maximum and this maximum value is $f(a/2b)\sin \pi/2 = (a/2b)^n (a - b.a/2b)^n / n! = (a/2b)^n (a^n)/2^{2n} n! = \pi^n a^n / 2^{2n} n!$. Now, $\pi^n a^n / 2^{2n} n! \to 0$ as $n \to \infty$, so if we choose n large enough, we can ensure that $0 < f(x)\sin x < 1/5$, and so

$$0 < \int_0^\pi f(x)\sin x\,dx < \pi/5 < 1\,,$$

which is a contradiction to (3). This proves that π is irrational.

We now turn to the transcendence of e and π, which is considerably more difficult to establish than their irrationality.

A complex number is said to be algebraic if, as in the case of real numbers, it is a root of some non-constant polynomial with rational coefficients. The transcendence of e and π can be deduced from:

Theorem 1.3.8 (Lindemann). *If $A_1, \ldots, A_n; a_1, \ldots, a_n$ are algebraic numbers (possibly complex), $a_i \neq a_j$ for $i \neq j$, $A_i \neq 0$ for all i, then $\sum_{i=1}^n A_i e^{a_i} \neq 0$.*

For a proof of this result, the reader is referred to [101].

Corollary 1.3.9. *The number e is transcendental.*

Proof. Take arbitrary $A_1, \ldots, A_n \in \mathbb{Q}$, then $A_1 e + A_2 e^2 + \cdots + A_n e^n \neq 0$.

Corollary 1.3.10. *The number π is transcendental .*

Proof. If π is algebraic, then, as observed in Remark 1.2.13, $\pi\iota$ is algebraic (as ι obviously is) and hence by Theorem 1.3.8, $e^{\pi\iota} + e^0 \neq 0$, but $e^{\pi\iota} = -1$ and $e^0 = 1$; which gives a contradiction.

It is not known whether the numbers πe and $\pi + e$ are transcendental or algebraic. However , we have the following interesting:

Theorem 1.3.11 ([17]). *At least one of the numbers πe and $\pi + e$ is transcendental.*

Proof. Suppose to the contrary, that both πe and $\pi + e$ are algebraic. By Remark 1.2.13, the number $A = (\pi + e)^2 - 4\pi e = (\pi - e)^2$ is algebraic. It is easy to see that if α^2 is an algebraic number, then so is α. Thus $\pi - e$ and hence $(\pi + e) + (\pi - e) = 2\pi$ would be algebraic, which is not true.

§1.4. The Cantor ternary set

In this section we describe and establish some properties of the Cantor ternary set, which is a source of several counter examples in real function theory. As a preparation, we first discuss some results in point set topology.

Theorem 1.4.1. *Let O be an open and bounded subset of $\mathbb{R}$. Then O can be uniquely expressed as a disjoint union of (at most) countably many open intervals.*

Proof. Let $a \in O$. Since O is open, there exist an open interval $(a - \epsilon, a + \epsilon) \subseteq O$. Now consider the closed set $S = (-\infty, a] \cap O^{\,c}$ (here $O^{\,c}$ denotes the complement of O), i.e., the points not in O, but lying to the left of a. Let $\alpha = \sup S$, which we know belongs to S, since S is closed and bounded above. It follows that $(\alpha, a] \subseteq O$.

Similarly, let $\beta = \inf([a, \infty) \cap O^{\,c})$, so that β is the smallest element of $[a, \infty) \cap O^{\,c}$. Then again $[a, \beta) \subseteq O$. Hence $(\alpha, \beta) \subseteq O, \alpha, \beta \notin O$, i.e., (α, β) is the 'largest' open interval contained in O which contains the point a. It is called a *component interval* of O and is clearly uniquely determined by a (or indeed by any other point of (α, β) as the starting point instead of the point a).

Thus, O is a disjoint union of open intervals, the component intervals. To see that they are at most countable, we select a rational number in each of them. Being disjoint intervals, these rationals are distinct so that there is a 1–1 correspondence between these component intervals of O and a subset of the rationals.

Now let F be a closed, bounded nonempty set and let $\alpha = \inf F$, $\beta = \sup F$, so that $\alpha, \beta \in F$, since F is closed. The interval $[\alpha, \beta]$ is called *the smallest* *closed interval containing* F. Let $O = [\alpha, \beta] \setminus F$. It is easy to see that O is open. Indeed, $O = F^c \cap [\alpha, \beta] = F^c \cap (\alpha, \beta)$ (since $\alpha, \beta \notin F^c$), which is an open set, since F^c and (α, β) are both open. It follows, by Theorem 1.4.1 that O is a disjoint union of at most countably many open intervals. These are called *the complementary intervals of* F (five of them are shown in Figure 1.2).

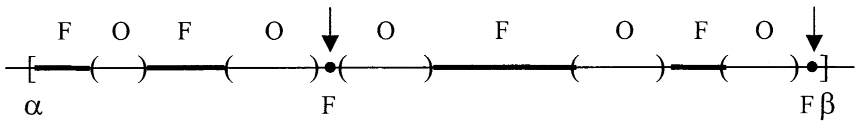

Figure 1.2

Since F is closed, F', the set of all limit points of F, is contained in F. We ask: When is $F \subseteq F'$, i.e., when is each point of F a limit point of F? Such closed subsets of $\mathbb{R}$ are called perfect sets. Now a point of F (indeed of any set) is either an *isolated point (i.e., a point such that there exists some open interval centred at the point which contains no point of* F *other than itself*) of F or a *limit point* of F. Thus, if F has no isolated point, then all points of F are limit points of F, i.e., $F \subseteq F'$ (i.e., F is dense in itself). Since $F' \subseteq F$ (F being closed), we get $F = F'$, i.e., F is perfect. We characterize all the isolated points of F in the following:

Theorem 1.4.2 *Let* F *be a closed, bounded, nonempty subset of* $\mathbb{R}$. *A point* γ *of* F *is an isolated point of* F *if and only if*
(i) γ *is a common end point of two complementary intervals of* F,
or *(ii)* $\gamma = \alpha$, *where* $(\alpha, \alpha + \delta)$ *is a complementary interval of* F,
or *(iii)* $\gamma = \beta$, *where* $(\beta - \delta', \beta)$ *is a complementary interval of* F.

Remark 1.4.3. The theorem says that either points of the type y shown in Figure 1.3, i.e.,

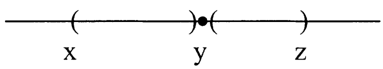

Figure 1.3

the common end points of the two complementary intervals (x, y) and (y, z) of F are isolated points of F or α is an isolated point of F if a complementary interval of F begins at α, i.e., is $(\alpha, \alpha + \delta)$

Figure 1.4

or β is an isolated point of F if a complementary interval of F ends at β, i.e., is $(\beta - \delta', \beta)$ and that there are no other isolated points of F.

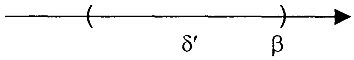

Figure 1.5

Proof of Theorem 1.4.2. The points mentioned in the theorem are clearly isolated points of F. Conversely, let γ be an isolated point of F. Then there exists an $\epsilon > 0$ such that $(\gamma - \epsilon, \gamma + \epsilon)$ has no point of F except γ, i.e., $(\gamma - \epsilon, \gamma) \subseteq O$, where O is $[\alpha, \beta] \smallsetminus F$, as defined above (proof to be modified for $\gamma = \alpha$) and $(\gamma, \gamma + \epsilon) \subseteq O$ (proof to be modified for $\gamma = \beta$), i.e., $(\gamma - \epsilon, \gamma)$ and $(\gamma, \gamma + \epsilon)$ are both subsets of two complementary intervals of F meeting at γ, as required.

For $\gamma = \alpha$ or $\gamma = \beta$, a similar argument yields the result.

Theorem 1.4.4. *The set S of all sequences of the type $(a_1, a_2, a_3, \dots)$ where $a_i = 0$ or 1, is uncountable.*

Proof. Indeed there is a one-to-one correspondence between S and all the real numbers in $[0, 1]$, except for at most countably many exceptions. In fact, writing each real number $r \in [0, 1]$ in its binary expansion , i.e., $r = .a_1 a_2 a_3 \dots$ $(a_i = 0, 1)$, we see that the mapping $r \leftrightarrow (a_1, a_2, a_3, \dots)$ is a one to one, onto mapping between S and $[0, 1]$, the exceptions mentioned being those reals which have two expansions, viz. certain rational numbers as for example $1/2 = .011111\dots = .100000\dots$, and being a subset of rational numbers, there are at most countably many of these.

We are now in a position to define and establish some properties of the curious Cantor ternary set $\mathcal{C}$. Let $F = [0, 1]$. Now remove the middle third open interval $(1/3, 2/3)$. From the two remaining intervals, again remove the middle third open intervals, i.e., remove $(1/9, 2/9)$ and $(7/9, 8/9)$, and so on. What remains, is the Cantor ternary set $\mathcal{C}$. Thus, for example, the points

$$0,\ 1\,;\ 1/3\,,\ 2/3\,;\ 1/9,\ 2/9,\ 7/9,\ 8/9\,;\dots$$

will never be removed and therefore belong to $\mathcal{C}$ (see Figure 1.6):

Figure 1.6

As $\mathcal{C}$ is a countable intersection of closed sets $F_1 = [0, 1]$, $F_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $F_3 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, ..., it follows that $\mathcal{C}$ is a closed set. Since no two removed intervals are of the type (x, y), (y, z), we see that $\mathcal{C}$ has no isolated point (see Theorem 1.4.2); in fact, we get:

Theorem 1.4.5. *The subset $\mathcal{C}$ of $\mathbb{R}$ is perfect, i.e., $\mathcal{C}' = \mathcal{C}$.*

To understand $\mathcal{C}$ more clearly, it is essential to look at the *ternary expansion* of real numbers. Let us therefore start first with a digression on the decimal expansion of real numbers $r \in [0,1]$.

Each r has a unique decimal expansion: $r = .r_1r_2r_3...$ $(r_i = 0,1,2,\dots,9)$, except those r for which $r_i = 0$ for all $i \geq i_0$, for some i_0 or for which $r_i = 9$ for all $i \geq i_0'$ for some i_0'. For example:

$$\begin{aligned} 1/2 &= .500000\dots &= .499999\dots\ , \quad &\text{i.e. } r_i = 0 \text{ or } 9 \text{ for } i \geq 2\ ; \\ 1/4 &= .250000\dots &= .249999\dots\ , \quad &\text{i.e. } r_i = 0 \text{ or } 9 \text{ for } i \geq 3\ ; \\ 1/125 &= .008000\dots &= .007999\dots\ , \quad &\text{i.e. } r_i = 0 \text{ or } 9 \text{ for } i \geq 4\ ; \end{aligned}$$

etc. Such numbers are a subset of the rational numbers and therefore countable. Note further that

- the numbers in the interval $[0,1/10]$ will have an expansion of the type $r = .0r_2r_3\dots$, beginning with a 0. For example, $1/10 = .09999\dots(= .10000\dots)$, $1/11 = .09090\dots$, $1/12 = .08333\dots$, etc. ,
- the numbers in the interval $[1/10,2/10]$ have an expansion of the type $r = .1r_2r_3\dots$, beginning with a 1. For example $1/10 = .1000\dots$ (this is in $[0,1/10]$ as well as in $[1/10,2/10]$), $1/5 = .1999\dots$ $(= .2000\dots)$ etc. ,

and so on. Finally ,

- the numbers in the interval $[9/10,1]$ have an expansion of the type $r = .9r_2r_3\dots$, beginning with a 9.

Next we divide each of $[0,1/10]$, $[1/10,2/10]$, ... , $[9/10,1]$ into ten equal parts again. Take the interval $[0,1/10]$, which gets divided into $[0,1/100]$, $[1/100,2/100]$, ..., $[9/100,10/100]$. Here the numbers in the first part have a 0 in the second place of their decimal expansion, those in the second part have a 1 in the second place and so on; and similarly for the third place, ...

Keeping this in mind, we now work in ternary expansions, so that we have just three digits $0,1,2$. Here each number in $[0,1/3]$ will begin its ternary expansion with a 0, while numbers in $[1/3,2/3]$ begin with a 1, and numbers in $[2/3,1]$ with a 2. Note that $1/3$ and $2/3$ both fall into two categories.

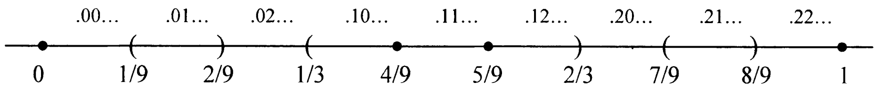

Figure 1.7

Next when we divide $[0,1/3]$ into three equal parts, the numbers in $[0,1/9]$ will have a 0 in the second place, those in $[1/9,2/9]$ a 1 in the second place and those in $[2/9,1/3]$ a 2 in the second place and so on; and so on for the third, fourth, ..., place. The full chart, up to nine parts, is as in Figure 1.7. Looking at the above chart, we easily get the following: