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TEXTS AND READINGS IN MATHEMATICS 30

# An Expedition to Geometry

S. Kumaresan
and
G. Santhanam

HINDUSTAN BOOK AGENCY

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Dedicated to My Parents

**G. Santhanam**

Dedicated to My Students

**S. Kumaresan**

# Contents

# Preface

This book is written as an attempt to entice the students of mathematics to the world of geometry. In spite of the great strides made in modern geometry at research level, there is no introductory book which gives modern aspects of geometry at undergraduate/graduate level. Our book is an attempt to fill this vacuum. In this endeavor, we have taken great care in the selection of topics and their treatment. Our guiding principles are

1. to cultivate geometric intuition in readers,
2. to give a panoramic view of various facets of geometry,
3. to give a modern treatment of intuitively appealing classical results, and
4. to employ as much as possible the methods which are close to the ones adopted in contemporary geometry.

Some of the material in this book has already been used in the Mathematics Training and Talent Search (MTTS) Programme and at various other places. The response to the course has been very enthusiastic both from students and teachers. Many of them insisted on our bringing out the privately circulated notes in book form. This book is thus an outcome of the cumulative effects of such requests.

Unlike some books on geometry, our work takes a holistic view of geometry. It introduces the readers to axiomatic, algebraic, analytic and differential geometry.

The first chapter introduces non-Euclidean geometry in an informal and engaging way. Chapters 2 to 6 put the geometries introduced in the first chapter on a rigorous footing. These may be considered as an explication of the Kleinian view of geometry *a la* Erlangen Programme. Discerning readers will find that we have gone beyond the Kleinian view of geometry in some of the topics in these chapters.

One of the hallmarks of this book is a completely rigorous discussion on the non-Euclidean geometries: Poincaré upper half plane or hyperbolic plane and the spherical plane. The general public, including the students of mathematics, looks at non-Euclidean geometry with some awe. We hope to show that these geometries are as *natural* as Euclidean geometry.

The spaces studied in chapters on Euclidean geometry, hyperbolic geometry and spherical geometry all are spaces of constant curvature. The eighth chapter "Theory of Surfaces" introduces the reader to the spaces of variable curvature in a geometric way. It is our firm belief that the reader who understands these chapters will be better prepared to plunge into modern differential geometry.

The most important features of our discussion are: (i) proving various results about triangles in these geometries which bring a perspective about them and (ii) affine and projective classification of conics. We are confident that diligent readers will notice that at many places our treatment is original and geometric. To the best of our knowledge, some of the topics, notably, transitive groups on conics, areas of geodesic triangles in $\mathbb{H}^2$ and $S^2$, two-point homogeneity of $\mathbb{E}^2$, $\mathbb{H}^2$ and $S^2$ and the fact that the set of distance preserving maps (isometries) is essentially the same as the set of length-preserving maps of these spaces appear for the first time in a book at this level in an accessible form with complete details. (See Remark 7.6.2.)

Chapters 2–5 can be used as a one-semester course at the undergraduate level and Chapters 5–8 can be used for a one-semester course at graduate level. The whole book can be covered in a one year course on geometry at a leisurely pace.

We take this opportunity to record our thanks to the Resident Faculty who have used the preliminary set of notes at MTTS camps and offered us their suggestions. We thank Akhil Ranjan and Amber Habib for their valuable comments. We also thank the participants of the MTTS camps for their overwhelming response to a course on geometry which was based on an earlier version of the manuscript. We thank Jugal Verma whose repeated enquiries about the manuscript goaded us to complete our project of writing this book. The book owes its birth to them! We also record our sincere thanks to Ajit Kumar for drawing all the figures in the book.

The first author acknowledges the invitation by the Department of Mathematics, Indian Institute of Technology, Kanpur, during the final stages in the preparation of the manuscript. The second author would like to place on record his appreciation of the invitation by the Department of Mathematics, University of Mumbai, which facilitated the preparation of the manuscript.

The unsung heroes in the world of publishing of technical books are the referees who remain anonymous. We record our sincere and heartfelt thanks to the referee whose diligent and pains-taking efforts have made this book more readable and eradicated many vague or imprecise statements. Of course, we take the responsibility for all remaining mistakes and errors of commission and omission.

**A Few Words to the Readers:** Our primary aim has been to impart a feeling for geometry. Nowadays, geometry is not taught at colleges.

The students, more often than not, approach Geometry with a sense of trepidation. They find it difficult to relate to the abstract concepts in an intuitive and geometric way.

We have emphasized the geometric intuition throughout the book. Figures are included wherever needed to make the geometric ideas clear. It is possible that some of the readers may find this an overkill. Also, some may find it difficult to see how the geometric ideas and the rigour relate to each other. While we accept that the book may be difficult at some places, we are sure that mulling over the material will enhance the readers' appreciation of Geometry. We would be pleased to hear that our book renewed their interest in Geometry.

We have a reasonable number of exercises varying from computational problems to investigative or explorative open questions. We believe that most of them are accessible and some of them are challenging.

At many places, we have given more than one proof. Our purpose was to make the reader realize that if he thinks on his own, he may be able to discover proofs which differ from the ones we have given. In fact, the alternative proofs were discovered by us while revising the manuscript.

In order to maintain the flow of the arguments, we have relegated the reasons (which are answers to "why?" questions traditionally interspersed in the proofs) as separate paragraphs of less width.

**Prerequisites:** For Chapters 1 to 4, the reader is expected to have a reasonable knowledge of Linear Algebra and the basic ideas in Group Theory. An acquaintance with group action will be an added advantage. However, for the benefit of the readers, we have discussed group action in the appendix with concrete examples.

For Chapter 5, we assume knowledge of inner product spaces, though we review some of the important results. For Chapters 5 to 8, we assume some basic knowledge of calculus of several variables.

An explanation: We are asked by all who went through the preliminary versions the reason for the title, especially concerning the word 'expedition'. We reproduce the meaning of this word as in the Cambridge International Dictionary of English:

> **expedition** *n* An organized journey for a particular purpose.

Suggestions for improvements and inclusion of topics are most welcome and may be sent to `santhana@iitk.ac.in`

Authors

# Chapter 1

# Introduction

In this introduction, we take a pedestrian approach and introduce the reader to three rich plane geometries. The rest of the book will put almost all the results of this introduction on a rigorous footing.

The phrase plane geometries should conjure up an image of points, lines and the incidence relations between them in the mind of the reader. Like all concepts in modern mathematics, we start with a simple set of axioms which are abstractions of the ideal requirements which our objects should have. We start with a (non-empty set) $X$ whose elements are called *points*, a class $\mathbb{L}$ of special subsets of $X$ called *lines*, and an *incidence relation*, viz., a point $x$ is *incident on* $\ell \in \mathbb{L}$ if $x \in \ell$. We may also express this incidence relation by saying that $\ell$ is incident on $x$ or $\ell$ passes through $x$. Now we impose some "natural conditions" on this pair $(X, \mathbb{L})$. We require that any two distinct points determine a unique line. This means that given $x, y \in X$ with $x \neq y$ there exists a unique element $\ell \in \mathbb{L}$ such that $x, y \in \ell$. We also demand that any two distinct lines pass through at most one point. That is, given $\ell, \ell' \in \mathbb{L}$ the intersection $\ell \cap \ell'$ has at most one point. These requirements are enough for the time being. Any pair $(X, \mathbb{L})$ which satisfies these conditions is called a plane geometry. Let us look at some examples of this concept.

As a first example let us give you an uninspiring one. Let $X = \{x, y\}$ be a two element set and $\mathbb{L} := \{X\}$. Then it satisfies all our requirements and hence is a plane geometry. This shows that we must impose perhaps some other condition so that our plane geometry will be "rich". One such condition may be to require that $X$ has at least three elements. Hereafter we shall assume that our plane geometry satisfies this condition.

Before the reader gets all knotted up, let us give an example which is the one closest to his intuition and most "well understood". As $(X, \mathbb{L})$ take the "Euclidean plane" along with the lines in the plane, that is, the plane geometry which the reader has learnt in high school.

The second example is again an abstract one to test your staying power. Take a set of 7 elements, say, $X = \{1, 2, \ldots, 7\}$. As lines let us take $\mathbb{L} := \{\ell_i : 1 \le i \le 7\}$ where the $\ell_i$ are defined as follows:

$$\ell_1 = \{1,7,5\} \quad \ell_2 = \{1,6,3\} \quad \ell_3 = \{1,4,2\} \quad \ell_4 = \{2,7,6\}$$
$$\ell_5 = \{2,5,3\} \quad \ell_6 = \{3,7,4\} \quad \ell_7 = \{4,5,6\}.$$

On seeing this, you may be tempted to say that this is the reason you never liked mathematics. We share your views and sympathize with you. However, there is a perfectly geometric way in which mathematicians visualize this plane. Look at Figure 1.0.1 below and ponder over it. The lines as drawn over there are there just for "ornamentation's sake" to aid our imagination.

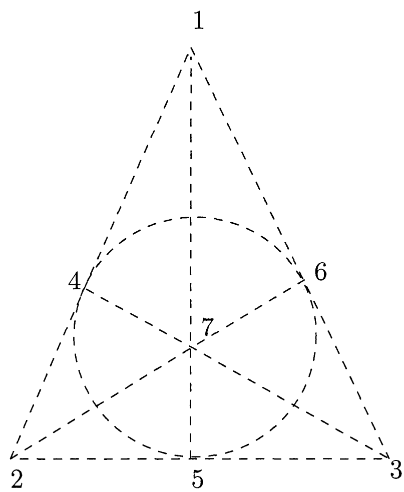


Figure 1.0.1 A Finite Plane

To the cognoscenti, this is nothing but the projective plane over the field of two elements.

After this esoteric example, let us look at a very concrete plane geometry. This time $X = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ is the upper half plane in $\mathbb{R}^2$. As lines, we take the collection $\mathcal{V}$ of vertical lines (in the usual Euclidean sense) and $\mathcal{C}$ the collection of semicircles of all possible radii with centres on the $x$-axis. Let $\ell_a := \{(a, y) : y > 0\}$ and

$C_{a,R} := \{(x,y) : y > 0, (x-a)^2 + y^2 = R^2\}$ for $a \in \mathbb{R}$ and $R > 0$, and

$$\begin{aligned} \mathcal{V} &= \{\ell_a : a \in \mathbb{R}\}, \\ \mathcal{C} &= \{C_{a,R} : a \in \mathbb{R}, R > 0\}. \end{aligned}$$

Thus $\mathbb{L} = \mathcal{V} \cup \mathcal{C}$. Note that the centre of $C_{a,R}$ is not a point of $X$! We invite you to check (at least convince yourself using coordinate geometry) that $(X, \mathbb{L})$ is a plane geometry. To check your understanding, answer this question: What is the unique line joining the following pair of points:

(a) $p = (1,1)$ and $q = (-1,1)$,

(b) $p = (0,1)$ and $q = (0,2)$?

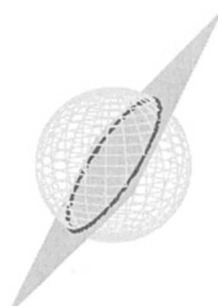

Figure 1.0.2 Great circle on $S^2$

As a final "example" (note the quotes), we take $\widetilde{X}$ to be the (surface of the) sphere $S^2$ of unit radius with centre at the origin in $\mathbb{R}^3$. That is, $S^2 := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Here, as the set $\tilde{\mathbb{L}}$ of lines $\tilde{\ell}$, we take all the great circles $\tilde{\ell}$ got as the intersection of a plane passing through the origin with $S^2$. (See Figure 1.0.2.) In particular, given any two points on $S^2$ which are not antipodal, that is, which are not diametrically opposite to each other, we take the plane passing through these two points and the origin and take its intersection with $S^2$. Thus given $p$ and $q (\neq \pm p)$, we have a unique 'line' joining them. But the shrewd reader must have observed that if we take $p$ and $q = -p$, then there are far too many lines (namely great circles) passing through the points $p$ and $-p$. (See Figure 1.0.3.) Thus $(\widetilde{X}, \tilde{\mathbb{L}})$ is not a plane geometry according to our definition.

Figure 1.0.3 Many great circles connecting the poles on $S^2$

Modern mathematicians do not sit and cry over this kind of set-up. They are crafty and get around this in an ingenious way. We pretend that we cannot distinguish between $p$ and $-p$. That is, as far as we are concerned, $p$ and $-p$ are one and the same point. Thus we take $X$ to be the set of pairs $[p,-p]$: $X := \{[p,-p]; p \in S^2\}$.

In terms of equivalence relations, the set $X$ is the quotient set of $S^2$ with respect to the equivalence relation: $x \equiv y$ iff $x = \pm y$ for $x, y \in S^2$.

We take $\ell$ to be the image of any $\tilde{\ell}$ (defined above) in $\widetilde{X}$: Thus,

$$\ell = [\tilde{\ell}] := \{[x,-x] : x \in \tilde{\ell}\}$$

and

$$\mathbb{L} = \{\ell = [\tilde{\ell}] \colon \tilde{\ell} \in \tilde{\mathbb{L}}\}.$$

We leave to you the very illuminating task of verifying that $(X, \mathbb{L})$ is a plane geometry. If you have difficulty in carrying out this detail, you may still proceed and come back later (after the discussion at the end of this chapter).

Now if you ask us to give a geometrically visualisable picture (in $\mathbb{R}^2$ or $\mathbb{R}^3$), we are in trouble. In such cases, what a mathematician does is this: if he wants to say something about $X$, then he looks at its analogue in $S^2$, checks it there and comes back to $X$. This principle is illustrated in a paragraph below.

Having done these examples, you might wonder what all these things lead to. We try to answer this in an oblique way in the rest of this chapter.

We agree to say that two lines $\ell$ and $\ell'$ in a plane geometry are *parallel* if either $\ell = \ell'$ or $\ell \cap \ell' = \emptyset$. Armed with this definition, let us look at some examples of parallel lines in the three examples above.

In the usual plane $X = \mathbb{R}^2$, if $\ell$ is the $x$-axis, then lines parallel to $\ell$ are given by $\ell' = \{(x, y) : y = \text{ constant}\}$.

In the second example where $X$ is the "upper half plane", if we take $\ell$ to be the vertical line $\{(0, y) : y > 0\}$, then any line $\ell_a = \{(a, y) : y > 0\}$ where $a \in \mathbb{R}$ is chosen arbitrarily is parallel to $\ell$. Can you think of some $C_{a,R}$ parallel to $\ell$? Can you find what are the lines parallel to $C_{a,R}$? If you cannot solve this, do not despair, it is kind of solved below.

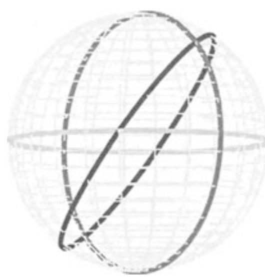

Figure 1.0.4 Any two great circles in $S^2$ meet each other

In the third example, $\ell'$ is parallel to $\ell$ iff $\ell = \ell'$. That is to say, there are no nontrivial pairs of parallel lines in this plane geometry! To verify this, let us look at the 'lines' in $S^2$. Consider any two lines $\tilde{\ell}$ and $\tilde{\ell}'$ such that $\tilde{\ell} = P \cap S^2$ and $\tilde{\ell}' = P' \cap S^2$ for some planes $P$ and $P'$ passing through the origin. Since $P$ and $P'$ are planes through the origin, they intersect along a (usual Euclidean) line through the origin. This line in turn intersects the sphere $S^2$ at two antipodal points $x$ and $-x$. Thus the images $\ell$ of $\tilde{\ell}$ and $\ell'$ of $\tilde{\ell}'$ intersect at the point $[x, -x] \in X$. (See Figure 1.0.4.) Thus any two lines intersect in this plane geometry!

Just as an aside, we invite the reader to check that the esoteric example of a plane consisting of 7 points and 7 lines also has this property.

All these examples should inevitably goad us into thinking of the (controversial) Euclid's parallel postulate. Euclid, in his definition of plane geometry, postulated that $(X, \mathbb{L})$ has the property that given a line $\ell$ and a point $p$ not on it, there exists a unique line $\ell'$ passing through $p$ parallel to $\ell$.

Let us see what happens in our three examples. In the usual plane, his postulate is true. In fact, if we develop this via linear algebra his postulate becomes a theorem. Those of you who know about cosets of subspaces of a vector space should know that any two cosets are either the same or they do not have any intersection.

In the second example, something fantastic happens. Take as $\ell$ any semicircle with the centre on the $x$-axis and a point $p = (p_1, p_2) \in X \subset \mathbb{R}^2$ not on the line $\ell$ as in Figure 1.0.5. Take as $\ell' = \{(p_1, y) : y > 0\}$. Then the line $\ell'$ is parallel to $\ell$. (That they seem to have a point on the $x$-axis in common is irrelevant as the $x$-axis is not a part of our plane $X$.) You can also convince yourself that you can draw an infinite number of semicircles passing through $p$ which are parallel to $\ell$.

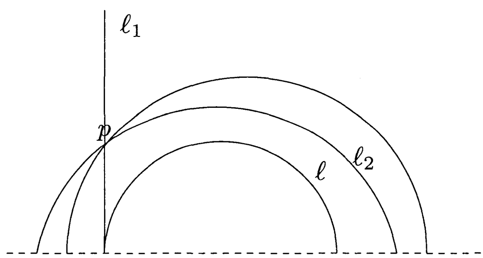


Figure 1.0.5 Lines in a hyperbolic plane

In the third example, given $\ell$ and a point $p \notin \ell$ as we have seen earlier it is not possible to find a line $\ell'$ parallel to $\ell$ and passing through $p$.

Thus our three planes exhibit all possible variations of Euclid's parallel postulate:

1. In the usual plane, given a line and a point $p$ not on the line there exists a *unique* line $\ell'$ passing though $p$ and parallel to $\ell$. Any plane geometry having this property is called a *Euclidean geometry.*

2. In the second example (where $X$ is the upper half plane), given a line $\ell$ and a point $p$ not on it, there exist *infinitely many* lines $\ell'$ parallel to the given line passing through the point $p$. A plane with this property is known as a *hyperbolic plane.*

3. In the last example (in which $X$ is the quotient set of the sphere), given a line $\ell$ and a point $p$ not on it, there exists *no* line through

$p$ which is parallel to $\ell$. A plane exhibiting this phenomenon is known as an *elliptic plane.*

The last two are hence known as *Non-Euclidean Geometries.*

Now you might raise the following point: "You started with some arbitrary $X$ and took some special class of curves as the lines in $X$. So anything can happen. What is the big idea?" Well, you are correct. But we have a reason for what we have done and which also explains why we chose these special curves as the lines.

If you agree with us that the intuitive notion of a line is that it is in some vague sense the "shortest" curve joining "nearby" points on it, then one can show that our curves are indeed lines in their respective planes, provided a proper interpretation of length of curves is given. One imitates the formula for arc-length in Euclidean geometry to define the length of a curve $c$ as follows:

$$\text{length of } c := \int_a^b (\dot{c}(t) \cdot \dot{c}(t))^{1/2} \varphi(c(t))\, dt.$$

Here $\varphi\colon X \to \mathbb{R}^+$ is a continuous function. Thus the tangent vector $\dot{c}(t)$ has as its length *not* the usual Euclidean length $(\dot{c}(t) \cdot \dot{c}(t))^{1/2}$ but $\varphi(c(t))(\dot{c}(t) \cdot \dot{c}(t))^{1/2}$. Thus the length of the tangent vector has a magnification factor depending upon its position. This kind of thing occurs naturally; e.g. When one studies the Lorentz metric and the Minkowski space-time in physics.

Finally two teasers: If $X = \mathbb{R}^3$, and lines are the usual lines as in three dimensional coordinate geometry, then $\mathbb{R}^3$ is also a "plane geometry" according to our definition. But a plane must surely be "two dimensional" and $\mathbb{R}^3$ is three dimensional. Is there any further condition to be imposed so that $\mathbb{R}^3$ is disqualified from being a "plane"?

The second one is as follows: Consider $\mathbb{R}^3$. As $X$ we take all the lines $L$ in $\mathbb{R}^3$ passing through the origin: $L := \{t(x, y, z) : t \in \mathbb{R}\}$ for a fixed non-zero $(x, y, z) \in \mathbb{R}^3$. As for lines we take the standard planes $P$ in $\mathbb{R}^3$ through the origin: $P := \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0\}$ for a fixed non-zero $(a, b, c) \in \mathbb{R}^3$. Then $(X, \mathbb{L})$ is a plane geometry whose points are "standard lines" in $\mathbb{R}^3$ passing through the origin and whose lines are "standard planes" in $\mathbb{R}^3$ passing through the origin. This is essentially the third example (involving $S^2$). In algebraic language, this is known as the projective plane over $\mathbb{R}$.

**Remark.** The purpose of this chapter is to introduce you quickly to axiomatic geometry as well as to excite and lure you into geometry. Hence we have taken care not to smother you with the most precise statements which may leave you cold. As said at the beginning, as you go along the book, we shall vindicate most of what is said in this chapter. We shall also plunge deeper into many of these geometries.

# Chapter 2

# Affine Geometry

## 2.1 Definition and Examples

**Definition 2.1.1** (Axiomatic Definition). An *affine plane* $\Pi$ is a nonempty set $X$, whose elements are called points of $\Pi$, and a class $\mathbb{L}$ of nonempty subsets of $X$, the elements of which are called the lines of $\Pi$ such that

[A1] given two distinct points $P$ and $Q$ in $X$, there exists a unique $\ell \in \mathbb{L}$ such that $P$ and $Q$ are in $\ell$,

[A2] there exist three distinct points $P_1$, $P_2$ and $P_3$ in $X$ such that these three points are not in the same $\ell \in \mathbb{L}$, and

[A3] given a line $\ell \in \mathbb{L}$ and a point $P \notin \ell$ there exists a unique $\ell_P \in \mathbb{L}$ such that $P \in \ell_P$ and $\ell \cap \ell_P = \emptyset$.

Let us now try to understand this axiomatic definition of an affine plane geometrically and recast it in geometric language.

**Definition 2.1.2.** Let $\Pi := (X, \mathbb{L})$ be an affine plane. We say that:

1. A point $P \in X$ lies on a line $\ell$ (or the line $\ell$ passes through $P$), if $P \in \ell$.
2. A set of points $\{P_1, P_2, \ldots, P_n\} \subseteq X$ is *collinear* if there exists a line $\ell \in \mathbb{L}$ such that $P_i \in \ell$ for all $i = 1, 2, \ldots n$.
3. Two lines $\ell$ and $m$ are *parallel* if either $\ell = m$ or $\ell \cap m = \emptyset$.

With these concepts, our geometric definition of an affine plane is

**Definition 2.1.3** (Geometric Definition). An affine plane $\Pi$ is a nonempty set $X$ of points and a class $\mathbb{L}$ of lines such that

[G1] given two distinct points $P, Q \in X$, there exists a unique line $\ell$ such that $\ell$ passes through $P$ and $Q$ (or the points $P$ and $Q$ lie on $\ell$),

[G2] there exist three non-collinear points in $X$ and

[G3] given a line $\ell \in \mathbb{L}$ and a point $P \notin \ell$ there exists a unique line $\ell_P$ in $\mathbb{L}$ passing through $P$ such that $\ell$ and $\ell_P$ are parallel. (*This axiom is known as the parallel postulate.*)

We derive some simple consequences of the definition and then look at some examples.

**Proposition 2.1.4.** *In an affine plane* $\Pi$*, two distinct lines meet at atmost one point.*

*Proof.* Let $\ell$ and $m$ be two lines meeting at two distinct points, say $P$ and $Q$. Since $P \neq Q$, there exists a unique line passing through both $P$ and $Q$, by A1 or G1. Hence $\ell = m$. □

**Proposition 2.1.5.** *In an affine plane* $\Pi$*, parallelism is an equivalence relation.*

*Proof.* Two lines $\ell$ and $m$ in an affine plane are parallel if either $\ell = m$ or $\ell \cap m = \emptyset$. Therefore a line $\ell$ is parallel to itself. This proves that parallelism is reflexive.

If a line $\ell$ is parallel to a line $m$, obviously $m$ is parallel to $\ell$. This proves symmetry.

We will now prove that parallelism satisfies transitivity. Let $\ell$, $m$ and $n$ be three lines in $\Pi$ such that $\ell \parallel m$ and $m \parallel n$. If $\ell = m$, then clearly $\ell$ is parallel to $n$. So we assume that $\ell \neq m$. If $\ell$ is not parallel to $n$, then these two lines will meet at a point, say $P$. Then the point $P$ either lies on $m$ or it does not lie on $m$.

Case 1: If the point $P$ lies on $m$, then, since $P$ lies on $\ell$ and $\ell \parallel m$, it follows that $\ell = m$, which is a contradiction.

Case 2: Assume that $P$ does not lie on $m$. Then the lines $\ell$ and $n$ pass through $P$ and are parallel to $m$. This violates the uniqueness part of the parallel postulate.

We therefore conclude that no such point $P$ can exist, i.e., $\ell \cap n = \emptyset$. This shows that parallelism is an equivalence relation. □

Let us now look at some examples.

**Example 2.1.6** (Affine Plane over $\mathbb{R}$)**.** Let $X = \mathbb{R}^2$ and $\ell \subseteq \mathbb{R}^2$ be a line in $\mathbb{R}^2$ iff

$$\ell = \{(x, y) \in \mathbb{R}^2 : ax + by = c\}$$

for some $(a, b) \neq (0, 0)$ and $c \in \mathbb{R}$.

We let $\mathbb{L}$ be the collection of all such subsets of $\mathbb{R}^2$. Then $\Pi = (X, \mathbb{L})$ is an affine plane.

*Proof.* Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two distinct points in $\mathbb{R}^2$. We want to find $(a, b) \neq (0, 0)$ in $\mathbb{R}^2$ and $c \in \mathbb{R}$ such that the points $P$ and $Q$ are solutions to the linear equation $ax + by = c$.

If you remember coordinate geometry, the line joining the points $(x_1, y_1)$ and $(x_2, y_2)$ is given by

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2},$$

for $y_1 \neq y_2$ and $x_1 \neq x_2$. If $y_1 = y_2 = c$ (respectively $x_1 = x_2 = c$), then the equation of the line is $y = c$ (respectively $x = c$).

After algebraic manipulations, the equation of the line is

$$-(y_1 - y_2)x + (x_1 - x_2)y = (x_1 - x_2)y_1 - (y_1 - y_2)x_1, \qquad (2.1.1)$$

so that $a = -(y_1 - y_2)$, $b = (x_1 - x_2)$ and $c = (x_1 - x_2)y_1 - (y_1 - y_2)x_1$. Note that (2.1.1) is the equation of the line joining $(x_1, y_1)$ and $(x_2, y_2)$ whether or not $x_1 = x_2$ or $y_1 = y_2$.

This verifies the first axiom.

The points $(0, 0)$, $(1, 0)$ and $(0, 1)$ are not collinear. (Verify.)

Now we verify the parallel postulate.

Let $\ell = \{(x, y) \in \mathbb{R}^2 : ax + by = c\}$ be a line in $\mathbb{R}^2$ and $P = (x_0, y_0)$ be a point not lying on the line $\ell$. Then the line

$$\ell_P := \{(x, y) \in \mathbb{R}^2 : ax + by = ax_0 + by_0\}$$

passes through the point $P$ and is parallel to $\ell$.(Why?)

> Reason: For, if $(x_0, y_0) \notin \ell$, then $ax_0 + by_0 \neq c$. Hence it follows that there cannot be a common point on $\ell$ and $\ell_P$.

We ask the reader to prove the uniqueness part. □

**Example 2.1.7.** Let $X = \{1, 2, 3, 4\}$ and $\mathbb{L}$ consist of all two element subsets of $X$. Then $\Pi = (X, \mathbb{L})$ is an affine plane.

*Proof.* The proof is left as an exercise. This plane is pictorially represented in Figure 2.1.1. □

Before some more examples, we ask

**Question 2.1.8.** Let $n \in \mathbb{N}$ be fixed. Let $X_n = \{1, 2, \ldots, n\}$ and $\mathbb{L}$ be the class of all two element subsets of $X$. Is $\Pi = (X_n, \mathbb{L})$ an affine plane?

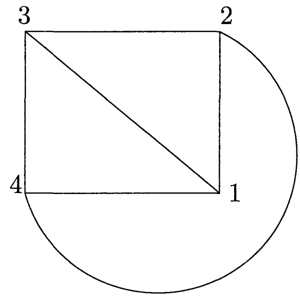


Figure 2.1.1 Affine Plane with 4 Points

1. $\Pi = (X_n, \mathbb{L})$ is an affine plane only for $n = 4$.

2. For $n > 4$, the parallel postulate is not valid. In fact, given a line $\ell$ and a point $P$ not lying on $\ell$, there are at least three lines parallel to $\ell$, passing through $P$. (Why?)

3. For $n = 3$, any two lines will meet at a point. This will violate the parallel postulate.

**Exercise 2.1.9.** Let $X = \{1, 2, 3\}$. Show that $X$ cannot be made into an affine plane.

**Exercise 2.1.10.** Let $\Pi = (X, \mathbb{L})$ be an affine plane. Prove that given a line $\ell$ in $\Pi$ there exists a point $P$ such that $P \notin \ell$.

**Exercise 2.1.11.** Show that given a point $P$ in an affine plane $\Pi$, there exist two distinct points $Q$ and $R$ such that the three points $P$, $Q$ and $R$ are not collinear. Hence prove that given a point $P$ there exists a line $\ell \in \mathbb{L}$ such that $P \notin \ell$.

**Notation:** Given two distinct points $A$ and $B$ in an affine plane $\Pi$, we let $\ell_{AB}$ or $AB$ denote the unique line joining the points $A$ and $B$.

The following is a version of Pasch's Axiom. (See also Ex. 2.10.16).

**Lemma 2.1.12.** *Let $\Pi = (X, \mathbb{L})$ be an affine plane. Let $A$, $B$ and $C$ be three non-collinear points. Let $\ell$ be a line distinct from the three lines $AB$, $BC$ and $CA$. Then $\ell$ meets two of the three lines.*

*Proof.* Note that at least one of $A, B, C$ will not lie on $\ell$. Otherwise, all lie on $\ell$ and hence they are collinear, a contradiction to our hypothesis. So we may assume, without loss of generality, that $\ell$ does not pass through $A$.

We first claim that $\ell$ has to meet one of the three lines. If not, $\ell$ is parallel to all of them. Now, we have two lines $AB$ and $AC$ through $A$ which are parallel to $\ell$, contradicting the parallel postulate. This contradiction proves our claim. Without loss of generality, we assume that $\ell$ meets $BC$.

We claim that $\ell$ meets $AB$ or $AC$. Suppose the claim is false. This means that the line $\ell$ is parallel to $AB$ and $AC$. Since parallelism is an equivalence relation, it follows that $AB$ is parallel to $AC$. But they meet at the point $A$. This forces us to conclude that $AB = AC$. Hence it follows that $A, B$ and $C$ are collinear. This contradicts our hypothesis. $\square$

**Proposition 2.1.13.** *Let* $\Pi = (X, \mathbb{L})$ *be an affine plane. Then every line in* $\Pi$ *has at least two points.*

*Proof.* Let $A$, $B$ and $C$ be three non-collinear points in $\Pi$. Therefore the lines $AB$, $BC$ and $CA$ are mutually non-parallel. See Figure 2.1.2.

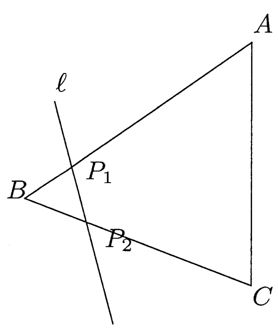


Figure 2.1.2 Prop. 2.1.13

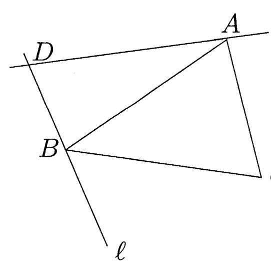


Figure 2.1.3 Prop. 2.1.13

Let $\ell$ be a line in $\Pi$. If $\ell$ is one of the above three lines, then it has two points and we are through. We assume that $\ell$ is not one of the three lines. It follows from Lemma 2.1.12 that the line $\ell$ meets at least two of the lines $AB$, $BC$ and $CA$.

So, we may assume without loss of generality that $\ell$ meets $AB$ and $BC$. Let $P_1 = \ell \cap AB$ and $P_2 = \ell \cap BC$. If $P_1 \neq P_2$, then we are through. Otherwise, we claim that $P_1 = P_2 = B$.

For, $P_1 = P_2$ is the point of intersection of $\ell$ with $AB$ and $BC$ so that $P_1 = P_2$ lies on $AB$ and $BC$. But $AB$ and $BC$ already meet at $B$. Hence the claim follows. Note that under the stated assumptions $B$ lies on $\ell$.

There are two possibilities: either $\ell$ meets $AC$ or it is parallel to $AC$.

If $\ell$ meets $AC$, say, at $E$, then $E \neq B$ since otherwise $B$ lies on the line $AC$, contradicting the non-collinearity of $A, B, C$. Hence, the two distinct points $B$ and $E$ lie on $\ell$ and hence we are through in this case.

Assume that $\ell$ is parallel to $AC$. Then, let $m$ be the line passing through the point $A$ such that $m \parallel BC$. (See Figure 2.1.3.) Then $\ell$ and $m$ meet at a unique point, say $D$. (Why?)

> Reason: For, otherwise, $m \parallel \ell$ and $\ell \parallel AC$ by assumption so that it follows that $m \parallel AC$. Since they meet at $A$, we conclude that $m = AC$. Our assumption that $m \parallel BC$ implies that $AC \parallel BC$. Since $AC$ and $BC$ have a point in common, this means that $AC = BC$, that is, $A, B, C$ are collinear.

We claim that $D$ is not any of these points $B$ or $C$: for, if $D = B$, then $m = AB$. But $AB$ and $BC$ are not parallel. Similarly $D \neq C$. Thus the line $\ell$ has at least two points.

*Another proof of the proposition.* Suppose $\{A\} \in \mathbb{L}$. We know that there are three non-collinear points $B$, $C$ and $D$ in $X$. Let $\ell_{BC}$ denote the line through the points $B$ and $C$. Similarly we have the lines $\ell_{CD}$ and $\ell_{BD}$. By our assumption, $\ell_{BD} \cap \ell_{CD} = D$ etc.,

Case 1: Let us assume that $A \notin \{B, C, D\}$. Then the point $A$ lies in atmost one of the lines $\ell_{BC}$, $\ell_{BD}$ and $\ell_{CD}$. (Why?) So we can assume that $A \notin \ell_{BC}, \ell_{BD}$. Then these two lines $\ell_{BC}$ and $\ell_{BD}$ passing through the point $B$ are parallel to the line $\{A\}$, violating the parallel postulate.

Case 2: Let us now assume that $A \in \{B, C, D\}$. Assume that $A = B$. Let $\ell$ be the line passing through $C$ and parallel to $\ell_{BD}$. Since the point $C$ does not lie on $\ell_{BD}$, $\ell \cap \ell_{BD} = \emptyset$. This shows that $\ell \parallel \{A\}$. We also have $\ell_{CD} \parallel \{A\}$, again violating the parallel postulate. □

**Example 2.1.14.** Let $X = \mathbb{R}^2$. For any $(a, b) \in \mathbb{R}^2$, we let

$$P_{(a,b)} := \{(a + x, b + x^2) \in \mathbb{R}^2 : x \in \mathbb{R}\},$$

a parabola with vertex at the point $(a, b)$. For any $c \in \mathbb{R}$, we denote by $\ell_c := \{(c, y) : y \in \mathbb{R}\}$ the vertical line $x = c$. Let $\mathbb{L}_{\mathbf{v}}$ be the collection of all such parabolas and vertical lines:

$$\mathbb{L}_{\mathbf{v}} = \{P_{(a,b)} : (a, b) \in \mathbb{R}^2\} \cup \{\ell_c : c \in \mathbb{R}\}.$$

Then $(X, \mathbb{L}_{\mathbf{v}})$ is an affine plane.

Let $(p_1, q_1)$ and $(p_2, q_2)$ be two distinct points in $X = \mathbb{R}^2$. If $p_1 = p_2$, then there is a unique line $\{(p_1, y) : y \in \mathbb{R}\}$ passing through these two