Transformation Groups

Symplectic Torus Actions and Toric Manifolds

Edited by
Goutam Mukherjee

With Contributions by
Chris Allday
Mikiya Masuda
P Sankaran

HINDUSTAN BOOK AGENCY

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Preface

This volume is updated and revised version of the main lectures delivered in the Winter School on Transformation Groups held in honor of Professor Amiya Mukherjee at the Indian Statistical Institute, Kolkata in December 1998 under the auspices and financial support of the Indian Statistical Institute, the National Board of Higher Mathematics, Chennai Mathematical Institute and the Council for Industrial and Scientific Research, India.

The aim of the school was to discuss the recent trends in the use of cohomological methods in the study of torus actions and toric manifolds and to expose young topologists to the recent developments in these areas.

A very interesting aspect of the cohomology theory of transformation groups is its interaction with the study of symplectic and Hamiltonian torus actions. Many of the results of the latter theory are cohomological. The importance of cohomology theory in the study of symplectic and Hamiltonian torus actions has been recognized for a long time and the usefulness of cohomology theory in the field continues today, significantly in the theory of toric varieties.

Chapter 1 is devoted to illustrate the cohomological methods used in the study of symplectic and Hamiltonian torus actions.

The basic theory of toric varieties was established in the early 70's by Demazure, Mumford etc., and Miyake-Oda. It says that there is a one-to-one correspondence between toric varieties and combinatorial objects called fans. Moreover, a compact non-singular toric variety together with an ample complex line bundle corresponds to a convex polytope through a map called the moment map. Chapter 2 is a brief introduction of the basic theory of toric varieties.

Finally, chapter 3 is intended to develop the theory of toric varieties, which is a bridge between algebraic geometry and combinatorics, from a topological point of view. This is done by studying new geometrical objects called toric manifolds, which generalize many results of toric varieties in a topological framework and produce nice applications relating topology, geometry and combinatorics.

Most of the techniques and proofs of results given in the notes are either new and have not appeared elsewhere in the literature or are written in a style which may be more accessible to the readers.

C. Allday, M. Masuda, G. Mukherjee, P. Sankaran.

About the Notes

This volume is updated and revised version of the main lectures delivered in the Winter School on Transformation Groups held at the Indian Statistical Institute, Kolkata in December 1998. The Chapter 1 of this volume is written by Professor Christopher Allday, Chapter 2 is written by Professor Parmeswaran Sankaran and Chapter 3 is written by Professor Mikiya Masuda. Dr. Goutam Mukherjee organizesd the school and acted as a coordinating editor.

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This volume is based on lectures delivered by Professor Christopher Allday, University of Hawaii, U.S.A, Professor Mikiya Masuda, Osaka City university, Japan and Professor Parameswaran Sankaran, Institute of Mathematical Sciences, Chennai, India, at the winter school on 'Transformation Groups' that was held at Indian Statistical Institute, Kolkata, India, from 8th december to 26th December, 1998. This was one of a series of meetings of topologists in India that are being organized at various Institutions in India since 1987 starting at NEHU, Shillong. I express my sincere thanks to Professor B. L. Sharma, for initiating this idea and being with us throughout.

I extend my warm and sincere thanks to Professor Chris Allday, Professor Mikiya Masuda and Professor Parameswaran Sankaran on my own behalf as well as on behalf of everyone else involved in the project, for accepting our invitation to visit Kolkata, for delivering such superb lectures at the school and writing notes for this volume. I express my sincere thanks to all other speakers who delivered lectures to make the audience prepared for the main lectures.

A course at this level cannot succeed without contributions from everyone involved in the project and I thank all my colleages and staff of ISI, Kolkata for their help in organizing the school. Specially I thank my friend and colleague Dr. A. C. Naolekar without his enormous help and cooperation it would not have been possible for me to achieve this goal. I wish to record my sincere thanks to Professor M. S. Raghunathan and Professor C. S. Seshadri for extending financial support. I also thank CSIR for providing financial support. I express my sincere thanks to Professor S. B. Rao, the then Director of ISI and Professor S. C. Bagchi, the then Professor-in-charge, Stat-Math division, ISI for their support. I also thank Hindustan Book Agency, for accepting this work for publication.

Goutam Mukherjee

We warmly thank Prof. Goutam Mukherjee for inviting us to speak in the winter school on Transformation Groups held at ISI, Kolkata, in December 1998 and for his interest in these notes. But for his sustained and patient efforts, these notes would not have been published. And we thank ISI, Kolkata, for the warm and generous hospitality extended to us during the winter school.

Chris Allday, Honolulu,
Mikiya Masuda, Osaka,
Parameswaran Sankaran, Chennai.

Chapter 1

Localization Theorem and Symplectic Torus Actions

1.1 Introduction

These notes are intended to be a brief introduction to the cohomology theory of transformation groups with applications to symplectic and Hamiltonian torus actions. More or less without exception – see, e.g., Theorem 1.6.5 – we consider only torus actions, and, often only circle actions. The main theorem of the subject is the Localization Theorem (Theorem 1.3.7), and, except in the more general statement and proof of this theorem, we use only cohomology with coefficients in a field of characteristic zero. Also, to further simplify the presentation, we prove the Localization Theorem only in the compact case, discussing other cases in remarks. As in the original Smith theory, the cohomology theory is useful too for studying elementary abelian p–group actions; but that will not be done here: see [4] for a more comprehensive introduction. For an introduction to compact Lie group actions in general, see [14], [28] or [56].

A very interesting aspect of the cohomology theory of transformation groups is its interaction with the study of symplectic and Hamiltonian torus actions. Many of the results of the latter theory are cohomological; and it is interesting to see which of them follow from the Localization Theorem and which of them do not. In these notes we give several results of each type. Along the way, in Section 1.5, we give the basic definitions of the theory of symplectic and Hamiltonian group actions. The definitions, while quite complicated for compact connected Lie group actions in general, become simple and elegant for circle actions.

The importance of cohomology theory in the study of symplectic and

Hamiltonian torus actions has been recognized for a long time – see, for example, [11] and [30] – and the usefulness of cohomology theory in the field continues today, significantly in the theory of toric varieties.

In Section 1.2 we discuss the Borel construction and some properties of the Leray–Serre spectral sequence which are used subsequently. Section 1.3 states and proves the Localization Theorem and gives some immediate corollaries. Section 1.4 concerns the consequences of Poincaré duality. Some of the basic results here are stated without proof, since the proofs are rather long; and they can be found in [4], for example. But we do give proofs of the Topological Splitting Principle and the Borel Formula: these show some of the power of the Localization Theorem. Section 1.5 introduces symplectic and Hamiltonian group actions with emphasis on Frankel's results. A recent theorem of Jones and Rawnsley is included also. In Section 1.6 we give the cohomology analogues of the definitions in the symplectic and Hamiltonian theory, and, again, we illustrate the power of the Localization Theorem with results such as Theorems 1.6.5 and 1.6.10. The brief and final Section 1.7 shows by example that a theorem of McDuff and recent theorems of Tolman and Weitsman and Giacobbe are not purely cohomological.

For the remainder of this introduction we give some of the basic terminology of transformation groups, some comments on the cohomology theory and the spectral sequence used in these notes, and some other assumptions and conventions.

Definition 1.1.1. *Let X be a Hausdorff topological space and let G be a compact Lie group.*

(1) An action of G on X on the left is a map $\Phi : G \times X \longrightarrow X$ such that $\Phi(1, x) = x$, for all $x \in X$, where $1 \in G$ is the identity, and $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$, for all $x \in X$ and all $g, h \in G$. Usually $\Phi(g, x)$ is written simply as gx; and so the two conditions become $1x = x$ and $g(hx) = (gh)x$ for all $x \in X$ and all $g, h \in G$.

An action on the right is defined similarly. In these notes, unless indicated otherwise, actions will be on the left.

(2) Given any action of G on X and given $x \in X$, the isotropy subgroup of x is $G_x = \{g \in G; gx = x\}$; and the orbit of x is $G(x) = \{gx; g \in G\}$

For any $y \in G(x)$, clearly, G_y is conjugate to G_x. And there is a homeomorphism $G/G_x \longrightarrow G(x)$ given by $gG_x \longmapsto gx$.

(3) *The fixed point set of an action of* G *on* X *is*

$$X^G = \{x \in X; gx = x, \textit{for all } g \in G\}$$
$$= \{x \in X; G_x = G\}.$$

(4) *An action of* G *on* X *is said to be free if* $G_x = 1$*, the trivial subgroup of* G*, for all* $x \in X$*: i.e., if for every* $x \in X$*,* $gx = x$ *implies that* $g = 1$*. The action is said to be almost-free if* G_x *is finite for all* $x \in X$*. And the action is said to be semi-free if* $X^G \neq \emptyset$ *and the action is free on* $X - X^G$*: i.e., if* $X^G \neq \emptyset$ *and* $G_x = 1$ *for all* $x \notin X^G$*.*

(5) *An action of* G *on* X *is said to be effective if* $gx = x$ *for all* $x \in X$ *implies that* $g = 1$*: i.e., if* $\bigcap_{x \in X} G_x = 1$*. In general,* $\bigcap_{x \in X} G_x$ *is called the ineffective kernel. Clearly it is a normal subgroup of* G*.*

The action is said to be almost-effective if the ineffective kernel is finite.

(6) *The relation* $x \sim y$ *if and only if* $y \in G(x)$ *is an equivalence relation on* X*. The equivalence class* $[x] = G(x)$*. The quotient space is denoted* X/G*, and it is called the orbit space. The quotient map* $\pi : X \longrightarrow X/G$ *is called the orbit map.*

The orbit space is Hausdorff, and the orbit map is open, closed and proper: see, e.g., [14], Chapter I, Theorem 3.1 .

(7) *The space* X *is called a* G*-space when there is given an action of* G *on* X*.*

For the existence and properties of tubes and slices see, for example, [14], Chapter II, Sections 4 and 5, and, for the smooth case, see [14], Chapter VI, Section 2, especially Corollary 2.4 .

To prove the Localization Theorem one wants a cohomology theory with the tautness property for closed sets. Since we shall consider only paracompact spaces, Alexander–Spanier cohomology is a good choice. The tautness property used is the following. Let X be a paracompact space and let $A \subseteq X$ be a closed subspace. Let $\mathcal{N}$ be the system of open neighborhoods of A, or the system of closed neighborhoods of A, directed downward by inclusion. Let $H^*(-;k)$ denote Alexander–Spanier cohomology with coefficients in an abelian group k. Then restriction induces an isomorphism

$$\varinjlim_{N \in \mathcal{N}} H^*(N;k) \xrightarrow{\sim} H^*(A;k).$$

See [76], Chapter 6, Section 6 or [15], Chapter II, Section 10 for the proof.

Thus, throughout these notes, the cohomology theory used will be Alexander–Spanier cohomology. Alexander-Spanier cohomology coincides with Čech cohomology for all spaces; and, for paracompact spaces, it coincides with sheaf cohomology and the cohomology theory given by the Eilenberg-MacLane spectrum. For paracompact locally contractible spaces, Alexander-Spanier cohomology coincides with singular cohomology. So singular theory works well for smooth actions on smooth manifolds or, more generally, on $G - CW$–complexes.

Because Alexander–Spanier cohomology, or, equivalently, sheaf cohomology , since all spaces will be paracompact, is used, in order to study the cohomology of fibre spaces, one needs the Leray spectral sequence (see [15]) or, as it is called in this context, the Leray–Serre spectral sequence. This is the same as the Serre spectral sequence when singular cohomology is used. [76], Chapter 9, is a good introduction. See also [66].

All spaces in these notes will be assumed to be Hausdorff. And the terms 'compact' and 'paracompact' will be assumed to include Hausdorffness as part of their definitions.

Algebraic notations and conventions(1) Elements in graded rings will be assumed to be homogeneous unless mentioned otherwise. Similarly, ideals in graded rings will usually be homogeneous. The prime ideals, $\mathfrak{p}$, which occur from time to time in Section 1.3, however, are not assumed to be homogeneous.

(2) If R is a commutative ring, M is a R–module, and $\mathfrak{p} \subseteq R$ is a prime ideal, then $R_\mathfrak{p}$, respectively $M_\mathfrak{p}$, denotes R, respectively M, localized at $\mathfrak{p}$, i.e., with respect to the multiplicative set $R - \mathfrak{p}$. In particular, if R is an integral domain, then $R_{(0)}$ is the field of fractions.

1.2 The Borel Construction

Let G be a compact Lie group. There is a universal principal G–space EG: that is, EG is a contractible paracompact space on which G acts