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hp-Version Discontinuous Galerkin Methods on Polygonal and Polyhedral Meshes

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To

Johanna, Shuheng, Elektra, and Kelly

for their patience while writing this volume.

Contents

Chapter 1
Introduction

The purpose of this volume is to present some recent developments concerning the mathematical aspects of discontinuous Galerkin finite element methods on general computational meshes consisting of polygonal/polyhedral (henceforth, jointly referred to as polytopic) element domains. We begin by providing some historical background and motivation for this work, as well as introducing some standard notation.

1.1 Background

Finite element methods (FEMs) represent an indispensable computational tool for the accurate, efficient, and rigorous numerical approximation of continuum models arising in engineering, physics, biology, and many other disciplines. Key reasons for the astounding success of FEMs is their applicability to general classes of partial differential equations (PDEs), simple treatment of complicated computational geometries and enforcement of boundary conditions, and ease of adaptivity including both local mesh subdivision (h-refinement) and local polynomial enrichment (p-refinement). Furthermore, from a mathematical point of view, tools are available for their rigorous error analysis, both in the a priori and a posteriori settings; this latter topic is of particular practical interest for both error quantification and automatic adaptive mesh design.

However, the exploitation of classical (conforming) FEMs for the numerical approximation of hyperbolic, or 'nearly' hyperbolic problems, and other strongly non-self-adjoint PDE problems is, generally speaking, unsatisfactory in the sense that the underlying numerical scheme lacks sufficient stability. This typically manifests itself through the production of spurious or unphysical oscillatory behaviour in the computed approximation in the vicinity of strong gradients in the analytical solution, for example, near boundary and internal layers. As a remedy, over the last

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40 years or so, Petrov-Galerkin, or more generally, stabilized variants of the standard FEM, have been devised; perhaps the most popular approach is the streamline upwind Petrov-Galerkin (SUPG) scheme, cf. [46, 110], for example. However, stabilized schemes often involve the determination of hard-to-evaluate user-defined parameters. While this topic is beyond the scope of the current volume, we refer to [43, 44, 51, 97], and the references cited therein.

At the other end of the spectrum, finite volume methods (FVMs) have been predominantly used in industrial software packages, especially in computational fluid dynamics (CFD), due to their efficiency of implementation, particularly on parallel computer architectures, as well as their robustness. While, in principle, these methods are typically second-order accurate, in practice, their convergence order may deteriorate on irregular and/or highly stretched meshes. Thereby, for reliable numerical predictions to be made by such methods, extremely fine meshes with a large number of degrees of freedom are required; this, in turn, leads to excessively long computing times. As an alternative approach, in recent years there has been significant interest in the development of high-order discretization methods. On a given computational mesh they allow for improved predictions of critical flow phenomena, as well as force coefficients, such as, for example, the lift, drag, or moment of a body immersed in a fluid. In particular, high-order methods are capable of achieving the same level of accuracy while exploiting significantly fewer degrees of freedom compared to classical FVMs.

In this volume, we focus on an extremely powerful class of arbitrary-order numerical schemes referred to as *discontinuous Galerkin finite element methods* (DGFEMs). Loosely speaking, DGFEMs can be considered as a hybrid between classical FEMs and FVMs. Indeed, in common with FEMs, DGFEMs approximate the underlying PDE solution by employing polynomials of arbitrary degree, defined over local element domains, but without the enforcement of any continuity constraints between neighbouring elements. Instead, elements are coupled via numerical flux functions in a similar manner to the design of FVMs. In the lowest-order case, i.e., when piecewise constant functions are employed, the corresponding DGFEM is equivalent to a cell-centred FVM, without a local recovery operator. Thereby, given the construction of DGFEMs as Galerkin procedures, rigorous error analysis is available for a variety of PDE problems.

The first DGFEM was introduced by Reed and Hill [148] for the numerical solution of the neutron-transport equation. This method was later analysed by Lesaint and Raviart [135] and by Johnson and Pitkäranta [127]; see, for example, [41, 66, 68, 69, 71, 72, 93], and the volume [73]. In the context of elliptic PDEs, Nitsche [143] introduced the idea of weakly imposing inhomogeneous essential boundary conditions for (classical) FEMs. This was subsequently studied by Baker [27] who proposed the first modern DGFEM for elliptic problems, later followed by Wheeler [173], Arnold [16], Baker et al. [28], and others; cf., also, the related penalty FEM studied in [23].

Over the last 20 years tremendous progress has been made on the development of both the analytical and computational aspects of DGFEMs for the numerical approximation of a wide variety of PDEs; see, for example, the recent mono-

graphs [73, 79, 117, 151] and the review articles [18, 65]. This progress has been stimulated by a number of important factors: firstly, the combination of FVM-type stability with the exploitation of high-order polynomials within the FEM setting means that DGFEMs can treat advection-dominated diffusion problems, without excessive numerical stabilization, in an accurate and efficient manner. Furthermore, the lack of continuity between neighbouring elements, allows for extremely general meshes to be employed; in particular, DGFEMs offer higher-order discretizations with a minimal computational stencil, irrespective of the element shape. On a more technical level, the simple communication via numerical fluxes at element interfaces afforded by DGFEMs allows for the natural incorporation of so-called hanging nodes, thereby simplifying local mesh refinement (*h*-refinement). Additionally, the mode of communication at elemental interfaces is independent of the order of the method which simplifies the use of schemes with different polynomial orders p in adjacent elements. This allows for the variation of the order of polynomials over the computational domain (p-refinement), which in combination with *h*-refinement leads to so-called *hp*-version approximations. Recent advances in domain decomposition techniques have highlighted that DGFEMs naturally admit Schwarz-type preconditioners, cf. [3–7, 11, 94]; see, also, [10, 13, 15] for the design of multigrid algorithms for DGFEMs. As a final remark, the level of generality offered by DGFEMs, in terms of both the method definition via numerical fluxes and the flexibility in the mesh design, has contributed to their use in practically relevant simulations for a wide variety of applications, ranging from CFD and electromagnetics to structural mechanics and mathematical biology; indeed, DGFEMs naturally treat multi-physics problems within a unified manner, cf. [120, 159].

On the other hand, many practitioners often conclude that DGFEMs are computationally expensive, since for a given mesh and polynomial order, DGFEMs lead to an increase in the number of degrees of freedom compared to classical FEMs with comparable accuracy, though it is important to note that such statements are typically made in the context of discretizing second-order self-adjoint elliptic PDEs. This is a rather simplistic argument, since it overlooks all of the key aforementioned and other potential advantages of DGFEMs in terms of their applicability, versatility, and mesh-flexibility. Indeed, as we shall see below, within the DGFEM framework, it is possible to employ the same underlying approximating space of discontinuous piecewise polynomials, irrespective of the structure of the PDE of interest and the type of computational mesh exploited. Moreover, the flexibility offered by different choices of numerical fluxes allows for the design of DGFEMs with desirable conservation properties of important quantities, such as, for example, mass, momentum, and energy. Furthermore, in the context of implicit discretizations, the size of the resulting linear systems can be reduced in such a manner that DGFEMs may be either competitive or, in some cases, cheaper to compute than the corresponding standard (conforming) FEM. Indeed, we first mention the pioneering work of Cockburn and his collaborators on the so-called hybridizable DGFEM (HDG), see, for example, [67, 74]. In this setting, additional unknowns are introduced on the boundary of each element within the underlying

computational mesh. Thereby, through a process of static condensation, a global matrix problem involving *only* the additional unknowns needs to be solved; the remaining unknowns are then recovered by solving local elementwise problems.

Secondly, we highlight the fact that, within the DGFEM framework, elemental polynomial bases can be constructed which contain less degrees of freedom than their FEM counterparts on quadrilateral/hexahedral elements. The essential idea is to construct a basis in the physical frame, without resorting to the use of local element mappings to a given reference element. In this way, spaces of polynomials of total degree p, denoted by $\mathscr{P}_p$, may be employed, irrespective of the shape of the element; see, for example, [32–34, 54]. We also refer to [35] where this technique was first used to exactly resolve curved boundaries. Thereby, the order of convergence of the underlying method is *independent* of the element shape; we refer to [17, 19] for a detailed discussion of this issue when element mappings are employed. Indeed, as noted in our recent work [54], when the underlying mesh consists of tensor-product elements, for example, quadrilaterals in 2D and hexahedra in 3D, the use of $\mathscr{P}_p$ polynomial spaces not only renders the underlying DGFEM more efficient than the standard DGFEM using tensor-product polynomials of degree p in each coordinate direction ($\mathscr{Q}_p$), but also more efficient than the corresponding standard FEM, as the polynomial degree p increases. Going one step further, the exploitation of DGFEMs using polynomial spaces defined in the physical frame, means that DGFEMs naturally allow for the use of computational meshes consisting of general polytopic elements; indeed, this is the principle topic of this volume.

Numerical methods defined on computational meshes consisting of polytopic elements, with, potentially, many faces, have gained substantial traction in recent years for a number of important reasons. Clearly, a key underlying issue for all classes of FEMs/FVMs is the design of a suitable computational mesh upon which the underlying PDE problem will be discretized. The task of generating the mesh must address two competing issues: on the one hand, the mesh should provide a good representation of the given computational geometry with sufficient resolution for the computation of accurate numerical approximations. On the other hand, the mesh should not be so fine that computational turn-around times are too high, or in some cases even intractable, due to the high number of degrees of freedom in the resulting FEM/FVM. Traditionally, standard mesh generators generate grids consisting of triangular/quadrilateral elements in 2D and tetrahedral/hexahedral/prismatic/pyramidal elements in 3D; these will, henceforth, be collectively referred to as *standard element shapes*. In the presence of essentially lower-dimensional solution features, for example, boundary/internal layers, anisotropic meshing may be exploited. However, in regions of high curvature, the use of such highly-stretched elements may lead to element self-intersection, unless the curvature of the geometry is carefully 'propagated' into the interior of the mesh through the use of (computationally expensive) isoparametric element mappings. These issues are particularly pertinent in the context of high-order methods, since in this setting, accuracy is often achieved by exploiting coarse meshes in combination with local high-order polynomial basis functions. Thereby, flexibility, in terms of

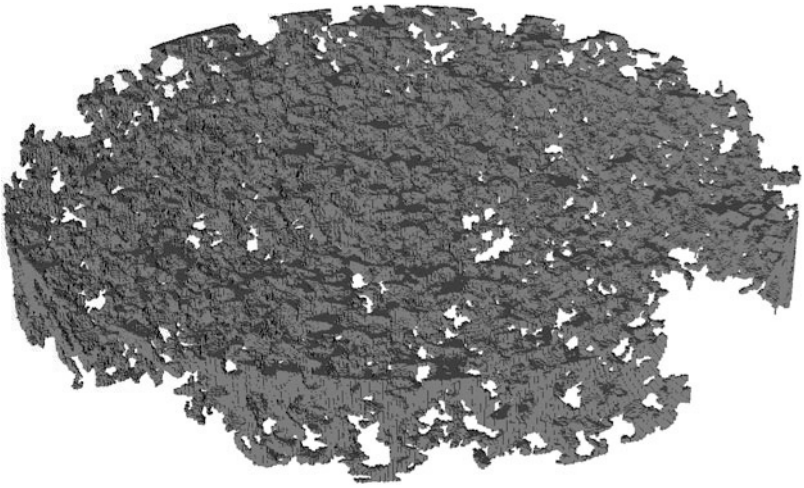

Fig. 1.1 Example of a porous scaffold used for in vitro bone tissue growth, cf. [12, 21, 22]

the shape of the elements admitted within a given coarse mesh, is crucial in this context for the efficient approximation of localized geometrical features present in the underlying geometry. Indeed, we highlight that the use of standard element shapes necessitates the exploitation of very fine computational meshes when the geometry possesses small details or microstructures. In such situations, an extremely large number of elements may be required for a given mesh generator to produce even a 'coarse' mesh which adequately describes the underlying geometry. As an example arising in biomedical applications, in Fig. 1.1 we show a finite element mesh, consisting of 3.2 million hexahedral elements, for a porous scaffold employed for in vitro bone tissue growth, cf. [12, 21, 22]. This computationally taxing mesh granularity is necessitated by the domain representation only. By dramatically increasing the flexibility in terms of the set of admissible element shapes present in the computational mesh, the resulting FEM/FVM can potentially deliver dramatic savings in computational costs. Indeed, allowing for polytopic element shapes, the number of elements can be substantially reduced without enforcing any domain approximation.

In the context of designing FEMs posed on meshes consisting of polytopic elements, a number of prominent techniques have been developed within the literature. For the discretization of PDEs in complicated geometries, Composite Finite Elements (CFEs) were originally proposed in the conforming setting by Hackbusch and Sauter [106, 107]; these techniques have been generalized to include DGFEMs in the series of articles [8, 12, 103]. We point out that CFEs are defined on general meshes consisting of polytopic elements generated as agglomerates of standard shaped elements. A closely related technique is the so-called agglomerated DGFEM [32–34]; this is very similar in spirit to the DGFEM CFE developed in [8], though the CFE methodology admits more general classes

of elemental shape functions. Another approach supporting general meshes is the recently introduced Hybrid High-Order method [80, 81, 83] which is related to the aforementioned HDG [75]. In the conforming setting, we also mention the Polygonal FEM [164] and the Extended FEM [98]; these two approaches achieve conformity by enriching/modifying the standard polynomial finite element spaces, in the spirit of the Generalized Finite Element framework of Babuška and Osborn [24]. Typically, the handling of non-standard shape functions carries an increase in computational effort. The recently proposed Virtual Element Method [2, 37, 39, 59], overcomes this difficulty, achieving the extension of conforming FEMs to polytopic elements, while maintaining the ease of implementation of these schemes; see, also, the closely related Mimetic Finite Difference method, cf. the monograph [38] and the references cited therein. We further refer to the volume [30] for a collection of review articles on the aforementioned techniques.

The ability to incorporate polytopic meshes offers a number of advantages also in the context of multilevel linear solvers, such as Schwarz-based domain decomposition preconditioners and multigrid. Indeed, a key difficulty in the implementation of the latter is the construction of a hierarchy of coarser meshes starting from a given fine one. The use of simple coarsening strategies may lead to the generation of 'holes' in the coarse meshes, and the poor approximation of fine scale geometric features; consequently, this can lead to a degradation in the performance of the resulting solver. This issue is trivially resolved when polytopic meshes are admitted, since hierarchies of coarser meshes can be constructed via agglomeration of fine-scale elements into coarser polytopes; see, for example, [9, 15, 102], and the references cited therein. Moreover, unstructured and/or hybrid meshes, consisting of mixed element shapes and nonconforming meshes containing hanging nodes, may be easily treated. DGFEMs are particularly pertinent in this context, as coarsening and refinement via element agglomeration and subdivision produce hierarchically-related approximation spaces. This is of crucial importance when projecting from one mesh to another, for example, in adaptive methods for evolution PDEs.

In conclusion, from a meshing point of view, the exploitation of general polytopic elements provides enormous flexibility. In addition to meshing complicated geometries using a minimal number of elements, polytopic elements are naturally suited to applications in complicated/moving domains, for example, in solid mechanics, fluid-structure interaction, geophysical problems including earthquake engineering and flows in fractured porous media, and mathematical biology. Indeed, general element shapes are often exploited as transitional elements in finite element meshes, for example, when fictitious domain methods, unfitted methods, or overlapping meshes are employed, cf., for example, [48–50, 125, 137]. The use of similar techniques in the context of characteristic-based/Lagrange–Galerkin methods is also relevant.

The principal aim of this volume is to provide a comprehensive mathematical introduction to the construction and analysis of DGFEMs on extremely general classes of meshes consisting of polytopic elements. Particular importance will be given to the key issue that, under mesh enrichment, shape-regular polytopes in $\mathbb{R}^d$, $d > 1$, may permit degenerate $(d-k)$-dimensional facets, $k = 1, \ldots, d-1$.

This issue does not typically arise when studying FEMs on standard shape-regular meshes consisting of simplicial/tensor-product elements. Indeed, shape-regular (cf. Definition 2 below for the precise meaning) d-dimensional polytopes with more than $d+1$ faces may admit arbitrarily small $(d-k)$-dimensional facets, $k = 1, \ldots, d-1$, relative to their diameter. This issue is intimately related to the correct choice of the (user-defined) discontinuity-penalization function present in the DGFEM discretization of second-order elliptic PDE problems. In the analysis presented below, stability and *hp*-version a priori error bounds will be established, which are sharp with respect to this type of degeneration. Additionally, under mesh refinement, the number of faces that each polytopic element possesses may not remain uniformly bounded; conditions under which this type of degeneration can be admitted will also be studied. While the following exposition is, naturally, focused on areas of interest of the authors, the concepts presented here are far-reaching; where applicable, we shall highlight potential extensions throughout this volume.

1.2 Overview and Scope

We assume that the reader is familiar with the derivation and construction of classical FEMs based on employing continuous piecewise polynomials defined over a given fixed computational mesh $\mathscr{T}_h$; for background, we refer, for example, to [42, 90, 126, 133]. The outline of this volume is as follows. The construction of DGFEMs starting from a local elementwise formulation with weakly imposed boundary conditions, together with the corresponding flux formulation for first-order hyperbolic problems, is covered in the first part of Chap. 2. On the basis of this flux formulation, the treatment of second-order PDEs is then also considered. The key theoretical tools needed to analyze DGFEMs on general polytopic meshes are outlined in Chap. 3; here, both *hp*-version inverse inequalities and approximation results are established. On the basis of these bounds, in Chap. 4 we consider the a priori error analysis of the so-called symmetric interior penalty DGFEM for the numerical approximation of pure diffusion problems, though we stress that the analysis naturally extends to include other DGFEM formulations. The analysis is extended to general second-order PDEs with nonnegative characteristic form in Chap. 5; here, we also consider the treatment of space-time DGFEMs for the numerical approximation of parabolic PDEs. The implementation aspects of DGFEMs on general polytopes are discussed in Chap. 6. Here the three main practical challenges, namely, mesh generation, construction of the elemental polynomial basis, and numerical integration, are discussed. To demonstrate the flexibility of this approach, in Chap. 7 we discuss automatic mesh refinement for DGFEMs posed on general agglomerated elements, based on exploiting dual-weighted-residual a posteriori error indicators. Finally, in Chap. 8 we summarize the work presented in this volume and outline potential future areas of research.