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Philosophical Logic: Current Trends in Asia

Proceedings of AWPL-TPLC 2016

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Preface

This new volume is a collection of selected papers presented at the Joint Conference of the Third Asian Workshop on Philosophical Logic and the Third Taiwan Philosophical Logic Colloquium (AWPL-TPLC 2016), held during 5–8 October 2016, at Department of Philosophy, National Taiwan University, Taiwan.

Ever since Gottlob Frege's *Begriffsschrift* (1879) and the pioneering works originating with the so-called new logic (to be compared with the then traditional Aristotelian syllogism), by George Boole, Georg Cantor, Giuseppe Peano, David Hilbert, Bertrand Russell and Alfred N. Whitehead, etc., logicians had concerned themselves with the establishment of the foundation of mathematics, which had led to various attempts to reduce arithmetic (and *a fortiori* mathematics) to logic. The new logic was thus entitled 'mathematical logic'. Russell during the 1910s and 1920s highlighted a more philosophical and methodological aspect of this new logic, suggesting that a logical theory not only builds up a solid foundation for mathematical reasoning, but also provides a comfortable framework for analysis of the nature of mathematical propositions and concepts. Basic concepts in mathematics and controversial issues regarding the foundation of mathematics can thus be explicitly articulated in a logical theory. Russell maintained that this methodology can be applied to philosophical propositions in general—the gist of his well-known logical atomism is that 'logic is fundamental in philosophy'.

It was soon realized that the framework of the new style of logical theory, taken as the orthodox or paradigm in the Fregean tradition, may not be comprehensible enough to deal with all kinds of philosophical propositions. It was well observed that certain types of propositions such as propositions concerning future events seemingly defy the basic logical laws of the orthodox logic, which suggests that the logic should be modified or revised. A variety of logical theories alternative to, or deviant from, the orthodox logic were proposed, such as Jan Łukasiewicz's three-valued propositional calculus (for future contingent propositions around 1917), Clarence Irving Lewis's strict implication calculus (for propositions with necessity/possibility, during the 1910s–1920s), Arend Heyting's intuitionistic logic (for a formal basis for Brouwer's intuitionism in the foundation of mathematics in 1930). In a more general sense, they were treated as the rivals of the so-called new logic, and gained the label 'non-classical logics', leaving the term 'classical logic' to the Fregean orthodox logic.

The flourishing of non-classical logics after late 1960s has proved that substantial analyses of certain philosophical propositions and the related concepts can be obtained based on the formulation of the correspondent non-classical logic and its semantical framework, such as modal logics for necessity/possibility and epistemic logic for knowledge/beliefs, to mention a few. This goal can be achieved typically by virtue of structural analyses of the models involved and aximatization, or theorization, of a group of related philosophical doctrines. The impact that non-classical logics have on philosophy is manifested by the fact that the term 'philosophical logic' has been widely used to cover the study of non-classical logics and the study of philosophical topics and concepts (typically, identity, existence, truth, predication, meaning, modality, etc.) that can be dealt with in the framework of non-classical logic. *The Journal of Philosophical Logic*, arguably the first journal mainly devoted to original works in philosophical logic, came into being in 1972 (the first volume of *The Journal of Symbolic Logic*, by comparison, was in 1936). And the first edition of *the Handbook of Philosophical Logic* (four volumes) was published during 1983–1989 (in comparison to the one volume *the Handbook of Mathematical Logic*, edited by Jon Barwise, North Holland, 1977). The second edition of *the Handbook of Philosophical Logic* has become an open-ended project, giving rise to an unlimited series of collection, and since 2001, 17 volumes have been published. This new edition of the Handbook has marked dramatic changes in the landscape of philosophical logic: New areas have been included by the establishment of a large number of new logics; while old areas were significantly enriched and expanded.

Since 2000, more and more logicians and philosophers in Asia area have paid great attention to philosophical logic. To promote mutual understanding and collaboration for future researchers in Asian on philosophical logic, a series of biennial conferences—*Asian Workshop on Philosophical Logic* (AWPLs), was initiated by Professor Hiroakira Ono (Japan Advanced Institute of Science and Technology, JAIST, Japan) and some others. *The First Asian Workshop on Philosophical Logic* (AWPL-2012) was held at JAIST in February 2012, followed by *The Second Asian Workshop on Philosophical Logic* (AWPL-2014) in April 2014 at Sun Yat-sen University, Guangzhou, China. A post-conference proceedings of AWPL-2014, *Modality, Semantics and Interpretations* (Ju Shier, Liu Hu and Hiroakira Ono, eds. 2015) was published as the first volume of the then newly established LIAA-book series ('Logic in Asia'), a subseries of Studia Logica Library, Springer. Almost at the same time, The Taiwan Philosophical Logic Colloquium (TPLCs), another series of biennial conferences in philosophical logic, hosted by the Department of Philosophy, National Taiwan University, Taiwan, with funding from the private sector, was established. A post-conference proceedings of TPLC-2014, *Structural Analysis of Non-classical Logics* (Syraya Chin-Mu Yang, Duen-Min Deng and Hanti Lin, eds.), was published as the second volume of LIAA-book series by Springer in 2015.

The aim of AWPLs and TPLCs is to provide a forum for dialogues amongst logic-minded philosophers and philosophically oriented logicians. The scope of AWPLs and TPLCs covers philosophical logic (in a broad sense), non-classical logics, algebraic logic, mereology, their applications to computer science, cognitive science, linguistics, game theory, and other social sciences, etc., and all kinds of semantics/logics relating to philosophical concepts and their applications in philosophical issues. It is dedicated to promoting both theoretical and empirical studies of logic (typically non-classical logics), with a close connection to disciplines that draw on diverse methods and approaches from philosophy, computer science, mathematics, psychology and linguistics.

The present volume, together with the two aforementioned volumes, *Modality, Semantics and Interpretations* and *Structural Analysis of Non-Classical Logics*, intends to flag a significant portion of the landscape of the development of philosophical logic in Asia at the early twenty-first century. We hope that these volumes can provide a useful and representative survey of the main fields to which distinguished logicians and philosophers in Asia, and perhaps in Australasian regions, have devoted their research.

In the opening chapter, "Representing and Completing Lattices by Propositions of Cover Systems," Robert Goldblatt continues his project of developing a theory of cover systems that encompasses non-distributive logics. In previous papers, he has used this to provide structural semantics for the logic of residuated ordered semigroups and quantales, intuitionistic modal logics, classical bilinear logic, relevant logic and the storage and consumption modalities of linear logic. The present article studies cover systems on the set of principal filters of a lattice and their role in lattice representations, obtaining presentations of the ideal completion and the MacNeille completion of the original lattice as lattices of propositions of a cover system. This is explored further for ortholattices, and for Heyting algebras in relation to Grothendieck topologies.

Hiroakira Ono's Chapter "A Uniform Algebraic Approach to Cut Elimination Via Semi-Completeness" introduces an algebraic condition 'semi-completeness' of sequent systems (due originally to Shoji Maehara, 1991). It is proved that the semi-completeness of a given system S without cut rule gives a sufficient condition of eliminating cut. Moreover, many of existing semantical proofs of cut elimination, using either Kripke semantics or algebraic one, can be naturally transformed into algebraic proofs of semi-completeness. The author concludes that semi-completeness offers a uniform algebraic way of understanding cut elimination, of both single- and multiple-succedent sequent systems, for a wide variety of non-classical propositional and predicate logics as well.

Chapter "Ancient Indian Logic, Pakṣa and Analogy" (by Jeffrey Paris and Alena Vencovská), provides a formalization which intends to capture the suggestion of B.K. Matilal, and earlier J.F. Staal, that the Indian Schema from Gotama's Nyāya–Sūtra should be understood in terms of an 'occurrence' relation linking events to their loci. It goes on to show that in consequence the Schema inherits a rational justification as analogical reasoning within Unary Pure Inductive Logic from the widely accepted principle of Atom Exchangeability, itself a property of the Carnap's Continuum of Inductive Methods.

Shih Ping Tung in Chapter "Provability and Decidability of Arithmetical Sentences", proves several results about the decidability, axiomatizability, recursive enumerability, etc., of certain rather basic families of arithmetic sentences. More specifically, it shows that the sets of all sentences of the form $\forall z \exists x \forall y\, f(x,y) - az \neq 0$, where $f(x,y) \in \mathbb{Z}[x, y]$ and $a \in \mathbb{Z}$, true in $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are axiomatizable, respectively. It follows that the sets of all sentences of the form $\exists z \forall x \exists y\, f(x,y) - az = 0$ true in $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are decidable, respectively. These results use earlier, seemingly more ground-breaking, work of the author's (from over 30 years ago) but nevertheless make a worthwhile contribution to the subject which may pave a new path to future work of researchers in this field.

In Chapter "On the Minimization Principle in the Boolean Approach to Causal Discovery", Jiji Zhang examines a Boolean approach to causal inference, which is rooted in John Mackie's celebrated INUS theory of causation and has been developed into several sophisticated data analysis methods for social scientists. The target of Zhang's criticism is the minimization steps in a most recent implementation of this approach, known as the method of Coincidence Analysis (CNA). Zhang presents prima facie counterexamples to the soundness of the minimization steps in CNA and discusses two possible responses to them. The author then argues that while one of the responses is viable, it renders the role of minimization much less substantial than it is usually intended to be.

Chapter "Contentual and Formal Aspects of Gentzen's Consistency Proofs", (by Ryota Akiyoshi and Yuta Takahashi) offers an analysis of Gentzen's second consistency proof for first-order arithmetic in 1936. Wilfried Sieg has recently highlighted two distinct notions of consistency proofs in Gentzen's series of work then: contentual correctness proofs (semantic manner in character, typically Gentzen's consistency proof in 1935); and formal correctness proofs (substantially a proof-theoretic approach, as shown in a 1938 paper). The authors show that Gentzen's 1936 proof is both contentual and formal. The connection between the contentual aspect of this proof and its formal aspect is specified and some consequences are noted.

Hao-Cheng Fu in Chapter "Saving Supervaluationism from the Challenge of Higher-Order Vagueness Argument" proposes a revised version of supervaluationism which could govern the puzzle of vagueness. Some plausible solutions to the problem of vagueness are reviewed, and shown to fail due to the phenomenon of higher order vagueness. Fu further argues that the proposed supervaluationism could be appealing if we can construct a dynamic model instead of the static model of supervaluationism.

Chapter "Cut Free Labelled Sequent Calculus for Dynamic Logic of Relation Changers", (by Ryo Hatano, Katsuhiko Sano, and Satoshi Tojo), provides a cut-free labelled sequent calculus **GDLRC** for van Benthem and Liu's dynamic logic of relation changers (**DLRC**, 2007), a variant of dynamic epistemic logic (**DEL**) that provides a general framework to capture many dynamic operators of **DEL** in terms of relation changing operation written by programs in propositional dynamic logic

(**PDL**). In contrast, proof theory for **DLRC** has not been well-studied except for a Hilbert-style axiomatization proposed in van Benthem and Liu's work. The authors further show that **GDLRC** is equipollent with the aforementioned Hilbert-style axiomatization.

Ryo Kashima in Chapter "On Second Order Propositional Intuitionistic Logics", studies two second-order propositional intuitionistic logics: the first one, with the full comprehension axiom; the other, including the constant domain axiom. The completeness theorems for these two logics with respect to corresponding Kripke models were proved by Sobolev (1977) and Gabbay (1974), respectively. Kashima here offers some slightly strong alternative proofs, using the technique of nested sequent calculi, and consider a closure condition on Kripke models, namely 'the domain of quantification is closed under any operation that is induced by a formula'. The author shows that this condition depends on the characterization of disjunction because at propositional level, a disjunction of the form 'A∨B', taking '∨' as primitive, and the formula (for all x)((A → x) →((B → x) → x)) induce different operations, whereas at second order level, A∨B can be defined by (for all x)((A → x) → ((B → x) → x)). Noticeably, this difference is critical in the argument on the constant domain condition in that if the language does not contain disjunction as primitive, the constant domain axiom is not required for the completeness with respect to constant domain models.

In Chapter "Classical Model Existence Theorem in Subclassical Predicate Logics. II", Jui-Lin Lee shows that there are some much weaker logics satisfying the classical model existence property (*CME*)—every consistent set has a classical model. By using weak deduction theorem, in propositional logics, Lee improves previous results and shows that some weak extensions of *BCI*/*BCIW* logic satisfy *CME*. Glivenko's Theorem for corresponding logics is also proved. In predicate logics, under such a weak propositional logic part, Lee uses the Herbrand-Henkin style approach (via prenex normal form theorem) and also the Hintikka style approach to construct weaker subclassical predicate logics which satisfy *CME*.

Ren–June Wang's Chapter "On Incorporating Reasoning Time into Epistemic Logic", introduces the notion of *reasoning-based knowledge*, a concept of knowledge taking reasoning time into account, in contrast with the *information-based knowledge*, which is normally formulated by the plain possible world semantics. Two formal systems, $\mathbf{tMEL}^{\mathbf{K}}$ and $\mathbf{tMEL}^{\infty}$, are proposed, with each of them having a device representing the information-based knowledge and reasoning-based knowledge, respectively. The author further applies $\mathbf{tMEL}^{\infty}$ to investigate the epistemic valid formula $\sim K(p \ \& \sim Kp)$, with a detour to discuss the Moore's paradox from the reasoning-based knowledge perspective.

Sakiko Yamasaki and Katsuhiko Sano in Chapter "Proof-Theoretic Embedding from Visser's Basic Propositional Logic to Modal Logic K4 via Non-Labelled Sequent Calculi" employ **G3**-style *non-labelled* sequent calculi to establish a proof-theoretic embedding from Visser's Basic Propositional Logic **BPL** into modal logic **K4** via a variant of Gödel–McKinsey–Tarski translation sending an atom P to $P \ \& \Box P$, where the logic **BPL** is obtained by dropping the requirement of reflexivity from Kripke semantics for intuitionistic logic. The authors first provide for **BPL** a **G3**-style non-labelled sequent calculus **G3BPL**, which enjoys cut elimination theorem and is proved to be sound and complete for intended Kripke semantics. Then they establish a proof-theoretic embedding where the above translation plays a key role in the proof for the direction of faithfulness, i.e. if the translation of a formula of **BPL** is provable in **K4** then the original formula is provable in **BPL**. By this proof-theoretic embedding, the authors also provide another syntactic proof of cut elimination theorem for **G3BPL** by reducing the admissibility of cut of **G3BPL** to that of a **G3**-style non-labelled sequent calculus for **K4**.

It is noteworthy that AWPL-TPLC 2016 organized two special workshops. The first one is a one-day plus workshop on Williamson's philosophy to honour Professor Timothy Williamson (Wykeham Professor of Logic, University of Oxford) for his outstanding contribution to philosophical logic (typically on identity, vagueness, knowledge first epistemology and metaphysics of modality). We regret that for the sake of the limitation of the space, papers based on talks at this workshop could not be included in this volume. A second workshop on mereology was organized to signify certain philosophical aspect of mereology. The remaining two chapters of this volume come out from presentations at this workshop.

In Chapter "Varieties of Parthood", Paul Hovda addresses formal patterns that emerge on the view that there are a variety of parthood relations, and that certain general principles govern these relations and connect them to one another. One such principle is shown to connect with the strong supplementation principle of classical mereology, and also to the classical notion of mereological fusion. The main line of development shows roughly that, for any given variety of parthood, relating objects in a given domain, for any larger domain that includes the given one, and where the new objects may have the old objects as parts in a new manner, then, on one general way of being a part, there is a 'unique' possible structure resulting, itself intimately related to classical mereology. The uniqueness is measured relative to a notion like the standard notion of isomorphism, but weaker, which we call 'quasi-isomorphism.'

In the final Chapter, "Infinite 'Atomic' Mereological Structures", Hsing–Chien Tsai shows that the strongest first-order atomic mereological theory which can be generated by axioms found in the literature, that is, General Extensional Mereology with the atomicity axiom, as well as a natural second-order extension of such a theory, that is, Classical Mereology with the atomicity axiom, still cannot secure an atomic domain, where a domain is atomic if it contains a collection of 'atoms' (an atom is something which has no proper parts) and each of its members is composed of some atoms and nothing else. These are further results of what have been done earlier by the author.

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Representing and Completing Lattices by Propositions of Cover Systems

Robert Goldblatt

Abstract Cover systems abstract from the properties of open covers in topology, and have been used to construct lattices of propositions for various modal and non-modal substructural logics. Here we explore cover systems on the set of principal filters of a lattice and their role in lattice representations. A particular system with finite covers gives a lattice of propositions isomorphic to the ideal completion of the original lattice. Relaxing the finiteness condition yields another cover system whose lattice of propositions gives a presentation of the MacNeille completion. This is analysed further for ortholattices. For Heyting algebras a stricter cover relation is shown to have the properties of a Grothendieck topology while its lattice of propositions is still the MacNeille completion.

1 Introduction

In several previous articles the author has used the notion of a *cover system* to develop structural semantics for propositional and quantifed versions of various logics. These include the logic of residuated ordered semigroups and quantales (Goldblatt 2006), intuitionistic modal logics (Goldblatt 2011), classical bilinear logic (Goldblatt 2011), relevant logic (Goldblatt 2011, Chap. 6), and the modalities ! and ? of linear logic (Goldblatt 2016).

Cover systems abstract from the properties of open covers in topology, and in a categorical setting have led to the development of the Kripke-Joyal semantics for intuitionistic logic in topoi (Mac Lane and Moerdijk 1992). This diverges from Kripke's own semantics by giving a non-classical interpretation to disjunction while still validating the distribution of conjunction over disjunction. In some of the above-mentioned articles we adapted cover systems to model logics that lack this distribution.

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The present article explores the use of cover systems to construct representations of lattices in general, including non-distributive ones. In the early days of lattice theory, Birkhoff and Frink (1948) defined a *meet-representation* of a lattice L as a map $a \mapsto |a|$ from elements a of L to certain sets $|a|$ which "sends lattice meets into set-intersections", i.e. $|a \sqcap b| = |a| \cap |b|$ where $\sqcap$ denotes meet and $\cap$ denotes intersection. They observed that any lattice can be given a faithful meet-representation by taking $|a|$ to be set of all filters of L containing a, or just the set of all principal filters containing a. Here we focus on the set S_L of principal filters. We construct a cover system that defines a closure operator j on subset of S_L such that a meet-representation results in which lattice joins are sent to *closures* of set-unions, i.e. $|a \sqcup b| = j(|a| \cup |b|)$.

So in this context the basic role of cover systems is to provide a way of defining closure operators. The lattice L is isomorphically embedded into a *complete* lattice of fixed points of the closure operator j. We call these fixed points *propositions* because of their role in modelling formulas of logical systems. The lattice $Prop(\mathscr{S}_L)$ of propositions is a *completion* of L: a complete lattice having a sublattice isomorphic to L. The first cover system we define has finite covers and makes $Prop(\mathscr{S}_L)$ isomorphic to the ideal completion of L. Dropping the finiteness condition produces another cover system whose lattice of propositions is a presentation of the MacNeille completion of L. We then analyse this situation further when L is an ortholattice or a Heyting algebra.

Here is an outline of the paper. After setting out the basic theory of closure operators and cover systems in Sect. 2, we proceed in Sect. 3 to define a cover system on S_L whose covers are finite and show that L embeds isomorphically into the resulting $Prop(\mathscr{S}_L)$. In Sect. 4 we show that if L is distributive then $\mathscr{S}_L$ has a property that forces $Prop(\mathscr{S}_L)$ to be a complete Heyting algebra. Then in Sect. 5 we show that if L itself is a Heyting algebra, then the embedding into $Prop(\mathscr{S}_L)$ preserves the Heyting implication operation. Section 6 proves that $Prop(\mathscr{S}_L)$ is isomorphic to the ideal completion of L, i.e. the complete lattice of all ideals of L. In Sect. 7 a different cover system $\mathscr{S}_L^m$ is defined by dropping the finiteness restriction on covers. It is shown that the resulting lattice of propositions $Prop(\mathscr{S}_L^m)$ is isomorphic to the MacNeille completion of L. Section 8 considers the case that L is an ortholattice and reviews the embedding of L into a complete lattice $Orth(S_L, \perp_L)$ of "bi-orthogonally closed" sets defined using an orthogonality relation $\perp_L$ on S_L. We show that $Orth(S_L, \perp_L) = Prop(\mathscr{S}_L^m)$. Finally, in Sect. 9 we return to the Heyting algebra case to define a more restricted cover system $\mathscr{S}_L^G$ that has additional properties making it a Grothendieck topology, but defines the same lattice of propositions as $\mathscr{S}_L^m$.

2 Closure Operators and Cover Systems

Let $(L, \sqsubseteq)$ be a *poset*, comprising a partial ordering $\sqsubseteq$ on a non-empty set L. We write $\bigsqcup X$ for the *join* (= least upper bound), and $\bigsqcap X$ for the *meet* (= greatest lower bound) of a set $X \subseteq L$, when these bounds exist. The smaller symbols $\sqcup$ and $\sqcap$ are used for the binary join and meet operations. The poset is a *lattice* if $a \sqcap b$ and $a \sqcup b$

exist for every $a, b \in L$, and so $\bigsqcup X$ and $\bigsqcap X$ exist for every finite non-empty X. A poset is *order-complete*, or just *complete*, if every subset has a meet, or equivalently if every subset has a join.

A function $j : L \to L$ is defined to be

- *monotone* if it is order-preserving, i.e. $a \sqsubseteq b$ implies $ja \sqsubseteq jb$;
- *inflationary* if $a \sqsubseteq ja$, for all $a \in L$;
- *idempotent* if $j(ja) = ja$, for all $a \in L$;
- a *closure operator* if it is monotone, inflationary and idempotent.

An element $a \in L$ is called *j-closed* if $ja = a$, i.e. if a is a *fixed point* under j. If $(L, \sqsubseteq)$ is a *complete* poset and j is a closure operator, the set L^j of j-closed elements is closed under meets $\bigsqcap X$, and so is order-complete under the same partial ordering and with same meet operation. The join operation $\bigsqcup^j$ in $(L^j, \sqsubseteq)$ is given by $\bigsqcup^j X = j(\bigsqcup X)$ for all $X \subseteq L^j$.

Our aim is to explore ways in which a given lattice can be represented as a lattice of j-closed elements for a closure operator j on some complete lattice $(L, \sqsubseteq)$ whose elements are *sets* and in which $\sqsubseteq$ is the set inclusion relation $\subseteq$, the meet $\bigsqcap \Theta$ is the set-intersection $\bigcap \Theta$, and the join $\bigsqcup \Theta$ is set-union $\bigcup \Theta$; hence in $(L^j, \subseteq)$ the meet will be $\bigcap \Theta$ and the join $j(\bigcup \Theta)$. In pursuit of that aim we consider structures of the form

$$\mathscr{S} = (S, \preccurlyeq, \lhd, \dots)$$

that fulfil the following description:

- $\preccurlyeq$ is a binary relation on set S that is a *preorder*, i.e. reflexive and transitive. We may write $y \succcurlyeq x$ when $x \preccurlyeq y$, and say that y *refines* x.
- $\lhd$ is a binary relation from S to its powerset $\mathscr{P}S$. When $x \lhd C$, where $x \in S$ and $C \subseteq S$, we say that x *is covered by* C, and write this also as $C \rhd x$, saying that C *covers* x or that C is an *x-cover*.

In such a structure, an *up-set* is a subset X of S that is closed upwardly under refinement: $y \succcurlyeq x \in X$ implies $y \in X$. For each $X \subseteq S$,

$$\uparrow X = \{y \in S : (\exists x \in X)\, x \preccurlyeq y\}$$

is the smallest up-set including X. For $x \in S$, we write $\uparrow x$ for $\uparrow\{x\} = \{y : x \preccurlyeq y\}$, the smallest up-set containing x.

The collection $Up(\mathscr{S})$ of all up-sets of $\mathscr{S}$ is a complete poset under the partial order $\subseteq$ of set inclusion, with the meet and join of any collection Θ of up-sets being its set intersection $\bigcap \Theta$ and union $\bigcup \Theta$, respectively. $Up(\mathscr{S})$ has least element $\emptyset$ and greatest element S.

Now for each subset X of S, define

$$j_\lhd X = \{x \in S : \exists C\, (x \lhd C \subseteq X)\}. \tag{1}$$

A property can be thought of as being *locally true* of x if x is covered by a set whose members that have this property, i.e. if there is some C such that $x \lhd C$ and each member of C has the property. In this sense, x belongs to $j_\lhd X$ just when the property of *being a member of* X is locally true of x. So $j_\lhd X$ can be thought of as the collection of "local members" of X. X is called *localised* if $j_\lhd X \subseteq X$, i.e. if every local member of X is an actual member of X.

The operation $j_\lhd$ is $\subseteq$-monotonic on subsets of S: $X \subseteq Y$ implies $j_\lhd X \subseteq j_\lhd Y$. To discuss further properties of $j_\lhd$ we will say that a subset Y of S *refines* a subset X, or X *is refined by* Y, if $Y \subseteq \uparrow X$, i.e. if every member of Y refines some member of X. Consider the following possible axioms, for an arbitrary $x \in S$:

- *Refinement*: if $x \preccurlyeq y$, then every x-cover can be refined to a y-cover, i.e. if $C \rhd x$, then there exists a C' such that $y \lhd C' \subseteq \uparrow C$.
- *Existence*: there exists an x-cover $C \subseteq \uparrow x$;
- *Refined Transitivity*: if $x \lhd C$ and for all $y \in C$, $y \lhd C_y$, then $\bigcup_{y \in C} C_y$ can be refined to an x-cover, i.e. there exists a C' such that $x \lhd C'$ and $C' \subseteq \uparrow \bigcup_{y \in C} C_y$.

Theorem 1 *(1) $\mathscr{S}$ satisfies the Refinement axiom iff $j_\lhd X$ is an up-set whenever X is an up-set.*
(2) $\mathscr{S}$ satisfies the Existence axiom iff $j_\lhd$ is inflationary on $Up(\mathscr{S})$.
(3) $\mathscr{S}$ satisfies Refined Transitivity iff $j_\lhd j_\lhd X \subseteq j_\lhd X$ for all up-sets X.

Proof (1) Let $\mathscr{S}$ satisfy Refinement. If X is an up-set and $y \succcurlyeq x \in j_\lhd X$, then there is an x-cover $C \subseteq X$, and this can be refined to y-cover $C' \subseteq \uparrow C \subseteq \uparrow X = X$. Hence $y \in j_\lhd X$. This shows that $j_\lhd X$ is an up-set.
Conversely, suppose $j_\lhd$ maps $Up(\mathscr{S})$ into itself. Let $C \rhd x$ and $x \preccurlyeq y$. Since $C \subseteq \uparrow C$ it follows that $x \in j_\lhd \uparrow C$. But $\uparrow C$ is an up-set, hence so is $j_\lhd \uparrow C$ by hypothesis, and therefore $y \in j_\lhd \uparrow C$. This means that there exists a y-cover $C' \subseteq \uparrow C$, confirming that the Refinement axiom holds.

(2) Let $\mathscr{S}$ satisfy Existence. Then if X is an up-set and $x \in X$, there is a C with $x \lhd C \subseteq \uparrow x \subseteq X$. Hence $x \in j_\lhd X$. This shows $X \subseteq j_\lhd X$ as required.
Conversely, if $j_\lhd$ is inflationary on $Up(\mathscr{S})$, then for any x we have $x \in \uparrow x \subseteq j_\lhd \uparrow x$. This implies that there is some C such that $x \lhd C \subseteq \uparrow x$, confirming Existence.

(3) Assume Refined Transitivity. If X is an up-set and $x \in j_\lhd j_\lhd X$, then there is an x-cover $C \subseteq j_\lhd X$, so for all $y \in C$ there is a y-cover $C_y \subseteq X$. Hence by assumption there exists a C' such that

$$x \lhd C' \subseteq \uparrow \bigcup_{y \in C} C_y \subseteq \uparrow X = X.$$

This implies that $x \in j_\lhd X$, confirming that $j_\lhd j_\lhd X \subseteq j_\lhd X$.
For the converse, let $x \lhd C$ and for all $y \in C$, $y \lhd C_y$. Define X to be the up-set $\uparrow(\bigcup_{y \in C} C_y)$. Then $y \lhd C_y \subseteq X$ and hence $y \in j_\lhd X$, for all $y \in C$. Thus $x \lhd C \subseteq j_\lhd X$, showing $x \in j_\lhd j_\lhd X$. So if $j_\lhd j_\lhd X \subseteq j_\lhd X$ we get $x \in j_\lhd X$, hence there is an x-cover $C' \subseteq X$, as required to prove Refined Transitivity. □

We now define a structure $\mathcal{S}$ to be a *cover system* if it satisfies the Refinement, Existence and Refined Transitivity axioms. From Theorem 1 we see that this is equivalent to requiring that $j_{\lhd}$ be a closure operator on the complete poset $(Up(\mathcal{S}), \subseteq)$ of up-sets of $\mathcal{S}$.

Drágalin (1979, p. 72) defined an operation $\mathbf{D}$ on *down*-sets of a preorder by taking a function Q assigning to each $x \in S$ a collection $Q(x)$ of subsets of S, and putting $\mathbf{D}Y = \{x \in S : \forall C \in Q(x), C \cap Y \neq \emptyset\}$. He gave conditions on Q ensuring that $\mathbf{D}$ is a closure operator, and interpreted $C \in Q(x)$ by saying that 'C is a path starting from the moment x'. Now if we define $x \lhd C$ to mean $C \in Q(x)$, then for any up-set X it follows that $S \setminus X$ is a down-set and $j_{\lhd}X = S \setminus (\mathbf{D}(S \setminus X)$, so in this sense Drágalin's approach is dual to that of cover systems. See (Bezhanishvili and Holliday 2016) for a detailed discussion of this relationship.

Typically we will be dealing with cover systems that satisfy a strong version of Refined Transitivity, namely the property

- *Transitivity*: if $x \lhd C$ and for all $y \in C$, $y \lhd C_y$, then $\bigcup_{y \in C} C_y$ itself is an x-cover.

A *proposition* in a cover system is an up-set X that is localised, i.e. $j_{\lhd}X \subseteq X$, hence $j_{\lhd}X = X$. In general, a set X is a proposition iff $X = \uparrow X = j_{\lhd}X$. $j_{\lhd}\uparrow X$ is the smallest proposition that includes an arbitrary X, and $j_{\lhd}\uparrow x$ is the smallest proposition containing the element x. The smallest proposition including an up-set X is just $j_{\lhd}X$, so in fact $j_{\lhd}$ maps $Up(\mathcal{S})$ onto the set $Prop(\mathcal{S})$ of all localised up-sets of a cover system $\mathcal{S}$. Indeed, $Prop(\mathcal{S})$ is precisely the earlier described set L^j of fixed points of j when $L = Up(\mathcal{S})$ and $j = j_{\lhd}$. Thus $Prop(\mathcal{S}) = Up(\mathcal{S})^{j_{\lhd}}$ and forms a complete lattice under the inclusion order, in which any collection Θ of propositions has meet $\bigsqcap \Theta = \bigcap \Theta$ and join $\bigsqcup \Theta = j_{\lhd} \bigcup \Theta$.

3 Finite Covers for Representing a Lattice

Let $(L, \sqsubseteq, \sqcup, \sqcap)$ be an arbitrary lattice, which we typically refer to just as L. We will construct a cover system $\mathcal{S}_L$ on the set of principal filters of L, and show that L can be isomorphically embedded into its lattice of propositions $Prop(\mathcal{S}_L)$. For each $a \in L$, let

$$[a) = \{c \in L : a \sqsubseteq c\}$$

be the *principal filter* of L *generated by* a. This generator is unique, as $[a) = [b)$ iff $a = b$.

From the definition of joins it is immediate that $[a) \cap [b) = [a \sqcup b)$, and more generally that

$$\bigcap_{a \in X} [a) = [\bigsqcup X) \tag{2}$$

whenever the join of a subset X exists in L.

Let S_L be the set of all principal filters of L. For each $a \in L$, let $|a| = \{x \in S_L : a \in x\}$ be the set of all principal filters containing a. In general, if $x \in S_L$ then $a \in x$ iff $[a) \subseteq x$, so $|a|$ is the up-set $\uparrow[a)$ in the poset $(S_L, \subseteq)$. In other words, if we define g_x to be the unique generator of principal filter x, i.e. $x = [g_x)$, then in $(S_L, \subseteq)$ we have $\uparrow x = |g_x|$.

The function $a \mapsto |a|$ maps L *injectively* into the set of up-sets of $(S_L, \subseteq)$, since if $|a| = |b|$ then $\uparrow[a) = \uparrow[b)$, hence $[a) = [b)$ and so $a = b$. This function from L into $Up(S_L, \subseteq)$ also preserves lattice meets, since $|a \sqcap b| = |a| \cap |b|$ and more generally

$$|\bigsqcap X| = \bigcap_{a \in X} |a| \tag{3}$$

whenever the meet of a subset X exists in L. However the function need not preserve lattice joins, for instance the principal filter $[a \sqcup b)$ will contain $a \sqcup b$ but may not contain either a or b, in which case $|a \sqcup b| \neq |a| \cup |b|$.

To overcome this we introduce a cover relation $\lhd$ on S_L. For $x \in S_L$ and $C \subseteq S_L$, define

$$x \lhd C \iff C \text{ is finite and } \bigcap C \subseteq x. \tag{4}$$

Then the structure $\mathscr{S}_L := (S_L, \subseteq, \lhd)$ satisfies the axioms of a cover system, as we now show:

- *Refinement*: let $C \rhd x$ and $x \subseteq y$. Then $\bigcap C \subseteq x \subseteq y$. So in this case we get a y-cover C' refining C just by putting $C' = C$.
- *Existence*: for $x \in S_L$, let $C = \{x\}$. Then $C \subseteq \uparrow x$ as $\uparrow x = \{y \in S_L : x \subseteq y\}$, and $\bigcap C = x$ so C is an x-cover included in $\uparrow x$ as required.
- *Transitivity*: let $x \lhd C$ and for all $y \in C$, $y \lhd C_y$. Put $C' = \bigcup_{y \in C} C_y$. Then for each $y \in C$, $C_y \subseteq C'$ and so $\bigcap C' \subseteq \bigcap C_y \subseteq y$, the last inclusion holding because $y \lhd C_y$. Hence $\bigcap C' \subseteq \bigcap C \subseteq x$. But C' is finite, being the union of finitely many finite sets, so we have $x \lhd C'$, as required to establish the (stronger than Refined) Transitivity axiom.

In the cover system $\mathscr{S}_L$, each up-set of the form $|a|$ is localised, i.e. $j_\lhd |a| \subseteq |a|$, and hence is a proposition. For if $x \lhd C \subseteq |a|$, then $a \in \bigcap |a| \subseteq \bigcap C \subseteq x$, and so $x \in |a|$. Thus the map $a \mapsto |a|$ embeds L into the complete lattice $Prop(\mathscr{S}_L)$ of propositions of the cover system $\mathscr{S}_L$. To show that this map preserves binary joins we observe first that if X is any subset of L that has a join $\bigsqcup X$ in L, then using (2), for all $x \in S_L$,

$$\bigcap\{[a) : a \in X\} \subseteq x \text{ iff } [\bigsqcup X) \subseteq x \text{ iff } \bigsqcup X \in x. \tag{5}$$

Consequently, if X is any *finite non-empty* subset of L, then

$$x \lhd \{[a) : a \in X\} \text{ iff } \bigsqcup X \in x. \tag{6}$$

From this it follows that for all $a, b \in L$,

$$j_{\lhd}(|a| \cup |b|) = |a \sqcup b|. \tag{7}$$

Proof Since $|a| \cup |b| \subseteq |a \sqcup b|$, we have $j_{\lhd}(|a| \cup |b|) \subseteq j_{\lhd}|a \sqcup b| \subseteq |a \sqcup b|$. Conversely, if $x \in |a \sqcup b|$, then $a \sqcup b \in x$, so by (6) it follows that $x \lhd \{[a), [b)\} \subseteq |a| \cup |b|$, hence $x \in j_{\lhd}(|a| \cup |b|)$.

Equation (7) confirms that the map $a \mapsto |a|$ preserves binary joins between L and $Prop(\mathscr{S}_L)$. Altogether then we have established the following.

Theorem 2 *Every lattice L has an embedding into the complete lattice $Prop(\mathscr{S}_L)$ of all localised up-sets of the cover system $\mathscr{S}_L$, by a lattice monomorphism that also preserves any existing infinite meets of L.* □

Note that as $j_{\lhd}$ is inflationary we have $S_L = j_{\lhd} S_L$ and this is the greatest element of $Prop(\mathscr{S}_L)$. But if L has a greatest element 1, then it belongs to all principal filters, so $|1| = S_L$ and the map $a \mapsto |a|$ preserves greatest elements.

$Prop(\mathscr{S}_L)$ always has least element $j_{\lhd}\emptyset = \{x \in S_L : x \lhd \emptyset\}$. By (4) $x \lhd \emptyset$ iff $L = \bigcap \emptyset \subseteq x$ iff the generator of the principal filter x is a least element of L. Thus $j_{\lhd}\emptyset = \emptyset$ if L has no least element, while if it does have least element 0, then $j_{\lhd}\emptyset = \{[0)\} = L = |0|$, so the map $a \mapsto |a|$ preserves least elements.

4 Distributive Lattices

The lattice representation just given will now be adapted to the case that the lattice L is *distributive*, meaning that it satisfies the equation

$$a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c),$$

and hence satisfies

$$a \sqcap (\bigsqcup X) = \bigsqcup \{a \sqcap b : b \in X\} \tag{8}$$

for any non-empty finite $X \subseteq L$.

A cover system $\mathscr{S}$ will be called *localic* if, for each point x of $\mathscr{S}$,

$$\text{every x-cover can be refined to an x-cover that is included in} \uparrow \text{x}, \tag{9}$$

i.e. if $x \lhd C$, then there an x-cover C' with $C' \subseteq \uparrow C$ and $C' \subseteq \uparrow x$.

Theorem 3 *If L is a distributive lattice, then its associated cover system $\mathscr{S}_L$ is localic.*

Proof Suppose $x \lhd C$ in S_L. If $C = \emptyset$, then C itself is an x-cover refining C and included in $\uparrow x$. Otherwise, when $C \neq \emptyset$, the set $X = \{a \in L : [a) \in C\}$ is finite and non-empty, and $\{[a) : a \in X\} = C \rhd x$, so $\bigsqcup X \in x$ by (6). Hence $g_x \sqsubseteq \bigsqcup X$, where g_x is the generator of x. Applying the distributive law (8), we see that

$$\bigsqcup\{g_x \sqcap a : a \in X\} = g_x \sqcap (\bigsqcup X) = g_x \in x.$$

Defining $C' = \{[g_x \sqcap a) : a \in X\}$ then gives $x \lhd C'$ by (6). But if $[a) \in C$, then since $[a) \subseteq [g_x \sqcap a)$ it follows that $[g_x \sqcap a) \in \uparrow C$. Hence $C' \subseteq \uparrow C$, i.e. C' refines C in $\mathscr{S}_L$. Also, as $x = [g_x) \subseteq [g_x \sqcap a)$ we get that $C' \subseteq \uparrow x$, completing the proof that $\mathscr{S}_L$ satisfies the localic condition. □

Combining this result with Theorem 5 gives

Corollary 1 *Every distributive lattice L can be isomorphically embedded into the lattice of all propositions of the localic cover system $\mathscr{S}_L$.* □

It turns out that when $\mathscr{S}$ is localic, the lattice $Prop(\mathscr{S}) = Up(\mathscr{S})^{j_\lhd}$ is not just distributive, but satisfies the law (8) for *every* subset X, including $X = \emptyset$. In general a complete lattice in which finite meets distribute over arbitrary joins as in (8) is called a *locale*. For example, the complete lattice $Up(\mathscr{S})$ of up-sets of any cover system is a locale, since in its case (8) amounts to the set-theoretic fact that

$$X \cap \bigcup \Theta = \bigcup\{X \cap Y : Y \in \Theta\}$$

for any up-set X and any collection Θ of up-sets.

Note that a complete distributive lattice need not be a locale: it may be that (8) holds for finite X but not in general:

Example 1 Let L be the set of closed subsets of the real line $\mathbb{R}$ under its standard topology. L is a complete lattice under the inclusion order, with $\bigsqcap \Theta = \bigcap \Theta$ and $\bigsqcup \Theta = \text{cl} \bigcup \Theta$, where cl denotes topological closure. In fact $\bigsqcup \Theta = \bigcup \Theta$ when Θ is a finite set of closed subsets, so this lattice is distributive. Each singleton subset $\{p\}$ belongs to L, but $\{p\}$ is not open and so $p \in \text{cl}(\mathbb{R}\setminus\{p\}) = \text{cl} \bigcup_{r \neq p}\{r\}$. Thus $\{p\} \sqcap (\bigsqcup_{r \neq p}\{r\}) = \{p\}$. However $\bigsqcup_{r \neq p}(\{p\} \sqcap \{r\}) = \emptyset$, so this is a failure of (8). □

The proof that $Prop(\mathscr{S})$ is a locale when $\mathscr{S}$ is localic involves further theory of closure operators which we now outline. A *nucleus* on a lattice L (Simmons 1978) is a closure operator j that satisfies $ja \sqcap jb \sqsubseteq j(a \sqcap b)$, and hence preserves meets because the reverse inequality holds by monotonicity of j. In fact a closure operator j is a nucleus iff for $a, b \in L$,

$$a \sqcap jb \sqsubseteq j(a \sqcap b). \tag{10}$$

Certainly (10) holds if j is $\sqcap$-preserving, by the inflationary property of j. Conversely, by using (10) twice (and commutativity of $\sqcap$) we reason that

$$ja \sqcap jb \sqsubseteq j(ja \sqcap b) \sqsubseteq jj(a \sqcap b) \sqsubseteq j(a \sqcap b).$$

Theorem 4 *A cover system $\mathscr{S}$ is localic iff $j_\lhd$ is a nucleus on $Up(\mathscr{S})$.*

Proof Take $X, Y \in Up(\mathscr{S})$. If $x \in X \cap j_\lhd Y$, then $x \in X$ and there is an x-cover $C \subseteq Y$, hence $\uparrow x \subseteq X$ and $\uparrow C \subseteq Y$. But if $\mathscr{S}$ is localic, there is a C' with

$$x \lhd C' \subseteq \uparrow x \cap \uparrow C \subseteq X \cap Y,$$

hence $x \in j_\lhd(X \cap Y)$. This shows that $X \cap j_\lhd Y \subseteq j_\lhd(X \cap Y)$, so $j_\lhd$ satisfies (10) on $Up(\mathscr{S})$.

Conversely, assume $j_\lhd$ is a nucleus on $Up(\mathscr{S})$. Let C be an x-cover. Since $C \subseteq \uparrow C$, we have $x \in j_\lhd \uparrow C$ and therefore $x \in \uparrow x \cap j_\lhd \uparrow C$. (10) on $Up(\mathscr{S})$ then implies $x \in j_\lhd(\uparrow x \cap \uparrow C)$, which implies that there is an x-cover included in both $\uparrow x$ and $\uparrow C$, showing that $\mathscr{S}$ is localic. □

Now if j is a nucleus on a locale L, then the complete lattice L^j of fixed points of j is also a locale (Macnab 1976, 1981). For, if $a \in L$ and $X \subseteq L$, then in L,

$$a \sqcap j(\bigsqcup X) \sqsubseteq j(a \sqcap \bigsqcup X) = j(\bigsqcup\{a \sqcap b : b \in X\}).$$

But $\bigsqcup\{a \sqcap b : b \in X\} \sqsubseteq a \sqcap (\bigsqcup X)$ in any lattice when the two joins involved exist, so $j \bigsqcup\{a \sqcap b : b \in X\} \sqsubseteq ja \sqcap j(\bigsqcup X)$ by monotonicity of j. It follows that if $a \in L^j$, then

$$a \sqcap j(\bigsqcup X) = j(\bigsqcup\{a \sqcap b : b \in X\}).$$

But when also $X \subseteq L^j$, this last equation expresses that finite meets distribute over arbitrary joins in L^j, i.e. L^j is a locale.

Consequently, if $\mathscr{S}$ is localic then the lattice $Prop(\mathscr{S})$ of fixed points of the nucleus $j_\lhd$ is a locale, and in particular if L is a distributive lattice then $Prop(\mathscr{S}_L)$ is a locale. Corollary 1 thus implies the well-known fact that any distributive lattice can be isomorphically embedded into a locale. We elaborate on this in Sect. 6.

5 Heyting Algebras

A *Heyting algebra* is a lattice with a least element and a binary "implication" operation $a \Rightarrow b$ satisfying

$$c \sqsubseteq a \Rightarrow b \text{ iff } c \sqcap a \sqsubseteq b. \tag{11}$$

This operation is thereby uniquely determined, since (11) implies that

$$a \Rightarrow b = \bigsqcup\{c \in L : c \sqcap a \sqsubseteq b\}. \tag{12}$$

Heyting algebras provide the standard algebraic semantics for intuitionistic propositional logic. They are always distributive.

Any locale becomes a Heyting algebra when $a \Rightarrow b$ is *defined* by equation (12). Conversely, a complete Heyting algebra satisfies distributivity of finite meets over arbitrary joins (8). So the notions of locale and complete Heyting algebra are equivalent. The complete distributive lattice of Example 1 is not a locale.

For any cover system $\mathscr{S}$ it is readily seen that the operation $\Rightarrow$ in the locale $Up(\mathscr{S})$, as given according to (12) by the equation

$$X \Rightarrow Y = \bigcup\{Z \in Up(\mathscr{S}) : Z \cap X \subseteq Y\},$$

in fact has the simpler characterisation

$$X \Rightarrow Y = \{x \in S : \uparrow x \cap X \subseteq Y\}. \tag{13}$$

This means that $x \in X \Rightarrow Y$ iff $x \preccurlyeq y \in X$ implies $y \in Y$, reflecting the Kripke semantics for intuitionistic implication.

For the case of a locale of the form $Prop(\mathscr{S})$, i.e. when $\mathscr{S}$ is localic, we turn to more theory of nuclei. Any nucleus j on a Heyting algebra L satisfies $a \Rightarrow jb = j(a \Rightarrow jb)$. Hence if $b \in L^j$, then $a \Rightarrow b \in L^j$ for all $a \in L$. In particular, L^j is closed under the operation $\Rightarrow$, and is a Heyting algebra under this same operation (Macnab 1976). Also, as (11) holds for all $c \in L$, it holds for all $c \in L^j$. Therefore if $a, b \in L^j$ we get that $a \Rightarrow b \in L^j$ and

$$a \Rightarrow b = \bigsqcup\{c \in L^j : c \sqcap a \sqsubseteq b\} = j \bigsqcup\{c \in L^j : c \sqcap a \sqsubseteq b\},$$

where $\bigsqcup$ is the join operation of L. In other words, when $a, b \in L^j$, the set $\{c \in L^j : c \sqcap a \sqsubseteq b\}$ has the same join in L as it does in L^j, and this join is the Heyting implication of a and b in both L and L^j.

This theory tells us that if $\mathscr{S}$ is localic, then if X and Y belong to the locale $Prop(\mathscr{S})$, and $X \Rightarrow Y$ is the set given by (13), then $X \Rightarrow Y \in Prop(\mathscr{S})$ and

$$X \Rightarrow Y = \bigcup\{Z \in Prop(\mathscr{S}) : Z \cap X \subseteq Y\} = j_{\lhd} \bigcup\{Z \in Prop(\mathscr{S}) : Z \cap X \subseteq Y\}.$$

Theorem 5 *Every Heyting algebra L can be embedded into the locale of all propositions of the localic cover system $\mathscr{S}_L$, by a Heyting algebra monomorphism.*

Proof Since every Heying algebra is distributive, we know from the previous two Sections that $\mathscr{S}_L$ is localic and the map $a \mapsto |a|$ is a lattice homomorphism embedding L injectively into the the locale $Prop(\mathscr{S}_L)$. It remains to show this map preserves the Heyting implication operations of L and $Prop(\mathscr{S}_L)$, i.e. that $|a| \Rightarrow |b| = |a \Rightarrow b|$, or equivalently by (13) that

$$\{x \in S_L : \uparrow x \cap |a| \subseteq |b|\} = |a \Rightarrow b|. \tag{14}$$

But for $x \in S_L$ we have $\uparrow x = |g_x|$ where g_x is the generator of the principal filter x, so $\uparrow x \cap |a| = |g_x| \cap |a| = |g_x \sqcap a|$. Hence $\uparrow x \cap |a| \subseteq |b|$ iff $|g_x \sqcap a| \subseteq |b|$ iff $g_x \sqcap a \sqsubseteq b$ iff (by (11)) $g_x \sqsubseteq a \Rightarrow b$ iff $a \Rightarrow b \in x$ iff $x \in |a \Rightarrow b|$. That proves (14) as required. □

6 Ideal Completion

The lattice of propositions $Prop(\mathscr{S}_L)$ can be seen as an alternative presentation of the *ideal completion* of the lattice L. We now explain this.

Recall that an *ideal* of a lattice L is any non-empty subset $I \subseteq L$ such that $a \sqcup b \in I$ iff $a, b \in I$. Equivalently, I is closed under the binary operation $\sqcup$ and closed downwards under the partial order of L: $b \sqsubseteq a \in I$ implies $b \in I$. The intersection of any collection of ideals is an ideal, so each subset J has a smallest ideal $(J]$ including it. This has

$$(J] = \{b \in L : b \sqsubseteq \bigsqcup X \text{ for some finite } X \subseteq J\}.$$

In particular, for each $a \in L$, the set $(a] = \{b \in L : b \sqsubseteq a\}$ is the *principal ideal generated by* a, the smallest ideal containing a.

The set $\hat{L}$ of all ideals of L forms a *complete* lattice when ordered by set inclusion $\subseteq$, with the meet and join of any collection Θ of ideals being its set intersection $\bigcap \Theta$ and the ideal $(\bigcup \Theta]$ generated by the union $\bigcup \Theta$, respectively.

Theorem 6 *The lattices* $(\hat{L}, \subseteq)$ *and* $(Prop(\mathscr{S}_L), \subseteq)$ *are isomorphic.*

Proof For each ideal I of L, define

$$X_I := \{x \in S_L : g_x \in I\} = \{[a) : a \in I\}.$$

Then the isomorphism is provided by the map $I \mapsto X_I$. To prove this we first have to show that X_I is a proposition of $\mathscr{S}_l$. That it is a $\subseteq$-up-set follows readily as if $x \in X_I$ and $x \subseteq y$, then $g_y \sqsubseteq g_x \in I$, hence $g_y \in I$ and so $y \in X_I$. To show that $j_{\lhd} X_I \subseteq X_I$, take any $x \in j_{\lhd} X_I$. Then there is a C with $x \lhd C \subseteq X_I$. Define $X_C := \{g_y : y \in C\}$. Since C is finite, so too is X_C. Since $C \subseteq X_I$ we get that $X_C \subseteq I$, and so $\bigsqcup X_C \in I$. But $C = \{[a) : a \in X_C\}$, so by (6), $\bigsqcup X_C \in x$. Hence $g_x \sqsubseteq \bigsqcup X_C$, forcing $g_x \in I$ and so $x \in X_I$.

This confirms that X_I is a localised up-set, and so $I \mapsto X_I$ is a map from $\hat{L}$ into $Prop(\mathscr{S}_L)$. In the reverse direction, take any $X \in Prop(\mathscr{S}_L)$ and define $I_X := \{a \in L : [a) \in X\}$. Then I_X is an ideal, since if $b \sqsubseteq a \in I_X$, then $[a) \in X$ and $[a) \subseteq [b)$, hence $[b) \in X$ so $b \in I_X$; and if $a, b \in I_X$ then as $[a) \cap [b) = [a \sqcup b)$, from (4) we get $[a \sqcup b) \lhd \{[a), [b)\} \subseteq X$, making $[a \sqcup b) \in j_{\lhd} X \subseteq X$ and hence $a \sqcup b \in I_X$. Thus indeed $I_X \in \hat{L}$. Since

$$I_{X_I} = \{a \in L : [a) \in X_I\} = \{a \in L : a \in I\} = I,$$

and

$$X_{I_X} = \{[a) : a \in I_X\} = \{[a) : [a) \in X\} = X,$$

we see that the maps $I \mapsto X_I$ and $X \mapsto I_X$ are mutually inverse, so each is a bijection.

Finally, it is straightforward that $I \subseteq I'$ iff $X_I \subseteq X'_I$, or equivalently that $X \subseteq X'$ iff $I_X \subseteq I'_X$, so these maps are mutually inverse order-isomorphisms between $(\hat{L}, \subseteq)$ and $(Prop(\mathscr{S}_L), \subseteq)$. □

Theorem 6 and the results of the Sect. 4 reproduce the well-known result of Stone (1937) that the ideal completion $\hat{L}$ of a distributive lattice satisfies the distribution law (8) for arbitrary subsets $X \subseteq \hat{L}$, so is a locale.

In the proof of Theorem 6, if $X = |a|$, then

$$I_X = \{b \in L : [b) \in |a|\} = \{b : b \sqsubseteq a\} = (a].$$

So the embedding $a \mapsto |a|$ of L into $Prop(\mathscr{S}_L)$ composes with the isomorphism $X \mapsto I_X$ from $Prop(\mathscr{S}_L)$ onto $\hat{L}$ to give the map $a \mapsto (a]$, which is the standard embedding of L into its ideal completion $\hat{L}$.

We can identify L with its isomorphic image under this embedding, by identifying a with $(a]$, and thereby view $\hat{L}$ as a complete extension of L. The image is *join dense* in $\hat{L}$ in the sense that every member of $\hat{L}$ is a join of members of the image. This is because each ideal I is the join of $\{(a] : a \in I\}$ in $\hat{L}$.

7 Covers for the MacNeille Completion

The MacNeille completion (MacNeille 1937) of a lattice (or poset) L is a complete lattice $\bar{L}$ containing an isomorphic image of L that is join dense and also *meet dense* in $\bar{L}$. Meet density means that each member of the completion is also a meet of members of the image. Possession of these two density properties characterises $\bar{L}$ uniquely up to isomorphism.

$\bar{L}$ can be constructed as follows. For each subset J of L, let J^u be the set of upper bounds of J and J^l the set of lower bounds. Composing these two operations, put $mJ = J^{ul}$, the set of lower bounds of the set of upper bounds of J. Then m is a closure operator on the complete lattice of all subsets of L under the inclusion ordering. $\bar{L}$ is defined to be the set of all m-closed (i.e. $J = J^{ul}$) subsets of L. It is a complete lattice under the ordering $\subseteq$, with $\bigsqcap \Theta = \bigcap \Theta$ and $\bigsqcup \Theta = m(\bigcup \Theta)$ in $\bar{L}$, where Θ is any collection of m-closed sets. Each m-closed set is an ideal of L, so $\bar{L}$ is a subset of $\hat{L}$. (See (Davey and Priestley 1990, pp. 40–44) for a comprehensive discussion of this construction.)

Any set of the form J^l is m-closed, including the set $\{a\}^l = \{b \in L : b \sqsubseteq a\} = (a]$ for each $a \in L$. The function $a \mapsto (a]$ is an order-invariant injection of $(L, \sqsubseteq)$ into $(\bar{L}, \subseteq)$ having the property that it preserves all joins and meets that exist in L.

Our purpose now is to define a new cover system $\mathscr{S}_L^m = (S_L, \subseteq, \blacktriangleleft)$, on the set of principal filters of L, whose lattice of proposition gives an alternative presentation of the MacNeille completion of L. The relation $\blacktriangleleft$ is obtained by dropping the finiteness condition on covers from (4), simply putting

$$x \blacktriangleleft C \text{ iff } \bigcap C \subseteq x. \tag{15}$$

The proof that the resulting structure $\mathscr{S}_L^m$ is a cover system is essentially the same as that for $\mathscr{S}_L$, without the requirement of finiteness in showing that $\mathscr{S}_L^m$ satisfies Transitivity. Note that since $\lhd$ is a subrelation of $\blacktriangleleft$ we have $j_\lhd X \subseteq j_\blacktriangleleft X$ in general, so any $j_\blacktriangleleft$-closed up-set is $j_\lhd$-closed. Hence $Prop(\mathscr{S}_L^m)$ is a subset of $Prop(\mathscr{S}_L)$. For each $a \in L$ the set $|a|$ is $j_\blacktriangleleft$-closed and so belongs to $Prop(\mathscr{S}_L^m)$. The map $a \mapsto |a|$ is an embedding of L into $Prop(\mathscr{S}_L^m)$ that preserves any meets of L by (3), and also preserves any join $\bigsqcup X$ that exists in L, i.e.

$$|\bigsqcup X| = j_\blacktriangleleft \bigcup\{|a| : a \in X\}. \tag{16}$$

To see this, observe that if $x \in |\bigsqcup X|$, then from (5), $x \blacktriangleleft \{[a) : a \in X\}$, and since $[a) \in |a|$, this implies $x \in j_\blacktriangleleft \bigcup\{|a| : a \in X\}$. The converse holds as $|\bigsqcup X|$ is a $j_\blacktriangleleft$-closed superset of $\bigcup\{|a| : a \in X\}$.

Lemma 1 *For any $X \subseteq S_L$ and $x \in S_L$, $x \in j_\blacktriangleleft X$ iff $\bigcap X \subseteq x$.*

Proof If $\bigcap X \subseteq x$, then $x \blacktriangleleft X \subseteq X$, hence $x \in j_\blacktriangleleft X$. For the converse, if $x \blacktriangleleft C \subseteq X$ for some C, then $\bigcap X \subseteq \bigcap C \subseteq x$. □

Theorem 7 *The lattices $(\bar{L}, \subseteq)$ and $(Prop(\mathscr{S}_L^m), \subseteq)$ are isomorphic.*

Proof The isomorphism is given by the maps $I \mapsto X_I$ and $X \mapsto I_X$ from Theorem 6 when restricted to $\bar{L}$ and $Prop(\mathscr{S}_L^m)$.

First we take an $I \in \bar{L}$ and show that $X_I \in Prop(\mathscr{S}_L^m)$. Since I is an ideal, it follows as before that X_I is a $\subseteq$-up-set. But whereas Theorem 6 showed that X_I is $j_\lhd$-closed, here we have to show the stronger property that it is $j_\blacktriangleleft$-closed. For this, let $x \in j_\blacktriangleleft X_I$. Then $\bigcap X_I \subseteq x$ by Lemma 1. Next we show that $g_x \in I^{ul}$: if $a \in I^u$, then for all $y \in X_I$ we have $g_y \in I$, hence $g_y \sqsubseteq a$ and so $a \in y$. This shows $a \in \bigcap X_I$, implying that $a \in x$ and so $g_x \sqsubseteq a$ as required to confirm $g_x \in I^{ul}$. But $I^{ul} = I$ as I is m-closed, giving $g_x \in I$ and thus $x \in X_I$. This proves that $j_\blacktriangleleft X_I \subseteq X_I$, making X_I $j_\blacktriangleleft$-closed and hence a member of $Prop(\mathscr{S}_L^m)$.

In the reverse direction, let $X \in Prop(\mathscr{S}_L^m)$. We show $I_X \in \bar{L}$. Let $a \in {I_X}^{ul}$. Then $\bigcap X \subseteq [a)$. For if $b \in \bigcap X$, then for all $c \in I_X$ we have $[c) \in X$, so $b \in [c)$, hence $c \sqsubseteq b$. This shows $b \in {I_X}^u$, hence $a \sqsubseteq b$ as $a \in {I_X}^{ul}$. So $b \in [a)$, as required to prove $\bigcap X \subseteq [a)$. From this Lemma 1 gives $[a) \in j_\blacktriangleleft X$. But X is $j_\blacktriangleleft$-closed, so $[a) \in X$ and thus $a \in I_X$. That proves that ${I_X}^{ul} \subseteq I_X$, making I_X m-closed and hence a member of $\bar{L}$.

The upshot is that the bijections $I \mapsto X_I$ and $X \mapsto I_X$ map $\bar{L}$ into $Prop(\mathscr{S}_L^m)$ and vice versa, respectively. Since they both preserve set inclusion, they provide the asserted order-isomorphism. □

A more abstract proof of this result can be obtained by showing that the image of the embedding $a \mapsto |a|$ is join and meet dense in $Prop(\mathscr{S}_L^m)$. For join density, if $X \in Prop(\mathscr{S}_L^m)$, then as X is an up-set it is equal to $\bigcup\{\uparrow x : x \in X\} = \bigcup\{|g_x| : x \in X\}$, hence as X is $j_\blacktriangleleft$-closed it is the join of $\{|g_x| : x \in X\}$ in $Prop(\mathscr{S}_L^m)$. For meet density we have $X = \bigcap\{|a| : X \subseteq |a|\}$. To show this, let x belong to the intersection on the right of the equation. Then $\bigcap X \subseteq x$, for if $a \in \bigcap X$, then $X \subseteq |a|$, hence by assumption $x \in |a|$, so $a \in X$. Since $\bigcap X \subseteq x$ we have $x \in j_\blacktriangleleft X$ by Lemma 1. But X is $j_\blacktriangleleft$-closed, so then $x \in X$ as required.

Thus $\{|a| : a \in L\}$ is join and meet dense in $Prop(\mathscr{S}_L^m)$, making the latter isomorphic to the MacNeille completion of L.

8 Ortholattices

Ortholattices have a unary *orthocomplementation* operation $a \mapsto a'$ and can be represented as a lattice of closed sets of a closure operator that is constructed from a binary *orthogonality* relation $\perp$. We now review how this works and then show that this closure operator is characterisable as the j-operator of the cover relation $\blacktriangleleft$ of (15).

An ortholattice can be defined as a lattice $(L, \sqcap, \sqcup, 0, ')$ with least element 0 and unary operation $'$ that is

- *antitone*: $a \sqsubseteq b$ implies $b' \sqsubseteq a'$;
- *involutary*: $a'' = a$; and
- *orthocomplemented*: $a \sqcap a' = 0$.

Then $0'$ is a greatest element, usually denoted 1, and has $a \sqcup a' = 1$. Ortholattices obey De Morgan's laws for distribution of complements over meets and joins. In particular $\bigsqcup X = (\bigsqcap_{a \in X} a')'$ whenever $\bigsqcup X$ exists.

Consider a structure $(S, \perp)$, where $\perp$ is a symmetric relation on S. For $X \subseteq S$, let $X^\perp = \{x \in S : x \perp y \text{ for all } y \in X\}$. This operation is antitone for inclusion: $X \subseteq Y$ implies $Y^\perp \subseteq X^\perp$. The symmetry of $\perp$ ensures that $X \subseteq X^{\perp\perp}$, and together with antitonicity this implies $X^{\perp\perp\perp} = X^\perp$.

X will be called *orthoclosed* if $X = X^{\perp\perp}$. Thus any set of the form $X^\perp$ is orthoclosed. The map $X \mapsto X^{\perp\perp}$ is a closure operator on the lattice of all subsets of S. Thus the set $Orth(S, \perp)$ of all orthoclosed sets is a complete lattice under the ordering $\subseteq$, with $\bigsqcap \Theta = \bigcap \Theta$ and $\bigsqcup \Theta = (\bigcup \Theta)^{\perp\perp}$, where Θ is any collection of orthoclosed sets. $Orth(S, \perp)$ has least element $S^\perp$ and is closed under the antitone operation $X \mapsto X^\perp$, which is involutory on orthoclosed sets. This operation is orthocomplemented, which makes $Orth(S, \perp)$ an ortholattice, precisely when $(S, \perp)$ has the following property (Hedlíková and Pulmannová 1991, Lemma 1.1):

$$\text{if } x \perp x \text{ then } x \perp y \text{ for all } y \in S. \tag{17}$$