Mao-Hong Yu

Unified Strength Theory and Its Applications

Second Edition

西安交通大学出版社
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Preface to the Second Edition

The first edition of *Unified Strength Theory and Its Applications* was published by springer in Berlin, Germany in 2004. In the August of this year, I was invited to make a closing lecture at the *International Symposium on Developments in Plasticity and Fracture: Centenary of M.T. Huber Criterion* which was held in World Historical and Cultural City Cracow, Poland. The title of this closing lecture is "The beauty of strength theory". In 2006, a review of the first edition of *Unified Strength Theory and Its Applications* was presented at *MATH* by P. Teodorescu, academician of Romania Academy of Science.

It is interesting that an ancient multifaceted seal gives me a new idea of creating a rhombicuboctahedron strength theory in 2007. This multifaceted seal was made in Western Wei Dynasty of China about 1500 years ago. The owner of this seal was a famous general. Now, it is exhibited at the Shaanxi History Museum. The multifaceted seal is similar to the rhombicuboctahedron mechanical model which was proposed first in my book *Twin-Shear theory and its Application* in 1998. The details of rhombicuboctahedron stress strength theory is shown in Chap. 13.

In recent years, the unified strength theory (UST) had been generalized and applied to different fields. UST was generalized to as the effective stress unified strength theory and unified strength theory for the equation of pore water pressure in soil mechanics. UST was generalized to as the three-parameter UST in rock mechanics. UST was also generalized to as five-parameter UST and UST fracture criterion in concrete mechanics. Six monographs are written in recent 10 years. These monographs are two trilogies. The first trilogy is Plasticity Trilogy, and they are as follows:

1. Generalized Plasticity: Both for Metals and Geomaterials. Berlin: Springer, 2006;
2. Structural Plasticity: Limit, Shakedown and Dynamic Plastic Analyses of Structures. Springer and ZJU Press, 2009;
3. Computational Plasticity: With Emphasis on the Application of the Unified Strength Theory. Springer and ZJU Press, 2012.

The second trilogy is Geo-mechanics Trilogy, and they are as follows:

1. Soil Mechanics: New Theory and New Results;
2. Rock Mechanics: New Theory and New Results;
3. Concrete Mechanics: New Theory and New Results.

These three monographs will be published in the next 3 years.

A large number of new results obtained by using the unified strength theory are summarized in these books. Therefore, the two chapters about the application of UST in the first edition are simplified to one chapter. On the other hand, six new chapters are added as follows:

Chapter 7—Principles for Comment, Formulation and Choice of the Strength Theory Function;

Chapter 10—Visualization of the Unified Strength Theory;

Chapter 11—Equivalent Stress of the Unified Strength Theory and Comparisons with other Theories;

Chapter 12—Economic Signification of the Unified Strength Theory;

Chapter 13—Rhombicuboctahedron Stress Strength Theory;

Chapter 14—The Beauty of Strength Theories.

The UST not only has rich content, but also contains major factors of the beauty of science. In addition, the unified strength theory contains a clear physical concept; unified mechanical model; simple and unified mathematic expression. The yield loci of UST covered the whole convex region from the lower bound to the upper bound. The UST can be used for most materials from metallic materials to geomaterials.

Acknowledgements I would like to acknowledge the support from:

Xi’an Jiaotong University Alumni Association of Hong Kong;

Xi’an Jiaotong University Alumni Association of Civil Engineering Department;

State Key Laboratory for Strength and Vibration of Mechanical Structures, Xi’an Jiaotong University, Xi’an, China.

I would like to express my sincere thanks to editors Kavitha Palanisamy, Parimelazhagan Thirumani and Na Xu, Springer and Ying Li, Xi’an Jiaotong University Press for their excellent editorial work on the second edition. Thanks are also due to my research assistants Jia-Yu Liang for the help of writing this book and Xia-Xia Wu for the collection of references from 2002 to 2017 in Chap. 17.

Mao-Hong Yu
Spring 2017

Preface to the First Edition

It has been 10 years since I presented the paper entitled “A new model and theory on yield and failure of materials under the complex stress state” at the Sixth Conference on Mechanical Behaviour of Materials held at Kyoto, Japan in 1991. The proceedings edited by Jono and Inoue were published by Pergamon Press in 1991. At that conference, Professor Murakami and I were invited to act as the chairperson and co-chairperson of a session, and I presented the paper at another session. Few days before the conference, I had given a seminar regarding the twin-shear strength theory and the unified strength theory at Nagoya Technological University. These were the first two presentations of the unified strength theory, although I had completed the research of the unified strength theory in 1990.

The paper “Twin-shear strength theory and its generalization” was published in the English edition of Sciences in China, the top journal in China, in 1985. The original generalized twin-shear strength theory was presented at the 16th International Theoretical and Applied Mechanics Congress held at Copenhagen in Denmark and MPA (Material Prüfungs Anstalt) at Stuttgart University, Germany in 1984. After this Congress I visited the MPA and School of Civil Engineering of Stuttgart University, and presented a seminar regarding the generalized twin-shear strength theory at MPA of Stuttgart University.

Professor Otto Mohr (1835−1918) has had worked at the Stuttgart University. Mohr was a very good professor, his lectures aroused great interest in his students. His lectures were always clear and logically constructed, and he always tried to bring something fresh and interesting to the students’ attention. The reason for his students’ interest in his lectures stemmed from the fact that he not only knew the subject thoroughly, but also had he done much in the creation of the science which he presented. The works of Mohr gave me very interesting and useful help to understand and study the strength theory.

The idea of twin-shear and the twin-shear yield criterion may be traced back to 1961. I presented two papers on the twin-shear yield criterion and its associated flow rules at Xi′an Jiaotong University and a conference on mechanics organized by the Association of Mechanics of Shaanxi Province in 1961. Thirty years elapsed from the twin-shear yield criterion to the generalized twin-shear strength theory and the unified strength theory. I am surprised that the progress in this field was so slow. Some hours are sufficient today to introduce the mathematical formulae of the twin-shear yield criterion, the generalized twin-shear strength theory, the unified yield criterion and the unified strength theory.

The limit surface of the twin-shear strength theory (Yu 1985) forms the upper (external) bound of all the convex limit loci on the deviatoric plane in stress space. No admissible convex limit surface may exceed the twin-shear limit surface. The single shear strength theory (Mohr–Coulomb 1900) forms the lower (inner) bound for all the possible convex failure surfaces coinciding with the Drucker postulation. The limit loci of the unified strength theory cover all regions of the convex limit loci and extend to the region of the nonconvex limit loci. It is better to convey the unified strength theory by the limit loci, as shown in Fig. 1.

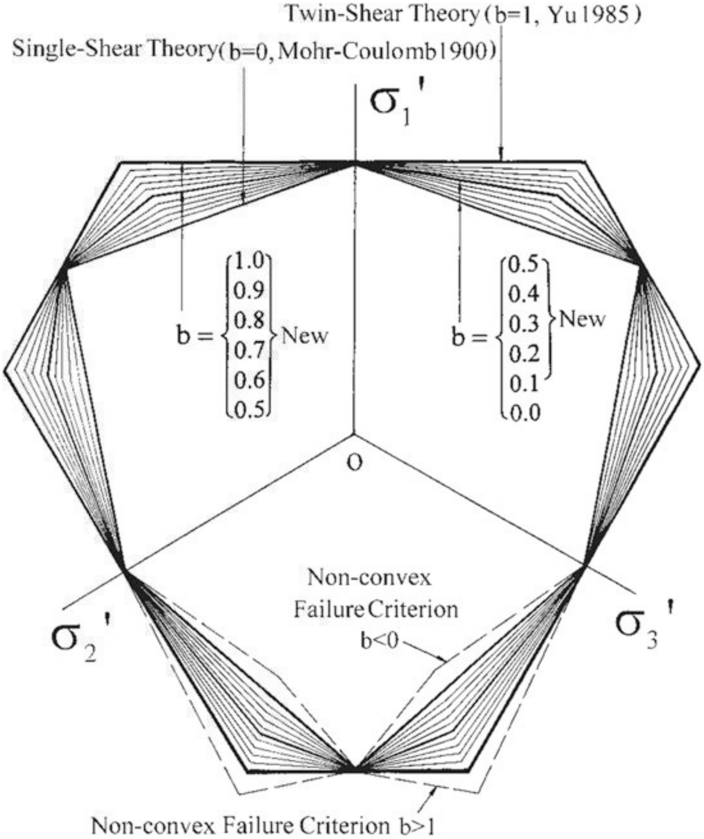

Fig. 1 Varieties of the unified strength theory on the deviatoric plane

The need to investigate the strength of materials under the complex stress states has stimulated research in a special problem known as strength theory or failure theory, which includes the yield criteria used in plasticity, the failure criteria used in rock mechanics, soil mechanics and concrete mechanics and materials models used in computational mechanics and the finite element method codes. The results of this research are described in a great number of papers scattered over many scientific and engineering journals and texts, and the proceedings of several conferences and symposia on the subject of strength of materials and structures. Moreover, the applications of strength theory appear in more fields. Research in strength theory is carried out not only for metallic materials, rock, soil and concrete, but also extends to polymers, ceramics, ice, glass, powder, energetic materials, biomaterials and other materials. It has attracted numerous research scientists from the areas of mechanics, mathematics, physics, materials sciences, geological sciences and many engineering fields. It has, therefore, become an interdisciplinary subject of academic and research interests. Research from different aspects and from different fields has greatly contributed to the continuous development of strength theories.

The study of the general patterns of materials strength with the variation of complex stress states is normally referred to as macroscopic strength theory, engineering strength theory or just strength theory. This subject is discussed in the framework of continuum and engineering applications in this book. A link among the yield criteria and failure criteria for various materials is provided by the study of the unified strength theory for isotropic materials. Sufficient information is now available to provide a useful and complete formulation of the strength theory of materials under complex stress states.

The contents of the book can logically be divided into four parts: theory, experiment, application and history. The unified yield criterion, extended unified yield criterion and the unified strength theory are described in Chaps. 3, 5 and 7. Experimental basics and verification are described in Chaps. 4, 6 and 8, after each respective theoretical chapter. Chapters 9 and 10 give the applications of the unified yield criterion and the unified strength theory. In order to present the total picture of the development of strength theory and to give the reader a complete overview of the achievements made by others in this field, a historical review of the development of strength theory is given in the Chap. 11. This review will help readers to better understand the strength theory. Readers who would prefer a historical orientation before they delve into the details of the subject may choose to begin with Chap. 11. In addition, more than 1200 references and bibliography regarding the strength theories and their applications are listed with brief introductions in the Chap. 12. Stress state analysis is discussed in Chap. 2. The description of the stress state may be found in a number of books covering mechanics of materials, solid mechanics, elasticity and plasticity. Only some basic formulae and figures as well as some new ideas are given here. Brief summaries and problems are given at the ends of most chapters.

In spite of the merits of the twin-shear strength theory and the unified strength theory, there are still a few limitations to be noted. For example, all other existing strength theories can be represented by a single equation, but the twin-shear strength theory and the unified strength theory need two equations. Although these are straightforward linear equations, a stress state condition is needed in order to decide which of the two equations is to be used. Moreover, in the case of triaxial tension, even though it is rarely encountered, a supplementary tension cutoff condition is required.

By the year 2000, the twin-shear strength theory and the unified strength theory had been included in over 70 monographs and textbooks. This shows that this new strength theory has gradually come to its stage of maturity. There are also many professors in various universities, who have made the twin-shear strength theory and the unified strength theory part of their courses on strength of materials, plasticity, mechanics of soils, plasticity of rocks and soils, nonlinear finite element analysis of concrete structures, engineering mechanics, soil dynamics.

The author would like to express his gratitude for the support of the National Natural Science Foundation of China (Grants nos. 5870402, 59779028, 59924033 and 50078046), the Ministry of Education of China, the China Academy of Launch Vehicle Technology and the Aircraft Strength Research Institute of China, as well as the National Key Lab for Mechanical Behavior of Materials at Xi'an Jiaotong University and the National Key Lab of Structural Strength and Vibration at Xi'an Jiaotong University.

Thanks are also due to Profs. Zhuang and He, Dr. Zhu HA, Yu F, Dr. Wei XY and Hu XR and others at Xi'an Jiaotong University for their support during the course of writing this book, and to my young brother Prof. Yu MZ for his help with German literature. He was awarded his doctor's degree at Karlsruhe University in Germany in 1986. I would also due to thank many professors from other universities and many research scientists and engineers from various institutions for their work in the research, experimental verification and application of the new strength theory. These include researchers from Tsinghua University, Zhejiang University, Beijing University, Tianjin University, North-Western Jiaotong University, North-East University, the University of Defense Science and Technology, the University of Hong Kong, The Hong Kong University of Science & Technology, The Hong Kong Polytechnic University, Nangyang Technological University in Singapore. the National University of Singapore, etc, as well as the Institute of Mechanics and the Institute of Rock and Soil Mechanics of the Chinese Academy of Science, the China Academy of Launch Vehicle Technology, the Aircraft Strength Research Institute of China, the Third Institute of Army, the Yangtze River Scientific Research Institute and the Northwestern Hydropower Investigation and Design Institute of the Ministry of Energy and the Ministry of Water Resources. The author would also like to acknowledge the support from all other individuals and universities, research organizations, journals and publishers. Some historical materials were taken from various journals, such as Applied Mechanics Reviews, Sciences in China (English edition) and Progress in Natural Science (English edition).

I would like also to express my sincere thanks to Ms. Mass, and Ms. King Editorial Department and International Engineering Department, Springer-Verlag, Germany, for their excellent editorial work on my manuscript.

Xi'an, China Mao-Hong Yu
Winter 2003

Review of the "Unified Strength Theory and Its Applications" Petre P. Teodorescu

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YU, Mao-Hong
Unified strength theory and its applications. (English) [B] Berlin: Springer.
xx,412 p. EUR 119.00/net; sFr 201.50; \sterling 91.50; \$139.00 (2004).
[ISBN 3-540-43721-5/hbk]

Strength theories focus on the limit states of stress and strain in order to compare them with admissible stresses and strains. Uniaxial experiments and results are no more sufficient, and two- or three-axial studies are needed. Because different materials have different mechanical behaviour under complex stress-strain states, yield criteria and failure criteria play an important rule. The goal of these theories is to ensure the safety of civil and mechanical structures. But—in general—such a theory can be applied to a small number of materials and states of stress and strain in their deterministic aspects, and does not cover the area of all problems which may arise, so that unified theories have been searched. Here, starting from the idea of twin-shear and twin-shear yield criterion, the author sets up a twin-shear strength theory and then a unified strength theory, the limit loci of which cover all regions of the convex limit loci and can be extended to the region of non-convex limit loci. The present book is not only a presentation of the theory, experiments, applications and history, but also a monograph on the own research of the author as it can be put in evidence by its contents, i.e.: 1. Introduction; 2. Stress states of elements; 3. Unified field criteria; 4. Verification of the yield criterion; 5. Extended unified field criterion; 6. Basic characteristics of strength of materials under complex stress; 7. Unified strength theory; 8. Experimental verification of strength theory; 9. Applications of the unified yield criterion; 10. The effects of failure criteria on structural analysis; 11. Historical reviews; 12. References and bibliography. Each chapter is followed by a summary and problems which concern the most important items. The last chapter contains a historical discussion and an exhaustive bibliography of more than thousand titles covering the interval 1638–2002.

The book is intended to a large community of readers and represents an important contribution to the field.

[**Petre P. Teodorescu** (Bucuresti)]

MSC 2000: *74-02 Research monographs (mechanics of deformable solids)
74COS-Small-strain, rate-independent theories
74K99 Thin bodies, structures; 74R20 An elastic fracture and damage
Keywords: limit states; complex stress; unified yield criterion; failure criteria
Cited in Zbl. reviews…

Comment on the “Unified Strength Theory” Kolupaev V A and Altenbach H

The development of Unified Strength Theory (UST) is an event in phenomenological material science. The model of UST provides a new family of material models. It contains a number of new models and highlights interrelations between known models. The UST-model can be fitted to different materials and is therefore suitable for the analysis of experimental results. The material parameters can be computed using results of only three experiments (e.g. tension, compression and torsion).

The following advantages of UST are to be pointed out:

1. The concept of understanding the stress components $\sigma_{ij} = (\sigma_i + \sigma_j)/2$ and $\tau_{ij} = (\sigma_i - \sigma_j)/2$ based upon polyhedral elements,
2. Extension of the stress state parameter according to Lode by “twin shear stress” parameters, $\mu_\tau = \tau_{12}/\tau_{13}$, $\mu_\tau = \tau_{12}/\tau_{13}$, $\mu_\tau + \mu'_\tau = 1$,
3. Fitting of the parameters $(b,\ \alpha)$ to various measured data found in literature, as well as recommendations for different types of materials,
4. Physical interpretation of the parameters,
5. Incorporation of the third deviatoric invariant I'_3 into the model,
6. Simple computation of the equivalent stress σ_{eq} as well as of the derivative $\partial\sigma_{eq}/\partial\sigma_{ij}$ everywhere except for singular points.

Kolupaev VA, Altenbach H (2009). “Strength hypotheses of Mao-Hong Yu and its generalisation”. In: Kuznetsov SA (Hrsg) 2nd Conference Problems in Nonlinear Mechanics of Deformable Solids, 8–11, December 2009. Kazan State University, Kazan, 10–12.

Contents

Notations

Stresses and Invariants

$\sigma_1, \sigma_2, \sigma_3$	Principal stresses
σ_{ij}	Stress tensor
σ	Normal stress
τ_{13}	Maximum principal shear stress
$\tau_{12},\ \tau_{23}$	Intermediate shear stress or minimum shear stress acting on the plane of a dodecahedral element
μ_σ	Lode stress parameter
$\mu_\tau \mu_\tau'$	Twin-shear parameter for stress state $\mu_\tau = \tau_{12}/\tau_{23},\ \mu_\tau' = \tau_{23}/\tau_{13}$
θ	Stress angle corresponding to the twin-shear parameter
τ_8	Octahedral shear stress
σ_8	Octahedral normal stress
I_1, I_2, I_3	Invariants of the stress tensor σ_{ij}
σ_m	Hydrostatic stress or mean stress
S_1, S_2, S_3	Deviatoric stresses
J_1, J_2, J_3	Invariants of the deviatoric stress tensor

Material Parameters

σ_y	Yield stress
σ_t	Uniaxial tensile strength
σ_c	Uniaxial compressive strength
b	Failure criterion parameter in the unified strength theory
α	Ratio of tensile strength to compressive strength
τ_y	Shear yield strength
β	Coefficient in the unified strength theory that represents the effect of the normal stress on failure
C_0	Cohesive strength

φ	Friction angle
E	Young's modulus
v	Poisson's ratio

Miscellaneous

M_r	Radial bending moment per unit length
M_θ	Circumferential bending moment per unit length
σ_r	Radial stress
σ_θ	Circumferential stress
σ_z	Axial stress
$\dot{W}$	Rate of deflection
$\dot{k}_r, \dot{k}_\theta$	Curvature rates
Q_i	Generalized stresses
ρ	Density of the material
u	Displacement
$\varepsilon_r, \varepsilon_\theta$	Radial and circumferential strain
ε_z	Longitudinal strain
ω_e	Limit rotating speed of disc
$[\sigma]$	Allowable tensile stress, $[\sigma] = {}^{\sigma_t}\!/_{n}$
p_e	Elastic limit pressure
p_p	Plastic limit pressure

Voigt-Timoshenko Conundrum and the Development of UST

W. Voigt 1901

Voigt made a lot of tests and concluded that it is impossible to formulate a single strength criterion which can be applied to various materials.

Voigt, W. (1901) Zur Festigkeitslehre. *Annalen der Physik*, 567-591.

S. P. Timoshenko 1953

"Voigt came to the conclusion that … it is impossible to devise a single theory for successful application to all kinds of structural materials."

Timoshenko SP (1953) *History of Strength of Materials*. McGraw-Hill, New York.

Encyclopedia of China 1985

"it is impossible to establish a unified strength theory for various materials." (This sentence had been deleted in the second edition in 2009)

The Editor Committee of Encyclopedia of China (1985) *Encyclopedia of China*. The Encyclopedia of China Press, Beijing.

1951 D. C. Drucker

The Drucker postulate and its associated convexity of yield surface are regarded as the fundamental law of plasticity.

Drucker DC (1951) A more fundamental approach to stress-strain relations. *Proc. of First U.S. National Cong. Appl. Mechanics*, ASME, 487–491.

1961 M-H Yu

Twin-shear Yield Criterion:

$$f=\sigma_1-\frac{1}{2}(\sigma_2+\sigma_3)=\sigma_y,\ \text{When}\ \ \sigma_2\le\frac{1}{2}(\sigma_1+\sigma_3)$$

$$f'=\frac{1}{2}(\sigma_1+\sigma_2)-\sigma_3=\sigma_y,\ \text{When}\ \ \sigma_2\ge\frac{1}{2}(\sigma_1+\sigma_3)$$

Yu MH (1961) Plastic potential and flow rules associated singular yield criterion. *Res. Report of Xi'an Jiaotong University*. Xi'an, China (in Chinese)

1985 M-H Yu

Twin-shear Strength Theory:

$$f=\sigma_1-\frac{\alpha}{2}(\sigma_2+\sigma_3)=\sigma_t,\ \ \text{When}\ \ \sigma_2\le\frac{\sigma_1+\alpha\sigma_3}{1+\alpha}$$

$$f'=\frac{\alpha}{2}(\sigma_1+\sigma_2)-\sigma_3=\sigma_t,\ \ \text{When}\ \ \sigma_2\ge\frac{\sigma_1+\alpha\sigma_3}{1+\alpha}$$

Yu MH, He LN, Song LY (1985) Twin shear stress theory and its generalization. *Science in China Series A*, 28(11), 1174-1183.

1991 M-H Yu, Unified Strength Theory (UST)

$$F=\sigma_1-\frac{\alpha}{1+b}(b\sigma_2+\sigma_3)=\sigma_t,\ \text{when}\ \ \sigma_2\le\frac{\sigma_1+\alpha\sigma_3}{1+\alpha}$$

$$F'=\frac{1}{1+b}(\sigma_1+b\sigma_2)-\alpha\sigma_3=\sigma_t,\ \ \text{when}\ \ \sigma_2\ge\frac{\sigma_1+\alpha\sigma_3}{1+\alpha}$$

Yu MH, He LN (1991) A new model and theory on yield and failure of materials under the complex stress state. *Mechanical Behavior of Materials-6* (ICM-6). Jono M and Inoue T eds. Pergamon Press, Oxford, (3):841–846.

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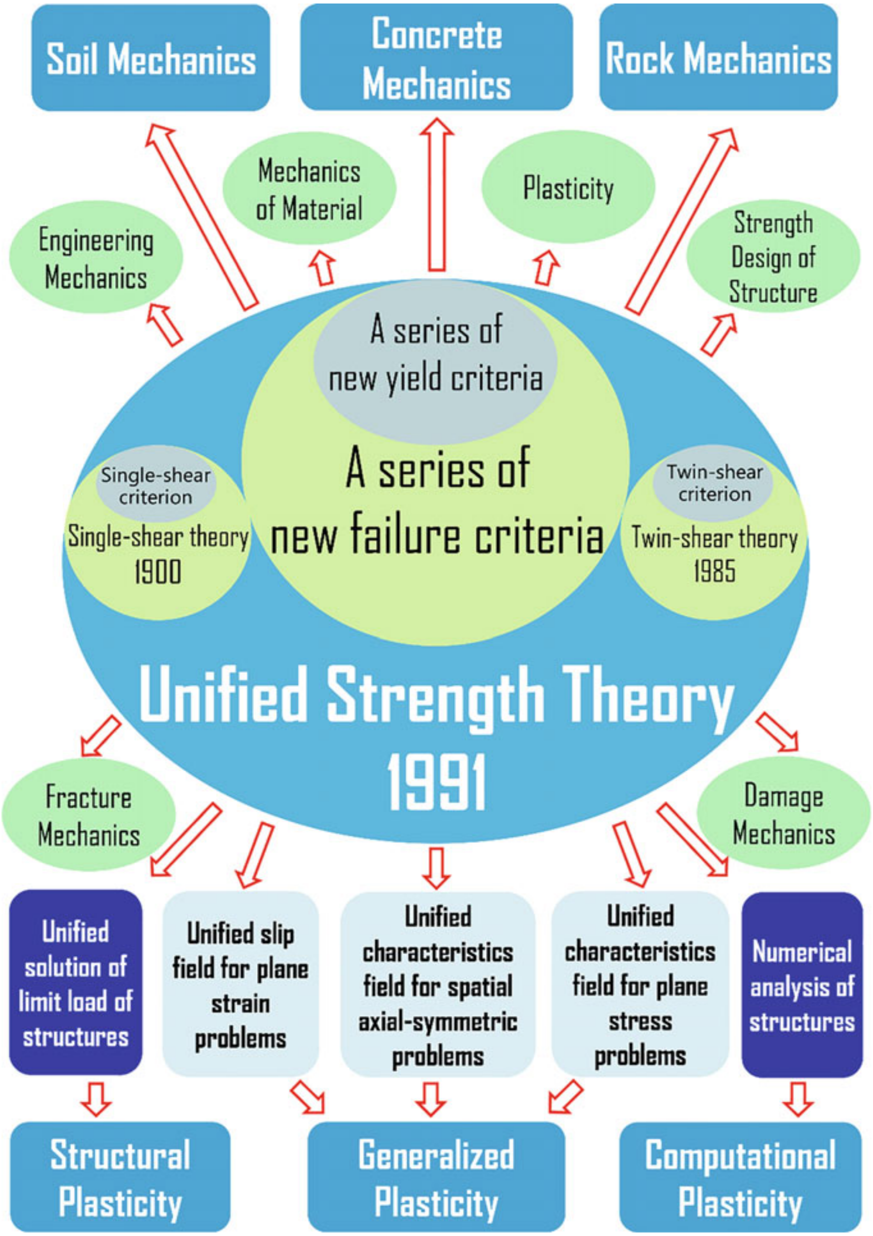

Soil Mechanics
Concrete Mechanics
Rock Mechanics
Mechanics of Material
Plasticity
Engineering Mechanics
Strength Design of Structure
A series of new yield criteria
Single-shear criterion
Single-shear theory 1900
A series of new failure criteria
Twin-shear criterion
Twin-shear theory 1985
Unified Strength Theory 1991
Fracture Mechanics
Damage Mechanics
Unified solution of limit load of structures
Unified slip field for plane strain problems
Unified characteristics field for spatial axial-symmetric problems
Unified characteristics field for plane stress problems
Numerical analysis of structures
Structural Plasticity
Generalized Plasticity
Computational Plasticity

Chapter 1
Introduction

1.1 Strength of Materials and Structures

Strength is an important concept in engineering and solid mechanics. It is a basic requirement of a variety of structures in mechanical engineering, civil engineering, aviation industry, aerospace industry and geotechnical engineering etc. Research on the strength of materials and structures as well as the strength design of various engineering structures requires increased knowledge of the strength of materials under complex stress states (Fig. 1.1).

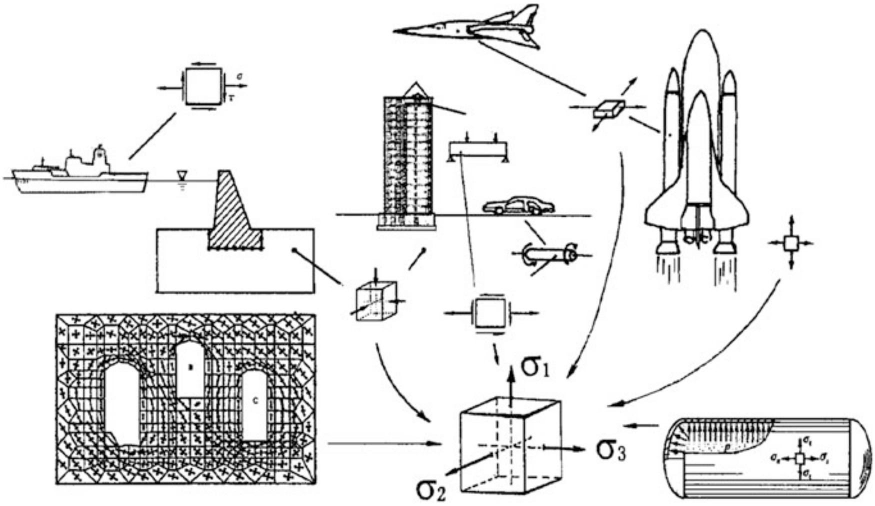


Fig. 1.1 Element in various structures

Strength is also very important in strength of materials, plasticity, soil mechanics, rock mechanics, concrete mechanics and solid mechanics.

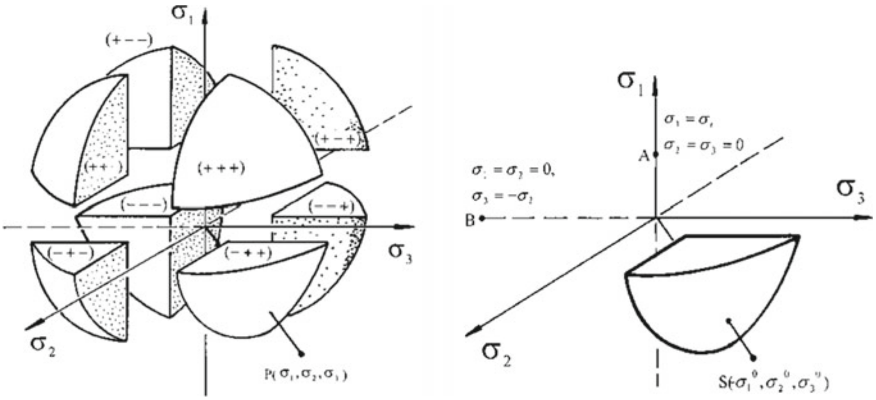
M.-H. Yu, *Unified Strength Theory and Its Applications*,
https://doi.org/10.1007/978-981-10-6247-6_1

1.2 Strength of Materials Under Complex Stress State

Most materials are acted under a complex stress state. They could be regarded as a point or a cubic element that is acted by three different combined stresses on each side. There are, in total, nine stress components on the three sides. Any changes of these components could cause a change in the strength of materials. Only six independent variables exist, according to stress symmetry. It is difficult to give a common solution for the strength of materials under the complex stress state, even if the question could be simplified to the three principal stresses when the materials are isotropic. The three-dimensional principal stresses acting on a cubic element are shown in Fig. 1.1. Sometimes they are referred to as the triaxial stresses or polyaxial stresses. The biaxial stresses and uniaxial stress are special cases.

The strength of materials is related to many factors such as the temperatures, loading rates and the stress state. Determining the rules governing the variation of the strength of materials with the stress state is a complex problem.

The three-dimensional principal stresses (σ_1, σ_2, σ_3) can be regarded as a three-dimensional space of principal stresses. If we take the tensile stress as positive while taking the compressive stress as negative, the stress state may combine the space stresses into various magnitudes and signs of stress combinations. The stress point P (σ_1, σ_2, σ_3) of different signs could combine up to eight quadrants of (+ + +), (+ + –), (+ – +), (+ – –), (– + +), (– + –), (– – +) and (– – –). A stress point could be situated anywhere within the three-dimensional space of the principal stresses (Fig. 1.2a).



(a) Eight quadrants (b) Tensile and compressive limit point

Fig. 1.2 Principal stress space

The need for careful consideration of strength theory may be illustrated by two examples. In the first example, the material is assumed to be a metallic material whose behaviour approximates the ideal elasto-plastic case. A uniaxial tensile test provides the yield strength σ_y. Now assume that a transverse compressive stress σ_2 of equal magnitude to the tensile stress σ_1 is also applied. In this case, the tensile stress σ_1 necessary to cause yielding is experimentally observed to be only about $\sigma_1 = 0.6\sigma_y$. The stress state of this condition is pure shear stress in which the three principal stresses are $\sigma_1 = 0.6\sigma_y$, $\sigma_2 = 0$ and $\sigma_3 = -0.6\sigma_y$. This result is easily verified by conducting a simple torsion test on a thin-walled tube. It will be discussed in Chap. 4. However, if a transverse tensile stress σ_2 of equal magnitude to the tensile stress σ_1 is applied, an experiment shows that the effect of the transverse stress on yielding is small or absent. The experiment could be done by pressurizing a thin-walled spherical vessel until yield, or by a combination of pressure and tension on a thin-walled tube. An additional experimental fact of interest is that it is difficult, and perhaps impossible, to cause a metallic material to yield if it is tested under simple hydrostatic stress, where $\sigma_1 = \sigma_2 = \sigma_3$, in either tension or compression. Hydrostatic tensile stress is difficult to achieve experimentally, but hydrostatic compressive stress consists of simply placing a sample of material in a pressurized chamber.

If the material is changed to a geomaterial, such as rock or concrete, the case is rather complex and provides the second example. The uniaxial tensile test and uniaxial compressive test could give us two limit points in the three-dimensional stress space, that is, *A* ($\sigma_1 = \sigma_t, \sigma_2 = 0, \sigma_3 = 0$) and *B* ($\sigma_1 = 0, \sigma_2 = 0, \sigma_3 = -\sigma_c$), as shown in Fig. 1.2b.

These two points could be simply expressed as (σ_t, 0, 0) and (0, 0, σ_c) in stress space. However, the strength of a material under a combination of complex stresses is far more complicated than that in a simple stress state. For example, the granite at Laxiwa Hydraulic Power Station situated on the Yellow River in China and the Three Gorges Power Station on the Yangtze River in China has tensile strength σ_t = 15–20 MPa and compressive strength σ_c = –(180–250) MPa. It could be failure by the combination of tensile stress σ_1 = 10 MPa, $\sigma_2 = 0$ and compressive stress σ_3 = –125 MPa. In this case, the maximum tensile stress σ_1 = 10 MPa is lower than the limit tensile strength σ_t = 15–20 MPa, and the minimum compressive stress σ_3 = –125 MPa is lower than the limit compressive strength σ_c = –(180–250) MPa. Furthermore, it could fail by a combination of triaxial compression ($-\sigma_1$, $-\sigma_2$, $-\sigma_3$) when σ_t = –100 MPa, σ_2 = –300 Mpa and σ_3 = –400 MPa. Such great changes make the research into the strength of materials under complex stress both complex and challenging. Similar changes in strength may be observed for other materials such as concrete, soil and polymers.

Strength theory is generally a question of a three-dimensional stress space. However, it is too complex to be determined from experiments.

The other difficulty is how to obtain the complex stresses, each of which can be controlled independently. The test facilities and technology are difficult to establish. The published test data regarding polyaxial tests are much less than those taken from uniaxial tests or biaxial tests. Moreover, we could not confirm the entire status of how the strength of materials changed by experiments because of the infinite numbers of feasible combinations. We require a theory to describe the rules of how the strengths of materials vary with the stress states. Therefore, the strength theory of materials under complex stress states, which is referred to as strength theory, was developed.

1.3 Definition of Strength Theory

Strength theory focuses on the rules of how the strength of materials varies with complex stress states. Strength theory provides us with the various yield criteria and failure criteria of materials under complex stress states. Strength theory function is a scalar function of stresses.

Because different materials have different mechanical behaviour, their uniaxial stress–strain curves are also different from each other. However, most of them can be represented by five kinds of curves, as shown in Fig. 1.3. They are brittle failure, hardening curve, ideal plastic, softening curve, and brittle-plastics. The relative strength is shown in the diagram. Therefore, they all have the strength limit point *S*. Before the materials reach the limit point *S* they all could be considered as linearly elastic.

The limit point *S* in Fig. 1.3 is a critical point (or limit point or peak point) where materials change from linearity to nonlinearity, from changes being reversible to irreversible and from no damage to failure. We define this limit point *S* as the strength limit point. Generalizing it to the three-dimensional complex stress state, we also define the strength limit point, the yield point, the peak point or the failure point. The mathematical expression of the strength limit point under certain three-dimensional stress states is generally called the failure criterion or the yield criterion. We also use it as a materials model in the computational mechanics and codes. All of areas comprise strength theory or failure theory.

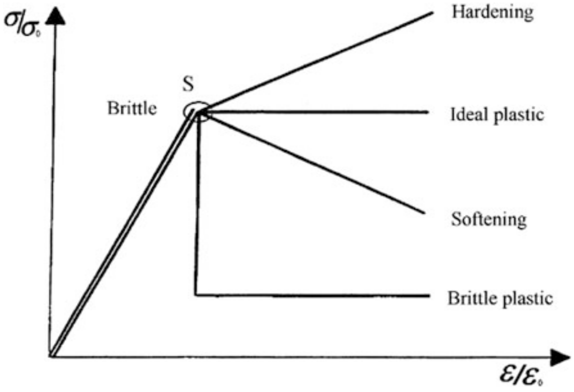


Fig. 1.3 Various types of stress–strain curves

The research of strength theory and its criteria constitute is an important part in solid mechanics. It provides a theoretical foundation for the mechanics of materials and the load-bearing capacity of structures, and to establish elasto-plastic constitutive relations. Moreover, strength theory has become a basic area of research on the mechanical behavior of materials and structures. It is an area of increasing focus. A great deal of yield criteria and failure criteria have been presented. In the past, these criteria were not systematically collected into a separate subject area. However, over the past century strength criteria have been developed and accumulated into an interrelated body of work. The author hopes this book will be helpful for the development of strength theory of materials and structures. The development of strength theory must promote the development of other related subjects, such as plasticity, soil and rock plasticity, solid mechanics, mechanics of materials, soil mechanics, plastic analysis of structures and nonlinear finite element methods.

1.4 Significance and Development of Strength Theory

Materials in nature and in engineering structures are often acted under complex stress states. Under certain conditions, such stresses may cause the yield and failure of materials, which can result in landslides, earthquakes, and damage or deformation of structures. Strength theories study the strength of materials with the variation of complex stress states and provide us with the criteria of yield and failure of materials under complex stress states.

On one hand, the strength theory is about the strength conditions under which materials would not fail due to complex stresses. Therefore, the study of strength theory is essential to the reliability and safety of structures. It is used for the structural analysis and the strength design of engineering structures.

On the other hand, the failure conditions are also indispensable for the study on plastic forming, high-impact penetration, explosion, landslide and earthquake. Strength theory is about the yield conditions and failure conditions of plastic deformation and failure of materials. As these are two aspects of the same subject, strength theory or the strength criterion can also be referred to as failure theory, or sometimes the yield criterion, or failure criterion or the material model.

Because of the generality and importance of this subject, it has attracted numerous research scientists from the areas of mechanics, mathematics, physics, materials sciences, geographic sciences and many other engineering fields. It has therefore become an interdisciplinary subject of academic and research interests. Research from different aspects and in different fields has greatly contributed to the continuous development of strength theories. The study of the general patterns of material strength with the variation of complex stress states is normally referred to as the macroscopic strength theory or the engineering strength theory, or just strength theory. This subject is discussed in the framework of continuum and engineering applications in the book.

The development of strength theory is slow. The research of Galileo on strength of materials (1638) may be regarded as the first research to maximum stress strength theory. Sometimes, the maximum normal stress strength theory was called the first strength theory in Russian and Chinese. The second strength theory (maximum strain strength theory), the third strength theory (maximum shear stress strength theory) and the forth theory (shear strain energy strength theory) appeared in 1686 (Marriott's paper was published posthumously), 1864 (Tresca) and 1904, 1913 (Huber 1904, von Mises 1913). Almost every theory appears over 50 years after the formal one. The single-shear strength theory appeared in 1900 (Mohr 1900, Coulomb 1773). The twin-shear strength theory (Yu M-H 1983, 1985) appeared 85 years later than the formal single-shear theory. This book should provide readers with engineering backgrounds with an easily understood introduction to the topic. However, from the twin-shear yield criterion presented in 1961 to the twin-shear strength theory in 1985, it was 24 years spent, it was 30 years from the twin-shear yield criterion (Yu M-H 1961) to the unified strength theory (Yu M-H 1991). The development was too slow!

Now the twin-shear yield criterion, the twin-shear strength theory and the unified yield criterion can be considered as parts of the framework of the unified strength theory. The unified strength theory also encompasses many well-known yield criteria and failure criteria as its special cases or linear approximations.

There are two aspects which impelled further development of strength theory. First, for the development of aviation and space-flight technology, nuclear electricity generating projects, large hydroelectric engineering, large electric power stations, chemical industry and machines, we must design structures and devices more reasonably to utilize materials effectively and to decrease the weight of the structure. This is an important and necessary task for engineers and scientists. For example, every 1 kg addition to the weight of the structure of an airplane or spaceship leads to a cost increase equal to the cost of 1 kg of gold. If the structural weight is too heavy, it will shorten a missile's flight 10s or 100s of kilometers. For large hydraulic and electric power plants, proper design could save funds and shorten construction times. So the economic importance is immense. The application area of strength theory is wider than before too. It is applied not only to traditional design of structural elasticity, but also extensively to analysis of the elasto-plasticity and limit-bearing capacity of structures, as well as to the research of fracture, damage, fatigue, creep, shear band formation, discontinuity bifurcation, crazing of polymers, meso-mechanics, plasticity processes, composite mechanics and material surface strength.

Second, the development of computer, computational methods and computing software has driven the advance of strength theory. Application of strength theory affects the results greatly, sometimes much more than the improvement of calculation methods. Therefore the research of strength theory gets more promotion. Strength theories (used for material models) have been implemented into most commercial FEM codes, such as ABAQUS, ADINA, ANSYS, ASAS, COSMOS, DIANA, MARC, MSC/NASTRAN, NON-SAP, PAFEC, PLAXIS (soil and rock plasticity), TITUS and DYTRAN. To the best of the author's knowledge, several users have implemented the twin-shear theory or the unified strength theory into FEM programs including the Aircraft Strength Research Institute of China, the Yangtze River Scientific Research Institute, the Nanjing Water Conservancy Scientific Research Institute, the Northwest Hydropower Investigation and Design Institute and some universities, users also include Nanyang Technological