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Resurgence, Physics and Numbers
edited by Frédéric Fauvet, Dominique Manchon,
Stefano Marmi and David Sauzin

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Preface

The conference "Resurgence, Physics and Numbers" took place in Pisa, at the Centro di Ricerca Matematica Ennio De Giorgi, on May 18–22, 2015. This meeting between mathematicians and theoretical physicists involved in resurgent functions, alien calculus and related combinatorics, with a component on Multiple Zeta Values, enabled an exchange of results in fields which had by that time become very active, and which have seen an outburst of publications since then.

The present volume contains contributions of invited speakers at this conference, reflecting the leading themes illustrated during the workshop. We express our deep gratitude to the staffs of the Scuola Normale Superiore di Pisa and of the CRM Ennio De Giorgi for their dedicated support in the preparation of this meeting; all participants could thus benefit from the wonderful and stimulating atmosphere in these institutions and around Piazza dei Cavalieri. We are also very grateful for the possibility of publishing this volume in the CRM series. We acknowledge with thankfulness the support of the CRM De Giorgi, of the laboratorio Fibonacci, of the ANR project "CARMA" and of the GDR 3340 "Renormalisation" (CNRS).

Pisa, December 2016

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Asymptotics, ambiguities and resurgence

Inês Aniceto

Abstract. The appearance of resurgent functions in the context of the perturbative study of observables in physics is now well established. Whether these arise from the related study of non-linear systems or the saddle-point perturbative analysis, one is left with an asymptotic series and the need of a non-perturbative completion, or transseries, which includes different non-perturbative phenomena. The complete understanding of resummation procedures and the resurgence of the non-perturbative phenomena can then lead to a systematic approach to obtain exact results such as strong-weak coupling interpolation, cancellation of ambiguities in the so-called Stokes directions, and more generally the study of analytic properties of the respective transseries solutions. These notes will give a general overview of how to set-up resurgence in simple examples, and how to proceed towards exact analytic results.

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1 Introduction and summary

Computing physical observables of a given quantum theory, can often only be performed via perturbation theory in either the weakly or strongly coupled regimes. Such perturbative expansions are however often divergent, with zero radius of convergence, and are defined only as asymptotic series

$$\langle \mathcal{O}(g) \rangle \simeq \sum_{k \geq 0} \mathcal{O}_k g^{-k}. \tag{1.1}$$

whose coefficients are factorially divergent at large order[1]

$$\mathcal{O}_k \sim \frac{\Gamma(k+\beta)}{A^{k+\beta}}, \quad k \gg 1. \tag{1.2}$$

It is well known that this divergence is connected to the existence of non-perturbative phenomena, unaccounted for in the perturbative analysis. Resurgence is a mathematical theory which allows us to effectively study this connection, and its consequences. Moreover, it allows us to construct a full non-perturbative solution from perturbative data. First introduced by Écalle in [1–3], modern day resurgence theory has developed in the last three decades into an elegant mathematical tool with a diverse set of applications [4–9]. [2]

Resurgent properties have been observed in a wide range of problems in mathematical physics. They appear for example in solutions of differential and finite difference equations (see *e.g.* the well studied cases of Painlevé I, II and Riccati non-linear differential equations [20–25]). In the contexts of quantum mechanics [26, 27] and quantum field theories [28], non-perturbative phenomena such as instantons [29] and renormalons [30], have long been known to exist beyond perturbation theory. In fact in quantum mechanical problems, it is the existence of asymptotic multi-instanton sectors which allow for a resurgent and unambiguous transseries solution to describe energy eigenvalues [31–33]. Since then, the asymptotic behaviour of perturbation theory and the resurgence behind it, was found in many different examples in physical systems, from quantum mechanics [19, 34–45], to large N gauge theories [23,24,46–63], quantum field theories [40,44,59,64–76], and string theory [13,22–24,48–51,77–87].

The aim of this paper is to present a simple, hands-on approach, on the use of resurgence in the study of asymptotic expansions with associated

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[1] A, β are numbers encoding the position and type of singularities of the related Borel transform.

[2] For recent reviews on resurgence, transseries and summability, see [8, 10–19].

non-perturbative phenomena. We will introduce some key ideas behind resurgence theory in Section 2. Concepts such as transseries, Borel resummation, Stokes phenomena and alien calculus will be presented with the help of simple examples, while the respective formal derivations are referenced for the interested reader.

In Sections 3 and 4 we show the tools of resurgence at work in two simple examples: the linear ordinary differential equation (ODE) governing the Airy function [10,49], and the non-linear ODE behind the so-called Müller-Israel-Stuart (MIS) theory [88,89]. Along the way different numerical methods are introduced, including convergence acceleration and resummation. This analysis closely follows the work of [19], together with some results of [42]. The milestones achieved for each example are:

1. Given our asymptotic expansion, we find (or make an educated guess for) the respective transseries solution, with sectors describing both perturbative and non-perturbative phenomena. The tools presented in Section 2 are then used to make predictions of the relations between those different sectors.
2. These predictions can be checked numerically to high precision. In cases where the non-perturbative data is not known, these predictions, together with the assumption that the transseries is resurgent, allow us to construct a full non-perturbative solution from perturbative data.
3. Finally, from the complete resurgent transseries solution, one can perform resummation methods to obtain exact results away from the asymptotic regime. Resurgence theory plays an essential role in deriving these results, such as the cancellation of ambiguities or strong/weak coupling interpolation.

Such linear and non-linear ODEs are the natural starting points to show resurgence at work. Resurgent techniques can also be applied to a much wider range of problems, following the same steps as described above. We finish in Section 5 with a summary of results and a discussion of open problems in mathematical physics, where resurgence can play a key role.

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2 General concepts and definitions

As a starting point, take the strong coupling regime of some observable $F(z)$, with coupling variable $z \gg 1$. We will further assume that using some procedure of perturbation theory (*e.g.* saddle-point analysis or recursion equation from differential equations) in this regime we obtain an expansion

$$F(z) \simeq \sum_{n=0}^{+\infty} \frac{F_n}{z^{n+1}}. \tag{2.1}$$

which is asymptotic, with zero radius of convergence. Moreover, we will take the large-order behaviour of the coefficients F_n to be growing factorially

$$F_n \sim n! \text{ for } n \gg 1. \tag{2.2}$$

This is a very common problem appearing in the study of observables of interacting theories, for example in QFTs, due to the factorial growth of the number Feynman diagrams at each loop order in perturbation theory [29]. Furthermore, it is often the case that perturbation theory and the associated series of the type (2.1), are the only results one can expect from the study of the observable in question. Given the asymptotic properties of the series, one may wonder how to make sense of the formal power series (2.1): this will be the main question we will address at present. This section follows largely the works [8,11,14,23,90,91].

As the preliminary step in our quest, one would like to know how to associate a value to (2.1) for each value of the coupling z. A well known process (see *e.g.* [49]) for asymptotic series is to perform an *optimal truncation*, which leads to very good approximations. Optimal truncation is the truncation of the series to the so-called least term: in the regime $z \gg 1$, the terms of (2.1) start by decreasing very rapidly, and only at some point (the least term) start increasing. In this truncation we want to keep terms such that

$$\frac{|F_n|}{|z|^{n+1}} \ll \frac{|F_{n-1}|}{|z|^n}, \text{ for } n \ll |z| \tag{2.3}$$

In fact we can keep terms such that $\frac{|F_n|}{|F_{n-1}|} \lesssim |z|$ for $n \leq N_{\text{op}}(z)$, thus obtaining an optimally truncated series

$$F_{\text{op}}(z) = \sum_{n=0}^{N_{\text{op}}(z)} \frac{F_n}{z^{n+1}}. \tag{2.4}$$

Note that the least term $N_{\text{op}}(z)$, will depend on the value of the coupling z for which the series (2.1) is being evaluated at. In performing the optimal truncation, we obtain an extremely accurate approximation of (2.1) and the error can be seen to be

$$\left|F(z) - F_{\mathrm{op}}(z)\right| \sim \mathrm{e}^{-Az}, \tag{2.5}$$

where A is a characteristic number of the problem in question. This is the first hint of the relation between the asymptotic behaviour of (2.1) and its non-perturbative origins: the error is non-analytic, $\mathrm{e}^{-Az} \sim 0$ if one expands around $z \sim \infty$.

One can improve on this error by performing certain resummation procedures to the divergent tail which was left out from the truncation procedure. One such framework to address the asymptotic properties and resummation of the series such as (2.1) is *Borel analysis*. In this framework we introduce the Borel transform, via the following rule

$$\mathcal{B}\left[\frac{1}{z^{\alpha+1}}\right](s) \equiv \frac{s^{\alpha}}{\Gamma(\alpha+1)}, \tag{2.6}$$

where $\Gamma(\alpha)$ is the gamma function. Performing this transformation to every term of the asymptotic series (2.1), we obtain the Borel transform associated to that series:

$$\mathcal{B}[F](s) = \sum_{k=0}^{+\infty} \frac{F_k}{\Gamma(k+1)} s^k. \tag{2.7}$$

This series is now convergent around the origin in $\mathbb{C}$, with some non-zero radius of convergence. Note that the rule (2.6) is not well defined for cases where the power of z^{-1} is non-positive (*i.e.* $\alpha \leq -1$). In an asymptotic series for $z \gg 1$ these terms are of finite number and must be excluded from the procedure of Borel analysis. They do not change the asymptotic properties of the series and can be easily re-inserted once the resummation procedure has been performed. The rule (2.6) which leads to the now convergent series (2.7) can be seen more naturally as applying an inverse Laplace transform to each of the terms in (2.1). Due to the divergent nature of the series (2.1), this procedure can only be seen as the inverse Laplace transform of $F(z)$ at a formal level. Nevertheless, given the convergent properties of (2.7) one can study the analytic properties of this second series and sum it around the origin in $\mathbb{C}$.

The Borel transform $\mathcal{B}[F](s)$ will naturally have singularities (defining its radius of convergence). To study the analytic properties of (2.7) one needs to locate these singularities in the complex s-plane (which we shall also call the *Borel plane*). Within the radius of convergence, the series (2.7) will define an analytic function, which can sometimes be guessed, but more often will be approximated numerically (as we shall see later). In directions $\arg s = \theta$ where there are no singularities, one may analytically continue this function along the ray $e^{i\theta}\mathbb{R}^+$ and define an inverse Borel transform - or *Borel resummation* of $F(z)$ along θ - by a Laplace transform

$$\mathcal{S}_\theta F(z) = \int_0^{e^{i\theta}\infty} ds\, \mathcal{B}[F](s)\, e^{-zs}. \tag{2.8}$$

The function $\mathcal{S}_\theta F(z)$ has the same asymptotic expansion as $F(z)$, and for each z will give a better approximation to the value of the asymptotic series (2.1) than the optimal truncation method (even if the function $\mathcal{B}[F](s)$ is only known as a numerical approximation).

But what happens when $\mathcal{B}[F](s)$ has singularities (poles and/or branch cuts) along a direction θ? How can we assign a value for $F(z)$ in this case? The Laplace transform (2.8) will be ill defined as we have singularities exactly on the direction of the integration contour. We then need to choose a contour which avoids the singularities. The most natural contours one can choose give rise to the so-called lateral Borel resummations:

$$\mathcal{S}_{\theta^\pm} F(z) \equiv \mathcal{S}_{\theta\pm\varepsilon} F(z), \quad \text{for} \quad \varepsilon \sim 0^+. \tag{2.9}$$

Different integration contours give rise to functions with the same asymptotic behaviour but which differ by non-analytic exponentially suppressed terms. Thus choosing different contours gives rise to a non-perturbative ambiguity and it is said that $F(z)$ is non-Borel summable along these singular directions θ.

As a simple example, assume that the first singularity of $\mathcal{B}[F](s)$ along a certain direction θ is at $s = A$, and it is a simple pole

$$\mathcal{B}[F](s)|_{s\simeq A} \sim \frac{1}{s-A}. \tag{2.10}$$

The difference between the two lateral Borel resummations is

$$\mathcal{S}_{\theta^+} F(z) - \mathcal{S}_{\theta^-} F(z) \propto \oint_{s=A} ds \frac{e^{-zs}}{s-A} = e^{-Az}. \tag{2.11}$$

Once again the non-analytic term e^{-Az} appears from the analysis of the asymptotic behaviour of (2.1), and the characteristic value A shows up as the leading singularity of the Borel transform (2.7).

The existence of different singular directions $\arg s = \theta_i$ on the Borel plane associated to the original series (2.1) leads to a family of sectorial analytic functions $\{\mathcal{S}_\theta F(z)\}$ all with the same asymptotic behaviour, and which differ by non-analytic terms. In order to understand how to "connect" each of these sectors, one needs to understand the behaviour of the Borel transform around each singularity for all singular directions θ_i, so we can learn how to jump across the direction θ_i and reach a different sector. The singular directions θ_i are called *Stokes lines*. Learning about the behaviour of the Borel transform along Stokes lines will lead to the construction of an unambiguous result for the resummed function, even along these singular directions: a procedure that known as ambiguity cancellation. To perform a thorough and systematic analysis of what happens at Stokes lines we now turn to resurgence, and the realm of simple resurgent functions.

2.1 Transseries, resurgence and discontinuities

The fact that the resummations in the different sectors $\{\mathcal{S}_\theta F\}$ differ by non-perturbative terms hints to the fact that the full solution associated with the observable $F(z)$ should be some non-perturbative completion of the asymptotic series (2.1), into what is called a transseries. Indeed, in the calculation of energies in quantum mechanics, the transseries is an essential step in the cancellation of ambiguities: the non-perturbative terms are given by instanton sectors, and A is the instanton action related to the probability of tunneling between (possibly complex) saddles of the potential (see *e.g.* [26,27,32–36,40,43,92]).

A transseries is a formal series expansion both in the original variable $z \gg 1$ and also in the non-analytic terms. We will work with the so-called log-free height-one transseries, where the expansion is on transmonomials $z^\alpha \mathrm{e}^{S(z)}$ with $\alpha \in \mathbb{R}$ and $S(z)$ is some particular convergent series (more intricate examples where $S(z)$ is in itself a transseries, with compositions of exponentials and logarithms, can be also studied, see *e.g.* [12]). In its simplest form, the transseries has the form

$$\mathcal{F}(z,\sigma) = \sum_{\ell=0}^{+\infty} \sigma^\ell F^{(\ell)}(z) \in \mathbb{C}\left[\!\left[z^{-1}, \sigma \mathrm{e}^{-Az}\right]\!\right], \tag{2.12}$$

where $F^{(0)}(z)$ is just the perturbative series (2.1) and

$$F^{(\ell)}(z) = \mathrm{e}^{-\ell A z}\Phi_\ell(z), \quad \ell \geq 1, \tag{2.13}$$

with $\Phi_\ell(z)$ generally an asymptotic series as well

$$\Phi_\ell(z) = z^{\beta_\ell} \sum_{k=0}^{+\infty} \frac{F_k^{(\ell)}}{z^k}. \tag{2.14}$$

In most cases of interest, the leading behaviour of the asymptotic series, given by z^{β_ℓ} is of the form $\beta_\ell = -\ell\,\beta$, and we shall assume this form from now on, unless otherwise stated. The transseries (2.12) is a one-parameter transseries: it appears when one has non-perturbative sectors such as instantons, which are exponentially suppressed with the same associated action A, and where ℓ is the instanton number. Note that from calculations such as (2.11) we know that the sectorial functions will differ by exponentially small terms e^{-Az}, but a more careful analysis of these differences would show that (in the case of simple resurgent functions, as defined below) for each suppressed contribution there is an asymptotic expansion associated with it. The parameter σ in (2.12) is the transseries parameter, which for each particular wedge of the complex plane, selects distinct non-perturbative completions to the original series (2.1).

There are extensions of the above transseries (2.12) to include more parameters σ_i. In fact, if a given observable has different non-analytic contributions $\mathrm{e}^{-A_i z}$, for $A_i \neq A$ (and typically also different from the multi-instanton contributions already included in the one-parameter case $A_i \neq \ell A$), one should expect a new transseries parameter for each non-analytic term appearing.

An asymptotic expansion $F(z)$ (such as $F(z)$ in (2.1) or any of the $\Phi_\ell(z)$) is said to be a simple resurgent function if its Borel transform only has simple poles or logarithmic branch cuts as singularities. Taking ω as a singularity, the Borel transform around this singularity will be of the form[3]

$$\mathcal{B}[F](s)\big|_{s=\omega} \sim \frac{a_\omega}{2\pi\mathrm{i}\,(s-\omega)} + \Psi(s-\omega)\,\frac{\log(s-\omega)}{2\pi\mathrm{i}} + \zeta_{\mathrm{hol}}(s-\omega)\,, \tag{2.16}$$

where $a_\omega \in \mathbb{C}$ and Ψ, ζ_{hol} are analytic around the origin. Moreover $\Psi(s)$ will be related to a function $G_1(z)$ by the inverse Borel transform

$$\Psi(s) = \mathcal{B}[G_1](s). \tag{2.17}$$

[3] Many times the Borel transform is not exactly of the shape (2.16), but instead it has square root branch cuts. Nevertheless we will still be in the realm of simple resurgent functions if the $\mathcal{B}\left[z^\gamma F\right](z)$ has the behaviour (2.16) where γ is commonly the "degree" of the branch cut. Typically this is related to a factorial growth of the coefficients F_k in (2.1) which differs from the factorial growth "removed" by the Borel transform. For example, assume in (2.1) that $F_n \sim \Gamma(n+1-\gamma)$ for some γ when $n \gg 1$. Then $\mathcal{B}[F](s)$ in (2.7) has coefficients which grow as

$$\frac{F_k}{\Gamma(k+1)} \sim \frac{\Gamma(k+1-\gamma)}{\Gamma(k+1)} \neq 1 \quad \text{if } \gamma \neq 0,\ k \gg 1. \tag{2.15}$$

On the other hand $\mathcal{B}\left[z^\gamma F\right](s) = \sum_{k\geq\gamma} F_k s^{k-\gamma}/\,\Gamma(k+1-\gamma)$ will have the expected behaviour (2.16). For a detailed analysis on this see [19].

Normally the function $G_1(z)$ is also known as a series and requires a resummation procedure as well. A transseries (2.12) will have resurgent properties if the coefficients of different sectors in (2.13) $F_k^{(\ell)}$ and $F_r^{(\ell')}$ will be related for ℓ close to ℓ'. This can be seen directly at the level of the Borel transforms by noticing that the type of relation (2.16) will relate Φ_ℓ to $\Phi_{\ell'}$ in the same way that F and G_1 are related by (2.16) and (2.17). In other words, if we take $F = \Phi_\ell$ for some particular ℓ, and analyse its behaviour on some particular singularity, we will see that the function G_1 in (2.17) will be $\Phi_{\ell'}$ for ℓ close to ℓ'. In fact, the value of ℓ' will directly depend on the singularity we are analysing. These relations can be checked via the so-called large-order relations, which will be exemplified later on.

To highlight how the behaviour (2.16) is related to the non-perturbative jump (2.11), assume that the Borel transform (2.7) has only one singularity $s = \omega_1$ of the type (2.16), in some direction θ of the complex Borel plane (see Figure 2.1). The difference between lateral Borel resummations will be given by the integration over the Hankel contour C_ω around the branch cut starting at ω_1, as defined on the right of Figure 2.1:

$$
\begin{aligned}
(\mathcal{S}_{\theta^+} - \mathcal{S}_{\theta^-})\, F(z) &= \int_{C_\omega} ds\, \mathcal{B}[F](s)\, \mathrm{e}^{-s z} \\
&= -a_{\omega_1} \mathrm{e}^{-\omega_1 z} + \mathrm{e}^{-\omega_1 z} \int_{C_0} \frac{ds}{2\pi \mathrm{i}}\, \Psi(s) \log(s)\, \mathrm{e}^{-s z}.
\end{aligned} \tag{2.18}
$$

The last integration over C_0 (Hankel contour now centred at the origin) will return the discontinuity across the log cut $(-2\pi\mathrm{i})$ multiplied by the Laplace transform of the function $\Psi(s)$. Given that we have the identification (2.17), this last part is nothing more than the resummation of the function $G_1(z)$. We can write[4]

$$
(\mathcal{S}_{\theta^+} - \mathcal{S}_{\theta^-})\, F(z) = -\left(a_{\omega_1} + \mathcal{S}_{\theta^-} G_1(z)\right) \mathrm{e}^{-\omega_1 z} + \cdots . \tag{2.19}
$$

In the $\cdots$ we have included the possibility that the Borel transform of $G_1(z)$ would also have a singularity along the same direction θ. If $G_1(z)$ is an analytic function, there are no more contributions and $\mathcal{S}_{\theta^-} G_1(z) = \mathcal{S}_\theta G_1(z)$. But if $\mathcal{B}[G_1](s)$ has a singularity along the direction θ, say at

[4] If $G_1(z)$ is in itself asymptotic with singularities along direction θ, we need to choose a lateral resummation for $G_1(z)$, which will be directly linked to the choice of the discontinuity for the log branch cut chosen.

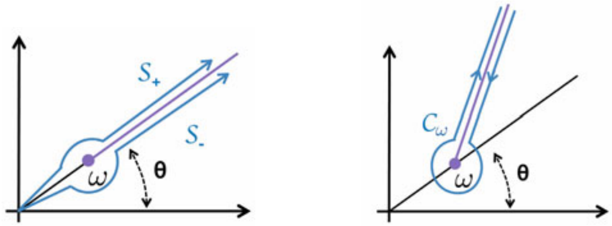

Figure 2.1. On the left: lateral Borel resummation contours around singularity $s = \omega$. On the right: Hankel contour around the branch cut starting at $s = \omega$.

$s = \omega_2$

$$\mathcal{B}\left[G_1\right](s)\big|_{s=\omega_2} \sim \frac{a_{\omega_2}}{2\pi\mathrm{i}\,(s-\omega_2)} + \mathcal{B}\left[G_2\right](s-\omega_2)\,\frac{\log(s-\omega_2)}{2\pi\mathrm{i}} + \text{holomorphic}, \tag{2.20}$$

then one can expect that this singularity will also contribute to the overall difference between lateral Borel resummations of $F(z)$, and the position of this singularity will naturally be at $s = \omega_1 + \omega_2$ (thus its contribution will be exponentially suppressed by $\mathrm{e}^{-(\omega_1+\omega_2)z}$ in (2.1)). This contribution can in fact be visible if we analyse the Borel transform $\mathcal{B}[F](s)$ at $s = \omega_1 + \omega_2 \equiv \omega$, and it will be of the same form as $\mathcal{B}[G_1](s)\big|_{s=\omega_2}$ up to an overall constant

$$\mathcal{B}[F](s)\big|_{s=\omega\equiv\omega_1+\omega_2} \sim \frac{C_2\,a_{\omega_2}}{2\pi\mathrm{i}\,(s-\omega)} + C_2\,\mathcal{B}\left[G_2\right](s-\omega)\,\frac{\log(s-\omega)}{2\pi\mathrm{i}} + \text{holomorphic}. \tag{2.21}$$

To reach this new singularity, coming from $B[G_1](s)$, there are in general different ways to analytically continue the paths of resummation to avoid the previous singularities (in this case only one), passing these singularities from above or from below. These different paths of analytic continuation to reach each singular point are encoded in the jump of $F(z)$ across the Stokes line, through a weighed average of such paths (see [14,90]).

The difference between lateral Borel resummations along a Stokes line defines the *discontinuity* of $F(z)$ across that line:

$$\left(\mathcal{S}_{\theta^+} - \mathcal{S}_{\theta^-}\right) F(z) \equiv -\mathcal{S}_{\theta^-} \circ \mathrm{Disc}_\theta F(z). \tag{2.22}$$

For the example shown above (where each asymptotic expansion $F(z)$ and $G_i(z)$ will only have one independent singularity each in the Stokes direction θ), this discontinuity will be given by the sum of the contributions of all the differences between lateral Borel resummations:

$$\begin{aligned} \mathrm{Disc}_\theta F(z) &= \left(a_{\omega_1} + G_1(z)\right) \mathrm{e}^{-\omega_1 z} + C_2 \left(a_{\omega_2} + G_2(z)\right) \mathrm{e}^{-(\omega_1+\omega_2) z} + \cdots \\ &= \sum_{\omega_n \in \mathrm{Sing}_\theta} C_n \left(a_{\omega_n} + G_n(z)\right) \mathrm{e}^{-\sum_{j=1}^{n} \omega_j z}. \end{aligned} \tag{2.23}$$

In the above result, $\mathrm{Sing}_\theta = \{\omega_i\}$ is the collection of singularities appearing in all asymptotic expansions in the direction θ. The constants C_n reflect the weighed average of paths that encode the contribution of the singularities of other Borel transforms $\mathcal{B}[G_n](s)$ to the discontinuity of the original asymptotic series $F(z)$, where

$$\begin{aligned} \mathcal{B}[G_i](s)\big|_{s=\omega_{i+1}} \sim & \frac{a_{\omega_{i+1}}}{2\pi\mathrm{i}\,(s-\omega_{i+1})} \\ & + \mathcal{B}[G_{i+1}](s-\omega_{i+1}) \frac{\log(s-\omega_{i+1})}{2\pi\mathrm{i}} \\ & + \text{holomorphic}. \end{aligned} \tag{2.24}$$

We have assumed that the singular behaviour of the Borel transform of $F(z)$ at $\omega \equiv \sum_{j=1}^{n} \omega_j$ (for $n > 1$), originated solely from the behaviour of $\mathcal{B}[G_{n-1}]$ at $s = \omega_n$ (and equivalently for the $G_i(z)$). More generally, the Borel transform of the asymptotic expansion $F(z)$ will have a singular behaviour at $\omega \equiv \sum_{j=1}^{n} \omega_j$

$$\begin{aligned} \mathcal{B}[F](s)\big|_{s=\omega\equiv\sum_{j=1}^{n}\omega_j} \sim & \frac{C_{0,n}\, a_{\omega_n}}{2\pi\mathrm{i}(s-\omega)} \\ & + C_{0,n}\, \mathcal{B}[G_n](s-\omega) \frac{\log(s-\omega)}{2\pi\mathrm{i}} \\ & + \text{holomorphic}, \end{aligned} \tag{2.25}$$

where the $C_{0,n}$ can have contributions from the singularities of all the sectors $G_i(z)$, as well as a contribution not associated to any of these. In this case, (2.22) and (2.23) still hold true, but the explicit form of the coefficients $C_n \equiv C_{0,n}$ will be more involved. For the Borel transforms of $G_i(z)$ we also expect the behaviour

$$\begin{aligned} \mathcal{B}[G_i](s)\big|_{s=\omega\equiv\sum_{j=i+1}^{n}\omega_j} \sim & \frac{C_{i,n}\, a_{\omega_n}}{2\pi\mathrm{i}\,(s-\omega)} \\ & + C_{i,n}\, \mathcal{B}[G_n](s-\omega) \frac{\log(s-\omega)}{2\pi\mathrm{i}} \\ & + \text{holomorphic}, \end{aligned} \tag{2.26}$$

and an expression similar to (2.23) can be found for their discontinuity.

In the interest of systematically determining the general explicit formulas for the discontinuity across Stokes directions,[5] we now turn to *alien calculus*. The discontinuous jump across Stokes lines, given by (2.22), naturally defines another operator called the Stokes automorphism $\underline{\mathfrak{S}}_\theta$

$$\mathcal{S}_{\theta^+} = \mathcal{S}_{\theta^-} \circ \underline{\mathfrak{S}}_\theta = \mathcal{S}_{\theta^-} \circ (\mathbf{1} - \mathrm{Disc}_\theta)\,. \tag{2.27}$$

The Stokes automorphism acts on the set of simple resurgent functions (which forms a subalgebra of $\mathbb{C}\left[\!\left[z^{-1}\right]\!\right]$), and will induce a differentiation operation on the same algebra via exponentiation (see *e.g.* [90]):[6]

$$\underline{\mathfrak{S}}_\theta = \exp\left\{\underline{\Delta}_\theta\right\}. \tag{2.28}$$

The operator $\underline{\Delta}_\theta$ is called a directional differentiation, and can be decomposed into components which depend only on each of the singularities existing in the direction θ. The Stokes automorphism in the direction θ becomes

$$\underline{\mathfrak{S}}_\theta = \exp\left\{\sum_{\omega_i \in \mathrm{Sing}_\theta} \mathrm{e}^{-\omega_i z} \Delta_{\omega_i}\right\}. \tag{2.29}$$

These *alien derivatives* Δ_ω are a differentiation (obey Leibnitz rule, as shown below in a simple example) and have the following properties:[7] for a resurgent function $F(z)$

- if ω is not a singular point in the Borel plane, then $\Delta_\omega F = 0$;
- if ω is the only (or the first) singular point in the direction θ of Borel plane, then (2.16) holds true, and Δ_ω is related to the algebraic structure of the Borel transform at the singular point (shedding the functional structure)
$$\mathcal{S}_\theta\left(\Delta_\omega F\right) = -a_\omega - \mathcal{S}_\theta G \tag{2.30}$$
or equivalently $\Delta_\omega F(z) = -a_\omega - G(z)$;

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[5] Each term in (2.23) can be directly determined by the analysis of singularities of Borel transforms for each sector $F(z)$ and $G_i(z)$. Nevertheless, it is extremely valuable to have an approach which uses the information that these are simple resurgent functions, with a set of singularities in each singular direction θ, to systematically construct a general formula for the discontinuity.

[6] In an equivalent way, the automorphism $T: f(x) \to f(x+1)$ which defines translations also induces a differentiation via $T = \exp\left(\frac{d}{dx}\right)$, which can be checked by a Taylor expansion of this exponential.

[7] These properties can be checked by expanding the exponential in (2.29) and taking into consideration the different paths of analytic continuation one can take to reach a given singularity, see *e.g.* [14].

- if we have a collection of singular points $\omega \in \{\omega_1,\, \omega_1 + \omega_2, \cdots \cdots,\, \sum_i \omega_i, \cdots\}$ on the Borel plane, then for $\omega \equiv \sum_{i=1}^{n} \omega_i$ the alien derivative Δ_ω will be given by

$$\Delta_\omega F = -\sum_{s=1}^{n} \frac{1}{s} \sum_{0=m_0<m_1<\cdots<m_s=n} \prod_{r=0}^{s-1} C_{m_r,m_{r+1}} \left(a_{\omega_n} + G_n\right), \tag{2.31}$$

where the $C_{m,n}$ are defined in (2.25) and (2.26).

The exponential factors appearing in (2.29) are an essential part of the construction of the jump across the Stokes line, as they will be responsible for the exponential weights appearing in (2.23). Another definition which will be of importance is the pointed alien derivative

$$\dot{\Delta}_\omega = \mathrm{e}^{-\omega z} \Delta_\omega. \tag{2.32}$$

If we expand the exponential in (2.29) we find

$$\begin{aligned} &\underline{\mathfrak{S}}_\theta F(z) \\ &= F(z) + \sum_{\substack{r \geq 1 \\ \omega_{n_i} \in \mathrm{Sing}_\theta}} \frac{1}{r!} \mathrm{e}^{-(\omega_{n_1}+\omega_{n_2}+\cdots+\omega_{n_r})z} \Delta_{\omega_{n_1}} \Delta_{\omega_{n_2}} \cdots \Delta_{\omega_{n_r}} F(z) . \end{aligned} \tag{2.33}$$

The jump of $F(z)$ across the Stokes direction θ is then

$$\begin{aligned} &\mathcal{S}_{\theta^+} F - \mathcal{S}_{\theta^-} F \\ &= \sum_{\substack{r \geq 1 \\ \omega_{n_i} \in \mathrm{Sing}_\theta}} \frac{1}{r!} \mathrm{e}^{-(\omega_{n_1}+\omega_{n_2}+\cdots+\omega_{n_r})z} \mathcal{S}_{\theta^-} \left(\Delta_{\omega_{n_1}} \Delta_{\omega_{n_2}} \cdots \Delta_{\omega_{n_r}} F(z) \right) . \end{aligned} \tag{2.34}$$

Take the example given above where $F(z)$ has a singularity in the Borel plane at $s = \omega_1$, each of the higher sectors $G_i(z)$ have also one singularity in the Borel plane at $s = \omega_{i+1}$, with behaviour around the singularities given by (2.16) and (2.24), respectively. We can directly write the nonzero alien derivatives acting on these functions:

$$\Delta_{\omega_1} F(z) = -a_{\omega_1} - G_1(z) , \;\; \Delta_{\omega_{i+1}} G_i(z) = -a_{\omega_{i+1}} - G_{i+1}(z) . \tag{2.35}$$

The set of all singularities appearing in the direction θ is $\mathrm{Sing}_\theta = \{\omega_i,\ i \in \mathbb{N}\}$. It is not hard to see that the only terms in (2.33) acting non-trivially on $F(z)$ are

$$\begin{aligned}
&\underline{\mathfrak{S}}_{\theta} F(z) \\
&= F(z) + \Big(\mathrm{e}^{-\omega_1 z} \Delta_{\omega_1} + \frac{1}{2!} \mathrm{e}^{-(\omega_1+\omega_2) z} \Delta_{\omega_2} \Delta_{\omega_1} \\
&\qquad + \frac{1}{3!} \mathrm{e}^{-(\omega_1+\omega_2+\omega_3) z} \Delta_{\omega_3} \Delta_{\omega_2} \Delta_{\omega_1} + \cdots \Big) F(z) \\
&= F(z) + \sum_{n \geq 1} \frac{(-1)^n}{n!} \mathrm{e}^{-\sum_{i=1}^{n} \omega_i z} \left(a_{\omega_n} + G_n(z) \right).
\end{aligned} \tag{2.36}$$

Comparing with (2.23) and taking into consideration (2.27), we find an agreement with the expression found before for the discontinuity of $F(z)$ with the identification of the constants $C_n = \frac{(-1)^n}{n!}$.

2.2 Some properties of the alien derivative revisited

There are two major properties of the alien operator Δ_ω extremely useful in the study of the Stokes phenomena occurring across singular directions: Δ_ω is a differentiation, and the pointed alien derivative $\dot{\Delta}_\omega$ as defined in (2.32) commutes with the natural derivative $\left[\dot{\Delta}_\omega, \frac{d}{dz}\right] = 0$. In this subsection we follow [14] and analyse these two properties in more detail.

The alien derivative operator is indeed a differentiation, in the sense that it obeys Leibnitz rule. Let $F(z)$ and $G(z)$ be simple resurgent functions. Then

$$\Delta_\omega \left(F(z)\, G(z) \right) = \left(\Delta_\omega F \right)(z)\, G(z) + F(z) \left(\Delta_\omega G(z) \right). \tag{2.37}$$

This can be clearly seen at the level of the respective Borel transforms for simple examples. Start by noting that the product of two simple resurgent functions will correspond to the convolution of their Borel transforms

$$\mathcal{B}[F\,G](s) = \mathcal{B}[F] * \mathcal{B}[G](s) = \int_0^s d\zeta\, \mathcal{B}[F](\zeta)\, \mathcal{B}[G](s-\zeta). \tag{2.38}$$

For simplicity take the case of the Borel transforms being simple poles at $s = \omega$ for both functions F, G:

$$\mathcal{B}[F](s) = \frac{a}{2\pi \mathrm{i} (s-\omega)}, \quad \mathcal{B}[G](s) = \frac{b}{2\pi \mathrm{i} (s-\omega)}. \tag{2.39}$$

The alien derivatives acting on each of these resurgent functions give non-zero results at $s = \omega$: $\Delta_\omega F(z) = -a$, $\Delta_\omega G(z) = -b$. One can easily see that a resummation (2.8) of each of these Borel transforms will lead to the relation $\mathcal{S}_\theta F(z) = \frac{a}{b}\mathcal{S}_\theta G(z)$. The Borel transform of the product of the two functions will be:

$$\begin{aligned}\mathcal{B}[F\,G](s) &= \frac{a\,b}{(2\pi\mathrm{i})^2}\int_0^s d\zeta\,\frac{1}{\zeta-\omega}\,\frac{1}{s-\zeta-\omega}\\ &= \frac{a\,b}{(2\pi\mathrm{i})^2}\,\frac{1}{s-2\omega}\left(\int_0^s d\zeta\,\frac{1}{\zeta-\omega}+\int_0^s d\zeta\,\frac{1}{s-\zeta-\omega}\right)\\ &= \frac{2\,a\,b}{(2\pi\mathrm{i})^2\,(s-2\omega)}\,\log\left(1-\frac{s}{\omega}\right),\end{aligned} \tag{2.40}$$

where we assumed $|s| < |\omega|$. Note the appearance of a new pole at $s = 2\omega$, and a log cut at $s = \omega$. We shall firstly focus on the singular behaviour at $s = \omega$. From the above result we can now read

$$\mathcal{B}[F\,G](s) = \Psi\,(s-\omega)\,\frac{\log(s-\omega)}{2\pi\mathrm{i}} + \text{holomorphic}. \tag{2.41}$$

The function

$$\Psi\,(s) \equiv \mathcal{B}[H](s) = \frac{2\,a\,b}{(2\pi\mathrm{i})\,(s-\omega)} \tag{2.42}$$

corresponds to the Borel transform of a function $H\,(z)$, such that the only non-zero alien derivative acting on the product FG is given by $\Delta_\omega\,(F\,G)\,(z) = -H(z)$. Given the Borel transforms of F and G, it is not difficult to note that

$$\Psi\,(s) = a\,\mathcal{B}[G]\,(s) + b\,\mathcal{B}[F]\,(s). \tag{2.43}$$

Consequently for this simple example we have just shown that the operator Δ_ω obeys the Leibnitz rule:

$$\begin{aligned}\Delta_\omega\,(F\,G)\,(z) &= -a\,G(z) - b\,F(z)\\ &= \Delta_\omega F\,(z)\;G(z) + \Delta_\omega G\,(z)\;F(z).\end{aligned} \tag{2.44}$$

One could worry that the new singularity at $s = 2\omega$ for the Borel transform of the product $F\,G$ would give rise to a new non-zero alien derivative $\Delta_{2\omega}$. Given that neither Borel transforms of F or G have a singularity at this point, one expects that $\Delta_{2\omega}\,(F\,G) = 0$, as it is a differentiation. But if one analytically continues (2.40) past the first singularity $s = \omega$ we find that the residue at $s = 2\omega$ is non-zero. This is not an inconsistency however, because the alien derivative can be defined via a combination of different analytic continuations of the paths of integration avoiding the previous singularities (in our case $s = \omega$).[8]

Another important property of the alien derivative is its commutation relations with the usual derivative $d_z \equiv \frac{d}{dz}$. Firstly note that d_z commutes with the lateral Borel resummations. Using (2.7)

$$\begin{aligned} d_z \mathcal{S}_{\theta^\pm} F(z) &= \int_0^{e^{\pm i\varepsilon}\infty} ds\, \mathcal{B}[F](s)\,(-s)\,e^{-sz} \\ &= -\int_0^{e^{\pm i\varepsilon}\infty} ds \sum_{k=0}^{+\infty} \frac{F_k s^{k+1}}{\Gamma(k+1)} = \mathcal{S}_{\theta^\pm}(d_z F)(z), \end{aligned} \tag{2.45}$$

and we conclude that $d_z \mathcal{S}_{\theta^\pm} = \mathcal{S}_{\theta^\pm} d_z$. From this result and (2.27) we easily obtain

$$d_z \mathcal{S}_{\theta^+} = d_z \mathcal{S}_{\theta^-} \circ \underline{\mathfrak{S}}_\theta = \mathcal{S}_{\theta^-} \circ \underline{\mathfrak{S}}_\theta\, d_z \;\Leftrightarrow\; \mathcal{S}_{\theta^-} d_z\, \underline{\mathfrak{S}}_\theta = \mathcal{S}_{\theta^-} \underline{\mathfrak{S}}_\theta\, d_z, \tag{2.46}$$

from which we see that d_z also commutes with the Stokes automorphism. Now using the definition of the Stokes automorphism in terms of the pointed alien derivatives, $\underline{\mathfrak{S}}_\theta = \exp\left(\sum_{\omega_i \in \mathrm{Sing}_\theta} \dot{\Delta}_{\omega_i}\right)$, we conclude that the pointed alien derivative commutes with the usual derivative

$$\left[\dot{\Delta}_\omega, \frac{d}{dz}\right] = 0. \tag{2.47}$$

With this relation it is now easy to determine the commutation relations of the usual derivative and the regular alien derivative Δ_ω:

$$\left[\Delta_\omega, \frac{d}{dz}\right] = -\omega\,\Delta_\omega. \tag{2.48}$$

These properties will allow us to find a connection between alien calculus and usual calculus, thus providing a way to determine the action of the alien derivative from the knowledge of the relevant transseries: this comes in the form of a set of equations called *bridge equations*.

2.3 Bridge equations

In the context of non-linear problems in ordinary differential equations, the transseries solution $\mathcal{F}(z,\sigma)$ (we are taking the simplest one-parameter example (2.12)) will obey a particular non-linear ODE in the variable

[8] Using (2.31), we can write $\Delta_{2\omega}(FG) = -C_{0,2} - \frac{1}{2} C_{0,1} C_{1,2}$, where the constants $C_{i,j}$ can be read from the local behaviour of $\mathcal{B}[FG](s)$ at $s = \omega$ and $s = 2\omega$ ($C_{0,1} = 1$ and $C_{0,2} = -a\,b$, respectively), as well as of $\Psi(s)$ at $s = \omega$ ($C_{1,2} = 2a\,b$). We then conclude that $\Delta_{2\omega}(FG) = 0$.

z. Given that the pointed alien derivative commutes with the usual derivative, $\left[\dot{\Delta}_\omega, d_z\right] = 0$, and that the transseries depends on two commuting parameters z and σ, $[d_z, d_\sigma] = 0$, one finds that $\dot{\Delta}_\omega F$ and $d_\sigma F$ will obey the same linearised ODE (in variable z).[9] As these are two complete solutions of the same ODE, it follows that they must be proportional

$$\dot{\Delta}_\omega F = S_\omega(\sigma) \frac{dF}{d\sigma}, \tag{2.49}$$

with the proportionality factor only allowed to depend on the parameter σ via some Taylor expansion:

$$S_\omega(\sigma) = \sum_{k=0}^{+\infty} S_\omega^{(k)} \sigma^k. \tag{2.50}$$

The equations (2.49) are Écalle's *bridge equations*. The coefficients in the expansion of the proportionality factor (2.50) will depend on the specific problem one is solving, *i.e.* the ansatz used for the transseries and the type of singularities in it.[10] The constants $S_\omega^{(k)}$ appearing in (2.50) are the well-known *Stokes coefficients* (or Stokes constants), which encode the Stokes phenomena across the singular Stokes directions. They naturally appear in the analysis of singularities in the Borel plane, as we will see in the examples below. Note that if the transseries has more than one-parameter (more than one singular direction) the bridge equations will reflect this (see for example Section 4 of [23], and [19]).

We shall now turn to some applications, and detail how to use resurgence in different examples of ODEs. In particular we will focus on the construction of transseries, the resurgent analysis of Borel transforms and analytic properties of the transseries solutions (such as varying $z \in \mathbb{C}$, performing strong-weak coupling interpolation and how to deal with the cancellation of ambiguities, see *e.g.* [42, 55, 60]). The first example we will discuss is of a linear ODE: the very well known example of the Airy function.

[9] This linearised ODE is directly obtained from the original ODE for the transseries $\mathcal{F}(z, \sigma)$.

[10] Given a particular transseries and using the bridge equations, many of these constants will in fact be zero.

3 The simplicity of linear differential equations: the Airy function

The Airy function example has been thoroughly studied from the point of view of resurgence and Stokes phenomena, being the quintessential example of these phenomena. It has been studied from the perspective of saddle-point analysis and hyperasymptotics (see [49,93–95] and references therein), and of resurgence techniques (see *e.g.* [10,49]). Presently, we will provide a brief analysis of the known results, together with the numerical checks and applications which can be performed. This is a very good setting to introduce many of these numerical checks, which can then be generalised to cases with more structure, such as the one studied in the following Section.

The linear ODE describing the Airy function is

$$Z''(\kappa) - \kappa\, Z(\kappa) = 0, \tag{3.1}$$

whose solutions can be written in integral form as

$$Z_\gamma = \frac{1}{2\pi \mathrm{i}} \int_\gamma du\, \mathrm{e}^{-V(u)}\,, \quad V(u) = -\kappa u + \frac{u^3}{3}. \tag{3.2}$$

The path γ is a contour chosen such that the integral converges. There are two homologically independent contours γ originating two independent solutions of (3.1), usually denoted by $Z_{\mathrm{A_i}}$ and Z_{Bi}. A general solution to (3.1) will be a linear combination of these two, forming a (two-parameter) transseries.

Given the integral form of the solution (3.2) we will analyse this problem perturbatively in two ways:

1. Take the solution (3.2) as a zero dimensional path integral and perform saddle-point analysis;
2. Construct a transseries solution and perform resurgent analysis directly from (3.1).

3.1 Saddle-point analysis

In order to construct explicit perturbative solutions of (3.2) as asymptotic expansions, one can perform saddle-point analysis. The saddle-points of the potential in (3.2) are

$$V'(u) = -\kappa + u^2 = 0 \;\Leftrightarrow\; u^\star_\pm = \pm\sqrt{\kappa}, \tag{3.3}$$

with $V\left(u^\star_\pm\right) = \mp\frac{2}{3}\kappa^{3/2}$. The leading contribution from each saddle to the exponential in the integrand of (3.2) can be found from the expansion